Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110510

Differential and Integral Equations for the 3-Variable Hermite-Frobenius-Euler and Frobenius-Genocchi Polynomials

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Received: 11 Jul. 2017, Revised: 15 Aug. 2017, Accepted: 21 Aug. 2017
Published online: 1 Sep. 2017

Abstract: The main goal of the present article is to derive some new classes of differential equations including partial and integro-differential equations for the 3-variable Hermite-Frobenius-Euler and Frobenius-Genocchi polynomials by use of the factorization method. We also perform a further investigation for aforementioned polynomials and derive corresponding homogeneous Volterra integral equations. The differential equations for these families of polynomials contain, as their special cases, the differential equations for some known special polynomials. Moreover, the inclusion of integral equations is a new and recent investigation which adds some extra attention to these polynomials.

Keywords: 3-variable Hermite-Frobenius-Euler polynomials; 3-variable Hermite-Frobenius-Genocchi polynomials; Differential equations; Integral equations.

1 Introduction

Several investigations have done to introduce and study classical and generalized forms of Apostol type polynomials systematically via various analytic means and generating functions method [16,10,1,2,14,8]. Very recently, Araci et. al. [3] introduced and studied a generalized class of 3-variable Hermite-Apostol type Frobenius-Euler polynomials systematically by use of generating method. The following class of polynomials is introduced by convoluting the 3-variable Hermite polynomials $H_n(x,y,z)$ [5] with the Apostol type Frobenius-Euler polynomials $\mathscr{F}_n^{(\alpha)}(x;u;\lambda)$ [19,17]. The convoluted special polynomials are important as they possess important properties such as recurrence and explicit relations, summation formulae, symmetric and convolution identities, algebraic properties etc. These polynomials are useful and possess potential for applications in certain problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics.

The results including explicit relations, summation formulae and symmetric identities related to the 3-variable Hermite-Apostol type Frobenius-Euler polynomials ${}_H\mathscr{F}_n^{(\alpha)}(x,y,z;u,\lambda)$ are derived in [3]. Here, we focus on establishing differential and associated integral equations related to these polynomials. We consider the following definitions:

Definition 1.1. For $\lambda \in \mathbb{C}$, $\lambda \neq 1$, the 3-variable Hermite-Frobenius-Euler polynomials(3VHFEP) ${}_HE_n^F(x,y,z;\lambda)$ are defined by the following generating function:

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HE_n^F(x,y,z;\lambda)\frac{t^n}{n!}. \tag{1}$$

Definition 1.2. For $\lambda \in \mathbb{C}$, $\lambda \neq 1$, the 3-variable Hermite-Frobenius-Genocchi polynomials (3VHFGP) ${}_HG_n^F(x,y,z;\lambda)$ are defined by the following generating

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function:

$$\left(\frac{(1-\lambda)t}{e^t-\lambda}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_HG_n^F(x,y,z;\lambda)\frac{t^n}{n!}. \quad (2)$$

Next, we consider some special cases of the 3-variable Hermite-Frobenius-Euler and Genocchi polynomials ${}_HE_n^F(x,y,z;\lambda)$ and ${}_HG_n^F(x,y,z;\lambda)$. We present these special cases in Table 1.

Table 1. Special cases of 3VHFEP and 3VHFGP

S.No.	Cases	Name of polynomial	Generating function
I.	$\lambda=-1$	3-variable Hermite-Euler polynomials [15,12]	$\left(\frac{2}{e^t+1}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_HE_n(x,y,z)\frac{t^n}{n!}$
	$\lambda=-1,\ z=0$	2-variable Hermite-Euler polynomials	$\left(\frac{2}{e^t+1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HE_n(x,y)\frac{t^n}{n!}$
	$\lambda=-1,\ x=2x,$ $y=-1;\ z=0$	Hermite-Euler polynomials	$\left(\frac{2}{e^t+1}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_HE_n(x)\frac{t^n}{n!}$
II.	$\lambda=-1$	3-variable Hermite-Genocchi polynomials	$\left(\frac{2t}{e^t+1}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_HG_n(x,y,z)\frac{t^n}{n!}$
	$\lambda=-1,\ z=0$	2-variable Hermite-Genocchi polynomials [6]	$\left(\frac{2t}{e^t+1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HG_n(x,y)\frac{t^n}{n!}$
	$\lambda=-1,\ x=2x,$ $y=-1;\ z=0$	Hermite-Genocchi polynomials	$\left(\frac{2t}{e^t+1}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_HG_n(x)\frac{t^n}{n!}$

We find that the 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ are defined by the following series representation:

$${}_HE_n^F(x,y,z;\lambda)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^F(\lambda)\,H_k(x,y,z), \quad (3)$$

which for $z=0$ becomes

$${}_HE_n^F(x,y,0;\lambda)=n!\sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{n-k}^F(\lambda)\,x^r\,y^{k-2r}}{(n-k)!\,r!\,(k-2r)}. \quad (4)$$

Similarly, we find that the 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ are defined by the following series representation:

$${}_HG_n^F(x,y,0;\lambda)=n!\sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]}\frac{G_{n-k}^F(\lambda)\,x^r\,y^{k-2r}}{(n-k)!\,r!\,(k-2r)}. \quad (5)$$

Particularly, the aforementioned forms of Hermite-Euler and Genocchi polynomials might be of great importance in several branches of pure and applied mathematics and physics, i.e. in various problems of quantum mechanics and of probability theory.

The generalized and multi-variable forms of special polynomials of mathematical physics have witnessed a significant evolution during the recent years. In particular, these polynomials provided new means of analysis for the solutions of large classes of differential equations often encountered in physical problems.

The study of differential equations is a wide field in pure and applied mathematics, physics and engineering. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion to bridge design, to interactions between neurons. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found applications. Recurrence relations have their origins in the attempt to model population dynamics. For example, the Fibonacci numbers were once used as a model for the growth of a rabbit population. In digital signal processing, recurrence relations can model feedback in a system, where outputs at one time become inputs for future time. Thus, they arise in infinite impulse response (IIR) digital filters. The linear recurrence relations are used extensively in both theoretical and empirical economics.

Definition 1.3. Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $\deg(p_n(x))=n,\ (n\in\mathbb{N}_0:=\{0,1,2,\dots\}$. The differential operators Φ_n^- and Φ_n^+ satisfying the properties

$$\Phi_n^-\{p_n(x)\}=p_{n-1}(x), \quad (6)$$

$$\Phi_n^+\{p_n(x)\}=p_{n+1}(x) \quad (7)$$

are called derivative and multiplicative operators, respectively and the polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ is called quasi-monomial, if and only if equations (1.6) and (1.7) are satisfied. Obtaining the derivative and multiplicative operators of a given family of polynomials give rise to differential equation such as

$$(\Phi_{n+1}^-\Phi_n^+)\{p_n(x)\}=p_n(x). \quad (8)$$

The method is known as the factorization method [9, 18, 11]. The main idea of the factorization method is to find the derivative operator Φ_n^- and multiplicative operator Φ_n^+. The factorization method can be equivalently treated as monomiality principle [4].

Integral equations arise in many scientific and engineering problems. Mathematical physics models, such as diffraction problems scattering in quantum mechanics, conformal mapping and water waves also contributed to the creation of integral equations. By using various methods, the differential and integral equations for some families of special polynomials are introduced, see [20,21,13,7,22]. The differential and integral equations satisfied by these special polynomials may be used to solve new emerging problems in different branches of science. Motivated by the usefulness and applications of multi-variable special families and their differential and integral equations, in this article, we derive some new classes of differential equations for the 3-variable Hermite-Frobenius-Euler and Frobenius-Genocchi polynomials. The integral equations for these polynomials are established. The corresponding differential and integral equations for certain special cases of these polynomials are also considered.

2 Recurrence relations and shift operators

In this section, we derive the recurrence relations and shift operators for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ and 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$. First, we derive the recurrence relation for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ by proving the following result:

Theorem 2.1. The 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ satisfy the following recurrence relation:

$$\begin{aligned}{}_HE_{n+1}^F(x,y,z;\lambda) &= \left(x-\tfrac{1}{1-\lambda}\right){}_HE_n^F(x,y,z;\lambda)+2ny \\ &\times {}_HE_{n-1}^F(x,y,z;\lambda)+3n(n-1)z\,{}_HE_{n-2}^F(x,y,z;\lambda) \\ &\times \tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}\binom{n}{k}{}_HE_k^F(x,y,z;\lambda)\;\; e_{n-k}^F(\lambda),\end{aligned} \tag{9}$$

where the numerical coefficients $e_k^F(\lambda)$ are related to Frobenius-Euler polynomials $E_k^F(x;\lambda)$ by following expansion:

$$e_k^F(\lambda) := -\sum_{i=0}^{k}\frac{1}{2^i}\binom{k}{i}E_{k-i}^F\left(\frac{1}{2};\lambda\right),\; e_0^F = -1,$$

$$e_1^F = -1-\frac{1}{1-\lambda}. \tag{10}$$

Proof. Differentiating both sides of generating relation (1) with respect to t and on simplification, we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_HE_{n+1}^F(x,y,z;\lambda)\tfrac{t^n}{n!} &= \left(x+2yt+3zt^2\right)\sum_{n=0}^{\infty}{}_HE_n^F(x,y,z;\lambda)\tfrac{t^n}{n!} \\ &+\tfrac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HE_k^F(x,y,z;\lambda)e_n^F(\lambda)\tfrac{t^{n+k}}{n!\,k!}.\end{aligned} \tag{11}$$

which on simplifying and applying Cauchy-product rule in the r.h.s. gives

$$\begin{aligned}\sum_{n=0}^{\infty}{}_HE_{n+1}^F(x,y,z;\lambda)\tfrac{t^n}{n!} &= \sum_{n=0}^{\infty}x\,{}_HE_n^F(x,y,z;\lambda)\tfrac{t^n}{n!} \\ &+\sum_{n=0}^{\infty}2ny\,{}_HE_{n-1}^F(x,y,z;\lambda)\tfrac{t^n}{n!}+\sum_{n=0}^{\infty}3n(n-1)z \\ &\times{}_HE_{n-2}^F(x,y,z;\lambda)\tfrac{t^n}{n!}+\tfrac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} \\ &\times{}_HE_k^F(x,y,z;\lambda)e_{n-k}^F(\lambda)\tfrac{t^n}{n!}.\end{aligned} \tag{12}$$

Equating the coefficients of same powers of t on both sides of above equation yields

$$\begin{aligned}{}_HE_{n+1}^F(x,y,z;\lambda) &= x\,{}_HE_n^F(x,y,z;\lambda)+2ny\,{}_HE_{n-1}^F(x,y,z;\lambda) \\ &+3n(n-1)z{}_HE_{n-2}^F(x,y,z;\lambda)+\tfrac{1}{1-\lambda}\sum_{k=0}^{n}\binom{n}{k} \\ &\times{}_HE_k^F(x,y,z;\lambda)e_{n-k}^F(\lambda).\end{aligned} \tag{13}$$

Solving the summation for $k=n$ in equation (13) and then using $e_0^F=-1$ in the resultant equation, we are led to assertion (9).

To find the shift operators for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$, we prove the following result:

Theorem 2.2. The shift operators for the 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ are given by

$${}_x\pounds_n^- := \frac{1}{n}D_x, \tag{14}$$

$${}_y\pounds_n^- := \frac{1}{n}D_x^{-1}D_y, \tag{15}$$

$${}_z\pounds_n^- := \frac{1}{n}D_x^{-2}D_z, \tag{16}$$

$${}_x\pounds_n^+ := \left(x-\tfrac{1}{1-\lambda}\right)+2yD_x+3zD_x^2+\tfrac{1}{1-\lambda}\textstyle\sum_{k=0}^{n-1}D_x^{n-k}\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}, \tag{17}$$

$$\begin{aligned}{}_y\pounds_n^+ &:= \left(x-\frac{1}{1-\lambda}\right)+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2 \\ &+\frac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{-(n-k)}D_y^{n-k}\frac{e_{n-k}^F(\lambda)}{(n-k)!}\end{aligned} \tag{18}$$

and

$$\begin{aligned}{}_z\pounds_n^+ &:= \left(x-\frac{1}{1-\lambda}\right)+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2 \\ &+\frac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{-2(n-k)}D_z^{n-k}\frac{e_{n-k}^F(\lambda)}{(n-k)!},\end{aligned} \tag{19}$$

where

$$D_x := \frac{\partial}{\partial x},\;\; D_y := \frac{\partial}{\partial y},\;\; D_z := \frac{\partial}{\partial z}\;\; \text{and}\;\; D_x^{-1} := \int_0^x f(\xi)d\xi.$$

Proof. Differentiating both sides of generating relation (1) with respect to x and then equating the coefficients of like powers of t on both sides of the resultant equation yields

$$\frac{\partial}{\partial x}\{{}_HE_n^F(x,y,z;\lambda)\} = n\,{}_HE_{n-1}^F(x,y,z;\lambda), \tag{20}$$

Consequently, we have

$$ {}_x\pounds_n^-\{{}_HE_n^F(x,y,z;\lambda)\} = \tfrac{1}{n}D_x\{{}_HE_n^F(x,y,z;\lambda)\} = {}_HE_{n-1}^F(x,y,z;\lambda), \tag{21}$$

which proves assertion (14).

Differentiating both sides of generating relation (1) with respect to y and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial y}\{{}_HE_n^F(x,y,z;\lambda)\} = n(n-1)\,{}_HE_{n-2}^F(x,y,z;\lambda). \tag{22}$$

The above equation can also be written as

$$\frac{\partial}{\partial y}\{{}_HE_n^F(x,y,z;\lambda)\} = n\,\frac{\partial}{\partial x}\{{}_HE_{n-1}^F(x,y,z;\lambda)\}, \tag{23}$$

which finally gives

$$\begin{aligned} {}_y\pounds_n^-\{{}_HE_n^F(x,y,z;\lambda)\} &= \frac{1}{n}D_x^{-1}D_y\{{}_HE_n^F(x,y,z;\lambda)\} \\ &= {}_HE_{n-1}^F(x,y,z;\lambda). \end{aligned} \tag{24}$$

Thus assertion (15) is proved.

Again, differentiating both sides of generating relation (1) with respect to z and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial z}\{{}_HE_n^F(x,y,z;\lambda)\} = n(n-1)(n-2)\,{}_HE_{n-3}^F(x,y,z;\lambda). \tag{25}$$

The above equation can also be written as

$$\frac{\partial}{\partial z}\{{}_HE_n^F(x,y,z;\lambda)\} = n\,\frac{\partial^2}{\partial x^2}\{{}_HE_{n-1}^F(x,y,z;\lambda)\}, \tag{26}$$

which finally gives

$$\begin{aligned} {}_z\pounds_n^-\{{}_HE_n^F(x,y,z;\lambda)\} &= \frac{1}{n}D_x^{-2}D_z\{{}_HE_n^F(x,y,z;\lambda)\} \\ &= {}_HE_{n-1}^F(x,y,z;\lambda). \end{aligned} \tag{27}$$

Thus yields assertion (16).

In order to derive the expression for raising operator (17), the following relation is used:

$$ {}_HE_k^F(x,y,z;\lambda) = ({}_x\pounds_{k+1}^-\,{}_x\pounds_{k+2}^-\cdots{}_x\pounds_{n-1}^-\,{}_x\pounds_n^-)\{{}_HE_n^F(x,y,z;\lambda)\}, \tag{28}$$

which in view of equation (21) can be simplified as:

$$ {}_HE_k^F(x,y,z;\lambda) = \frac{k!}{n!}D_x^{n-k}\{{}_HE_n^F(x,y,z;\lambda)\}. \tag{29}$$

Making use of equation (29) in recurrence relation (9), we find

$$\begin{aligned} {}_HE_{n+1}^F(x,y,z;\lambda) = \Bigg(\left(x-\frac{1}{1-\lambda}\right) + 2yD_x + 3zD_x^2 \\ + \frac{1}{1-\lambda}\sum_{k=0}^{n-1} D_x^{n-k}\,\frac{e_{n-k}^F(\lambda)}{(n-k)!}\Bigg)\,{}_HE_n^F(x,y,z;\lambda) \end{aligned} \tag{30}$$

which yields expression (17) of raising operator ${}_x\pounds_n^+$.

Now in order to derive the expression for raising operator (18), the following relation is used:

$$ {}_HE_k^F(x,y,z;\lambda) = ({}_y\pounds_{k+1}^-\,{}_y\pounds_{k+2}^-\cdots{}_y\pounds_{n-1}^-\,{}_y\pounds_n^-)\{{}_HE_n^F(x,y,z;\lambda)\}, \tag{31}$$

which in view of equation (24) can be simplified as:

$$ {}_HE_k^F(x,y,z;\lambda) = \frac{k!}{n!}D_x^{-(n-k)}D_y^{(n-k)}\{{}_HE_n^F(x,y,z;\lambda)\}. \tag{32}$$

Making use of equation (32) in recurrence relation (9), we find

$$\begin{aligned} {}_HE_{n+1}^F(x,y,z;\lambda) = \Big(\left(x-\tfrac{1}{1-\lambda}\right) + 2yD_x^{-1}D_y \\ + 3zD_x^{-2}D_y^2 + \tfrac{1}{1-\lambda}\sum_{k=0}^{n-1} D_x^{-(n-k)}D_y^{n-k} \\ \times \tfrac{e_{n-k}^F(\lambda)}{(n-k)!}\Big)\,{}_HE_n^F(x,y,z;\lambda), \end{aligned} \tag{33}$$

which yields expression (18) of raising operator ${}_y\pounds_n^+$.

Next, to find the raising operator ${}_z\pounds_n^+$, the following relation is used:

$$ {}_HE_k^F(x,y,z;\lambda) = ({}_z\pounds_{k+1}^-\,{}_z\pounds_{k+2}^-\cdots{}_z\pounds_{n-1}^-\,{}_z\pounds_n^-)\{{}_HE_n^F(x,y,z;\lambda)\}, \tag{34}$$

which in view of equation (27) can be simplified as:

$$ {}_HE_k^F(x,y,z;\lambda) = \frac{k!}{n!}D_x^{-2(n-k)}D_z^{(n-k)}\{{}_HE_n^F(x,y,z;\lambda)\}. \tag{35}$$

Making use of equation (35) in recurrence relation (9), we find

$$\begin{aligned} {}_HE_{n+1}^F(x,y,z;\lambda) = \Big(\left(x-\tfrac{1}{1-\lambda}\right) + 2yD_x^{-2}D_z + 3zD_x^{-4}D_z^2 \\ + \tfrac{1}{1-\lambda}\sum_{k=0}^{n-1} D_x^{-2(n-k)}D_z^{n-k} \\ \times \tfrac{e_{n-k}^F(\lambda)}{(n-k)!}\Big)\,{}_HE_n^F(x,y,z;\lambda) \end{aligned} \tag{36}$$

which yields expression (19) of raising operator ${}_z\pounds_n^+$.

Next, we derive the recurrence relation for the 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$ by proving the following result:

Theorem 2.3. The 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ satisfy the following recurrence relation:

$$\begin{aligned}{}_HG_{n+1}^F(x,y,z;\lambda) &= \left(x-\tfrac{n+1}{2(1-\lambda)}\right){}_HG_n^F(x,y,z;\lambda)\\ &+2ny{}_HG_{n-1}^F(x,y,z;\lambda)+3n(n-1)z{}_HG_{n-2}^F(x,y,z;\lambda)\\ &-\tfrac{1}{1-\lambda}\sum_{k=2}^{n+1}\binom{n+1}{k}{}_HG_{n-k+1}^F(x,y,z;\lambda)\ \ g_k^F(\lambda),\end{aligned} \tag{37}$$

where the numerical coefficients $g_k^F(\lambda)$ related to Frobenius-Genocchi polynomials $G_n^F(x;\lambda)$ given by following expansion:

$$g_k^F(\lambda):=\sum_{i=0}^{k}\frac{1}{2^i}\binom{k}{i}G_{k-i}^F\left(\frac{1}{2};\lambda\right),$$

$$g_0^F=0,\ \ g_1^F=\frac{1}{2}. \tag{38}$$

Proof. Differentiating both sides of generating relation (2) with respect to t and on simplification, we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_HG_{n+1}^F(x,y,z;\lambda)\tfrac{t^n}{n!} &= \left(x+2yt+3zt^2\right)\sum_{n=0}^{\infty}{}_HG_n^F(x,y,z;\lambda)\tfrac{t^n}{n!}\\ &-\tfrac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HG_n^F(x,y,z;\lambda)g_k^F(\lambda)\tfrac{t^{n+k}}{n!\,k!},\end{aligned} \tag{39}$$

which on further simplifying and applying Cauchy-product rule in the r.h.s. yields

$$\begin{aligned}\sum_{n=0}^{\infty}{}_HG_{n+1}^F(x,y,z;\lambda)\tfrac{t^n}{n!} &= \sum_{n=0}^{\infty}x\,{}_HG_n^F(x,y,z;\lambda)\tfrac{t^n}{n!}\\ &+\sum_{n=0}^{\infty}2ny\,{}_HG_{n-1}^F(x,y,z;\lambda)\tfrac{t^n}{n!}+\sum_{n=0}^{\infty}3n(n-1)z\\ &{}_HG_{n-2}^F(x,y,z;\lambda)\tfrac{t^n}{n!}-\tfrac{1}{1-\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\\ &\times{}_HG_{n-k}^F(x,y,z;\lambda)g_k^F(\lambda)\tfrac{t^n}{n!}.\end{aligned} \tag{40}$$

Equating the coefficients of same powers of t on both sides of above equation yields

$$\begin{aligned}{}_HG_{n+1}^F(x,y,z;\lambda) &= x\,{}_HG_n^F(x,y,z;\lambda)+2ny\,{}_HG_{n-1}^F(x,y,z;\lambda)\\ &+3n(n-1)z{}_HG_{n-2}^F(x,y,z;\lambda)-\tfrac{1}{1-\lambda}\sum_{k=0}^{n}\binom{n}{k}\\ &\times{}_HG_{n-k}^F(x,y,z;\lambda)g_k^F(\lambda).\end{aligned} \tag{41}$$

Now, replacing n by $n+1$ in the summation of above equation, we find

$$\begin{aligned}{}_HG_{n+1}^F(x,y,z;\lambda) &= x\,{}_HG_n^F(x,y,z;\lambda)+2ny\,{}_HG_{n-1}^F(x,y,z;\lambda)\\ &+3n(n-1)z{}_HG_{n-2}^F(x,y,z;\lambda)-\tfrac{1}{1-\lambda}\sum_{k=0}^{n+1}\binom{n+1}{k}\\ &\times{}_HG_{n+1-k}^F(x,y,z;\lambda)g_k^F(\lambda),\end{aligned} \tag{42}$$

which on solving the summation for $k=0,\ 1$ and then using $g_0^F=0$ and $g_1^F=\frac{1}{2}$ in resultant equation, we are led to assertion (37).

To find the shift operators for the Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$, we prove the following result:

Theorem 2.4. The shift operators for the 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ are given by

$${}_x\pounds_n^-:=\frac{1}{n}D_x, \tag{43}$$

$${}_y\pounds_n^-:=\frac{1}{n}D_x^{-1}D_y, \tag{44}$$

$${}_z\pounds_n^-:=\frac{1}{n}D_x^{-2}D_z, \tag{45}$$

$$\begin{aligned}{}_x\pounds_n^+ &:= \left(x-\frac{n+1}{2(1-\lambda)}\right)+2yD_x+3zD_x^2\\ &-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{k-1}\frac{g_k^F(\lambda)}{k!},\end{aligned} \tag{46}$$

$$\begin{aligned}{}_y\pounds_n^+ &:= \left(x-\frac{n+1}{2(1-\lambda)}\right)+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2\\ &-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-(k-1)}D_y^{k-1}\ \frac{g_k^F(\lambda)}{k!}\end{aligned} \tag{47}$$

and

$$\begin{aligned}{}_z\pounds_n^+ &:= \left(x-\frac{n+1}{2(1-\lambda)}\right)+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2\\ &-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-2(k-1)}D_z^{k-1}\ \frac{g_k^F(\lambda)}{k!},\end{aligned} \tag{48}$$

where

$$D_x:=\frac{\partial}{\partial x},\ \ D_y:=\frac{\partial}{\partial y},\ \ D_z:=\frac{\partial}{\partial z}\ \ \text{and}\ \ D_x^{-1}:=\int_0^x f(\xi)d\xi.$$

Proof. Differentiating both sides of generating relation (2) with respect to x and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial x}\{{}_HG_n^F(x,y,z;\lambda)\}=n\,{}_HG_{n-1}^F(x,y,z;\lambda), \tag{49}$$

Consequently, we have

$${}_x\pounds_n^-\{{}_HG_n^F(x,y,z;\lambda)\}=\tfrac{1}{n}D_x\{{}_HG_n^F(x,y,z;\lambda)\}={}_HG_{n-1}^F(x,y,z;\lambda), \tag{50}$$

which proves assertion (43).

Again, differentiating both sides of generating relation (2) with respect to y and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial y}\{{}_HG_n^F(x,y,z;\lambda)\} = n(n-1)\,{}_HG_{n-2}^F(x,y,z;\lambda). \quad (51)$$

The above equation can also be written as

$$\frac{\partial}{\partial y}\{{}_HG_n^F(x,y,z;\lambda)\} = n\,\frac{\partial}{\partial x}\{{}_HG_{n-1}^F(x,y,z;\lambda)\}, \quad (52)$$

which finally gives

$${}_y\pounds_n^-\{{}_HG_n^F(x,y,z;\lambda)\} = \frac{1}{n}D_x^{-1}D_y\{{}_HG_n^F(x,y,z;\lambda)\}$$

$$= {}_HG_{n-1}^F(x,y,z;\lambda). \quad (53)$$

Thus assertion (44) is proved.

Again, differentiating both sides of generating relation (2) with respect to z and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial z}\{{}_HG_n^F(x,y,z;\lambda)\} = n(n-1)(n-2)\,{}_HG_{n-3}^F(x,y,z;\lambda). \quad (54)$$

The above equation can also be written as

$$\frac{\partial}{\partial z}\{{}_HG_n^F(x,y,z;\lambda)\} = n\,\frac{\partial^2}{\partial x^2}\{{}_HG_{n-1}^F(x,y,z;\lambda)\}, \quad (55)$$

which finally gives

$${}_z\pounds_n^-\{{}_HG_n^F(x,y,z;\lambda)\} = \frac{1}{n}D_x^{-2}D_z\{{}_HG_n^F(x,y,z;\lambda)\}$$

$$= {}_HG_{n-1}^F(x,y,z;\lambda). \quad (56)$$

Thus yields assertion (45).

In order to derive the expression for raising operator (46), the following relation is used:

$${}_HG_{n-k+1}^F(x,y,z;\lambda) = ({}_x\pounds_{n+2-k}^-\,{}_x\pounds_{n+3-k}^-\cdots{}_x\pounds_{n-1}^-\,{}_x\pounds_n^-)$$

$$\{{}_HG_n^F(x,y,z;\lambda)\}, \quad (57)$$

which in view of equation (50) can be simplified as:

$${}_HG_{n-k+1}^F(x,y,z;\lambda) = \frac{(n+1-k)!}{n!}D_x^{k-1}\{{}_HG_n^F(x,y,z;\lambda)\}. \quad (58)$$

Making use of equation (58) in recurrence relation (37), we find

$${}_HG_{n+1}^F(x,y,z;\lambda) = \left(\left(x-\frac{n+1}{2(1-\lambda)}\right)+2yD_x+3zD_x^2\right.$$

$$\left.-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{k-1}\,\frac{g_k^F(\lambda)}{k!}\right){}_HG_n^F(x,y,z;\lambda), \quad (59)$$

which yields expression (46) of raising operator ${}_x\pounds_n^+$.

Now in order to derive the expression for raising operator (47), the following relation is used:

$${}_HG_{n+1-k}^F(x,y,z;\lambda) = ({}_y\pounds_{n+2-k}^-\,{}_y\pounds_{n+3-k}^-\cdots{}_y\pounds_{n-1}^-\,{}_y\pounds_n^-)$$

$$\times\{{}_HE_n^F(x,y,z;\lambda)\}, \quad (60)$$

which in view of equation (53) can be simplified as:

$${}_HG_{n-k+1}^F(x,y,z;\lambda) = \tfrac{(n-k+1)!}{n!}D_x^{-(k-1)}D_y^{k-1}\{{}_HG_n^F(x,y,z;\lambda)\}. \quad (61)$$

Making use of equation (61) in recurrence relation (37), we find

$${}_HG_{n+1}^F(x,y,z;\lambda) = \left(\left(x-\tfrac{n+1}{2(1-\lambda)}\right)+2yD_x^{-1}D_y\right.$$
$$\left.3zD_x^{-2}D_y^2-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-(k-1)}D_y^{k-1}\,\tfrac{g_k^F(\lambda)}{k!}\right) \quad (62)$$
$$\times\,{}_HG_n^F(x,y,z;\lambda),$$

which yields expression (47) of raising operator ${}_y\pounds_n^+$.

Next, to find the raising operator ${}_z\pounds_n^+$, the following relation is used:

$${}_HG_{n-k+1}^F(x,y,z;\lambda) = ({}_z\pounds_{n+2-k}^-\,{}_z\pounds_{n+3-k}^-\cdots{}_z\pounds_{n-1}^-\,{}_z\pounds_n^-)$$

$$\{{}_HE_n^F(x,y,z;\lambda)\}, \quad (63)$$

which in view of equation (56) can be simplified as:

$${}_HG_{n-k+1}^F(x,y,z;\lambda) = \tfrac{(n-k+1)!}{n!}D_x^{-2(k-1)}D_z^{k-1}\{{}_HG_n^F(x,y,z;\lambda)\}. \quad (64)$$

Making use of equation (64) in recurrence relation (37), we find

$${}_HG_{n+1}^F(x,y,z;\lambda) = \left(\left(x-\tfrac{n+1}{2(1-\lambda)}\right)+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2\right.$$
$$\left.-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-2(k-1)}D_z^{k-1}\,\tfrac{g_k^F(\lambda)}{k!}\right){}_HG_n^F(x,y,z;\lambda), \quad (65)$$

which yields expression (48) of raising operator ${}_z\pounds_n^+$.

In the next section, we derive some classes of differential equations for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ and 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$.

3 Differential, integro differential and partial differential equations

We derive the differential, integro-differential and partial differential equations for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ and 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$. First, we derive the differential equation for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ by proving the following result:

Theorem 3.1. The 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ satisfy the following differential equation:

$$\left(\left(x-\frac{1}{1-\lambda}\right)D_x+2yD_x^2+3zD_x^3+\frac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{n-k+1}\frac{e_{n-k}^F(\lambda)}{(n-k)!}-n\right){}_HE_n^F(x,y,z;\lambda)=0. \quad (66)$$

Proof. Use of expressions (14) and (17) of the shift operators in the the following factorization relation:

$${}_x\mathscr{L}_{n+1}^{-}\,{}_x\mathscr{L}_n^{+}\{{}_HE_n^F(x,y,z;\lambda)\}={}_HE_n^F(x,y,z;\lambda) \quad (67)$$

we are led to assertion (66).

Theorem 3.2. The 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ satisfy the following integro-differential equations:

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_y+2D_x^{-1}D_y+2yD_x^{-1}D_y^2+3zD_x^{-2}D_y^3+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{-(n-k)}D_y^{n-k+1}\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x\right){}_HE_n^F(x,y,z;\lambda)=0, \quad (68)$$

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_z+2yD_x^{-2}D_z^2+3D_x^{-4}D_z^2+3zD_x^{-4}D_z^3+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{-2(n-k)}D_z^{n-k+1}\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x^2\right){}_HE_n^F(x,y,z;\lambda)=0, \quad (69)$$

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_y+2D_x^{-2}D_z(1+yD_y)+3zD_x^{-4}D_yD_z^2+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{-2(n-k)}D_z^{n-k}D_y\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x\right){}_HE_n^F(x,y,z;\lambda)=0 \quad (70)$$

and

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_z+2yD_x^{-1}D_yD_z+3D_x^{-2}D_y^2(1+zD_z)+\tfrac{1}{1-\lambda}\textstyle\sum_{k=0}^{n-1}D_x^{-(n-k)}D_y^{n-k}D_z\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x^2\right){}_HE_n^F(x,y,z;\lambda)=0. \quad (71)$$

Proof. Use of expressions (15) and (18) of shift operators in the following factorization relation:

$${}_y\mathscr{L}_{n+1}^{-}\,{}_y\mathscr{L}_n^{+}\{{}_HE_n^F(x,y,z;\lambda)\}={}_HE_n^F(x,y,z;\lambda), \quad (72)$$

yields assertion (68).

Use of expressions (16) and (19) of shift operators in the following factorization relation:

$${}_z\mathscr{L}_{n+1}^{-}\,{}_z\mathscr{L}_n^{+}\{{}_HE_n^F(x,y,z;\lambda)\}={}_HE_n^F(x,y,z;\lambda), \quad (73)$$

yields assertion (69).

Use of expressions (15) and (19) of shift operators in the following factorization relation:

$${}_y\mathscr{L}_{n+1}^{-}\,{}_z\mathscr{L}_n^{+}\{{}_HE_n^F(x,y,z;\lambda)\}={}_HE_n^F(x,y,z;\lambda), \quad (74)$$

yields assertion (70).

Use of expressions (16) and (18) of shift operators in the following factorization relation:

$${}_z\mathscr{L}_{n+1}^{-}\,{}_y\mathscr{L}_n^{+}\{{}_HE_n^F(x,y,z;\lambda)\}={}_HE_n^F(x,y,z;\lambda), \quad (75)$$

yields assertion (71).

Theorem 3.3. The 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ satisfy the following partial differential equations:

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_x^nD_y+nD_x^{n-1}D_y+2D_x^{n-1}D_y+2yD_x^{n-1}D_y^2+3zD_x^{n-2}D_y^3+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^kD_y^{n-k+1}\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x^{n+1}\right){}_HE_n^F(x,y,z;\lambda)=0, \quad (76)$$

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_x^{2n}D_z+2nD_x^{2n-1}D_z+2yD_x^{2(n-1)}D_z^2+3D_x^{2(n-2)}D_z^2+3zD_x^{2(n-2)}D_z^3+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{2k}D_z^{n-k+1}\times\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x^{2(n+1)}\right){}_HE_n^F(x,y,z;\lambda)=0, \quad (77)$$

$$\left(\left(x-\tfrac{1}{1-\lambda}\right)D_x^{2n}D_y+2nD_x^{2n-1}D_y+2D_x^{2(n-1)}D_z+2yD_x^{2(n-1)}D_yD_z+3zD_x^{2(n-2)}D_yD_z^2+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^{2k}D_z^{n-k}D_y\tfrac{e_{n-k}^F(\lambda)}{(n-k)!}-(n+1)D_x^{2n+1}\right){}_HE_n^F(x,y,z;\lambda)=0 \quad (78)$$

and

$$\Big(\Big(x-\tfrac{1}{1-\lambda}\Big)D_x^nD_z+nD_x^{n-1}D_z+2yD_x^{n-1}D_yD_z+3D_x^{n-2}D_y^2 +3zD_x^{n-2}D_y^2D_z+\tfrac{1}{1-\lambda}\sum_{k=0}^{n-1}D_x^kD_y^{n-k}D_z\tfrac{e_{n-k}^F(\lambda)}{(n-k)!} -(n+1)D_x^{n+2}\Big)\,{}_HE_n^F(x,y,z;\lambda)=0. \tag{79}$$

Proof. On differentiation of integro-differential equations (68) and (69) n and $2n$ times with respect to x yield assertions (76) and (77), respectively.

Again, on differentiation of integro-differential equations (70) and (71) $2n$ and n times with respect to x yield assertions (78) and (79), respectively.

By making similar approach, we derive the differential, integro-differential and partial differential equations for the 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$:

Theorem 3.4. The 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ satisfy the following differential equation:

$$\Big(\Big(x-\frac{n+1}{2(1-\lambda)}\Big)D_x+2yD_x^2+3zD_x^3-\frac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^k \frac{g_k^F(\lambda)}{k!}-n\Big)\,{}_HG_n^F(x,y,z;\lambda)=0. \tag{80}$$

Theorem 3.5. The 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ satisfy the following integro-differential equations:

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_y+2D_x^{-1}D_y+2yD_x^{-1}D_y^2 +3zD_x^{-2}D_y^3-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-(k-1)}D_y^k\tfrac{g_k^F(\lambda)}{k!} -(n+1)D_x\Big)\,{}_HG_n^F(x,y,z;\lambda)=0, \tag{81}$$

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_z+2yD_x^{-2}D_z^2+3D_x^{-4}D_z^2 +3zD_x^{-4}D_z^3-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-2(k-1)}D_z^k\tfrac{g_k^F(\lambda)}{k!} -(n+1)D_x^2\Big)\,{}_HG_n^F(x,y,z;\lambda)=0, \tag{82}$$

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_y+2D_x^{-2}D_z(1+yD_y)+3zD_x^{-4}D_yD_z^2 -\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-2(k-1)}D_z^{k-1}D_y\tfrac{g_k^F(\lambda)}{k!} -(n+1)D_x\Big)\,{}_HG_n^F(x,y,z;\lambda)=0 \tag{83}$$

and

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_z+2yD_x^{-1}D_yD_z+3D_x^{-2}D_y^2(1+zD_z) -\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{-(k-1)}D_y^{k-1}D_z\tfrac{g_k^F(\lambda)}{k!} -(n+1)D_x^2\Big)\,{}_HG_n^F(x,y,z;\lambda)=0. \tag{84}$$

Theorem 3.6 The 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ satisfy the following partial differential equations:

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_x^nD_y+nD_x^{n-1}D_y+2D_x^{n-1}D_y+2yD_x^{n-1} D_y^2+3zD_x^{n-2}D_y^3-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{n-k+1}D_y^k\tfrac{g_k^F(\lambda)}{k!} -(n+1)D_x^{n+1}\Big)\,{}_HG_n^F(x,y,z;\lambda)=0, \tag{85}$$

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_x^{2n}D_z+2nD_x^{2n-1}D_z+2yD_x^{2(n-1)}D_z^2 +3D_x^{2(n-2)}D_z^2+3zD_x^{2(n-2)}D_z^3-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_z^kD_x^{2(n-k+1)} \tfrac{g_k^F(\lambda)}{k!}-(n+1)D_x^{2(n+1)}\Big)\,{}_HG_n^F(x,y,z;\lambda)=0, \tag{86}$$

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_x^{2n}D_y+2nD_x^{2n-1}D_y+2D_x^{2(n-1)}D_z+ 2yD_x^{2(n-1)}D_yD_z+3zD_x^{2(n-2)}D_yD_z^2-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_x^{2(n-k+1)} D_z^{k-1}D_y\ \tfrac{g_k^F(\lambda)}{k!}-(n+1)D_x^{2n+1}\Big)\,{}_HG_n^F(x,y,z;\lambda)=0 \tag{87}$$

$$\Big(\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_x^nD_z+nD_x^{n-1}D_z+2yD_x^{n-1}D_yD_z+3D_x^{n-2} D_y^2+3zD_x^{n-2}D_y^2D_z-\tfrac{n+1}{1-\lambda}\sum_{k=2}^{n+1}D_y^{k-1}D_x^{n-k+1}D_z \tfrac{g_k^F(\lambda)}{k!}-(n+1)D_x^{n+2}\Big)\,{}_HG_n^F(x,y,z;\lambda)=0. \tag{88}$$

Remark 3.1. Taking $\lambda=-1$ in Theorems 2.1, 2.2, 3.1, 3.2 and 3.3, we get the corresponding recurrence relation, shift operators and differential equations for the 3-variable Hermite-Euler polynomials ${}_HE_n(x,y,z)$, for this see [20].

Remark 3.2. Taking $\lambda = -1$ in Theorems 2.3, 2.4, 3.4, 3.5 and 3.6, we get the corresponding recurrence relation, shift operators and differential equations for the 3-variable Hermite-Genocchi polynomials ${}_HG_n(x,y,z)$. These results are new. For the lack of space, we omit these.

In the next section, we derive the integral equations for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ and 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$.

4 Volterra integral equations

To derive the volterra integral equations for the 3VHFEP and 3VHFGP, we prove the following results:

Theorem 4.1. The 3-variable Hermite-Frobenius-Euler polynomials ${}_HE_n^F(x,y,z;\lambda)$ satisfy the following homogeneous Volterra integral equation:

$$\begin{aligned}\Psi(x) = &-\tfrac{6(1-\lambda)}{e_3^F(\lambda)}\Big(3zn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\\ &+2yn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x\\ &+2yn(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\\ &+\Big(x-\tfrac{1}{1-\lambda}\Big)\Big(n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!}\\ &+n(n-1){}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big)\\ &-n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^3}{2!\,3!}\\ &-n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!}-n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x\\ &-{}_{\mathscr{H}}\mathscr{E}_{n}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big)+\int_0^x\Big(-\tfrac{6(1-\lambda)}{e_3^F(\lambda)}\Big(3z+2y\\ &(x-\xi)+\Big(x-\tfrac{1}{1-\lambda}\Big)\tfrac{(x-\xi)^2}{2!}\Big)-n\tfrac{(x-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.\end{aligned} \tag{89}$$

Proof. We first consider the fourth order differential equation for the 3VHFEP ${}_HE_n^F(x,y,z;\lambda)$ of the following form:

$$\Big(D_x^4+\tfrac{6(1-\lambda)}{e_3^F(\lambda)}\Big(3zD_x^3+2yD_x^2+\Big(x-\tfrac{1}{1-\lambda}\Big)D_x -n\Big)\Big)\,{}_HE_n^F(x,y,z;\lambda)=0. \tag{90}$$

Next, we find the following initial conditions:

$$\begin{aligned}{}_HE_n^F(x,y,0;\lambda) &= {}_HE_n^F(x,y;\lambda) = n!\sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{n-k}^F(\lambda)\ x^r\ y^{k-2r}}{(n-k)!\ r!\ (k-2r)}\\ &:= {}_{\mathscr{H}}\mathscr{E}_{n}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&\tfrac{d}{dx}{}_HE_n^F(x,y,0;\lambda) = n\ {}_HE_{n-1}^F(x,y,0;\lambda)\\ &= n(n-1)!\sum_{k=0}^{n-1}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{n-1-k}^F(\lambda)\ x^r\ y^{k-2r}}{(n-1-k)!\ r!\ (k-2r)}\\ &:= n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&\tfrac{d^2}{dx^2}{}_HE_n^F(x,y,0;\lambda) = n(n-1)\ {}_HE_{n-2}^F(x,y,0;\lambda)\\ &= n(n-1)(n-2)!\sum_{k=0}^{n-2}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{n-2-k}^F(\lambda)\ x^r\ y^{k-2r}}{(n-2-k)!\ r!\ (k-2r)},\\ &:= n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&\tfrac{d^3}{dx^3}{}_HE_n^F(x,y,0;\lambda) = n(n-1)(n-2)\ {}_HE_{n-3}^F(x,y,0;\lambda)\\ &= n(n-1)(n-2)(n-3)!\sum_{k=0}^{n-3}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{n-3-k}^F(\lambda)\ x^r\ y^{k-2r}}{(n-3-k)!\ r!\ (k-2r)}\\ &:= n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned} \tag{91}$$

respectively, where

$${}_{\mathscr{H}}\mathscr{E}_{s}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda) := s!\sum_{k=0}^{s}\sum_{r=0}^{[\frac{k}{2}]}\frac{E_{s-k}^F(\lambda)\ x^r\ y^{k-2r}}{(s-k)!\ r!\ (k-2r)} \tag{92}$$

for $s = n,\ n-1,\ n-2,\ n-3$.

Now, consider

$$D_x^4{}_HE_n^F(x,y,z;\lambda) = \Psi(x). \tag{93}$$

Integrating the above equation and by use of initial conditions (91), we have

$$\begin{aligned}&\tfrac{d^3}{dx^3}{}_HE_n^F(x,y,z;\lambda) = \int_0^x\Psi(\xi)d\xi+n(n-1)(n-2)\\ &\quad {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&\tfrac{d^2}{dx^2}{}_HE_n^F(x,y,z;\lambda) = \int_0^x\Psi(\xi)d\xi^2+n(n-1)(n-2)\\ &\quad {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&\tfrac{d}{dx}{}_HE_n^F(x,y,z;\lambda) = \int_0^x\Psi(\xi)d\xi^3+n(n-1)(n-2)\\ &{}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!}+n(n-1)\\ &{}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned}$$

$$\begin{aligned}&{}_HE_n^F(x,y,z;\lambda) = \int_0^x\Psi(\xi)d\xi^4+n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\\ &\quad\tfrac{x^3}{2!\,3!}+n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!}\\ &\quad+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+{}_{\mathscr{H}}\mathscr{E}_{n}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda),\end{aligned} \tag{94}$$

Use of above equations in differential equation (90), we find

$$\begin{aligned}\Psi(x) &= -\tfrac{6(1-\lambda)}{e_3^F(\lambda)}\Big(3z\Big(\int_0^x \Psi(\xi)d\xi \\ &+n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big) \\ &+2y\Big(\int_0^x \Psi(\xi)d\xi^2+n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x \\ &+n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big)+\Big(x-\tfrac{1}{1-\lambda}\Big) \\ &\times\Big(\int_0^x \Psi(\xi)d\xi^3+n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!} \\ &+n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big)\Big) \\ &+\tfrac{6n(1-\lambda)}{e_3^F(\lambda)}\Big(\int_0^x \Psi(\xi)d\xi^4+n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda) \\ &\times\tfrac{x^3}{2!\,3!}+n(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda) \\ &\times\tfrac{x^2}{2!}+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+{}_{\mathscr{H}}\mathscr{E}_{n}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big),\end{aligned} \tag{95}$$

which on simplifying and then integration of the resultant equation using the following formula:

$$\int_a^x f(\xi)\,d\xi^n=\int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}f(\xi)d\xi, \tag{96}$$

yields assertion (89).

Next, we derive the integral equation for the 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$. For this we prove the following result:

Theorem 4.2. The 3-variable Hermite-Frobenius-Genocchi polynomials ${}_HG_n^F(x,y,z;\lambda)$ satisfy the following homogeneous Volterra integral equation:

$$\begin{aligned}\Psi(x) &= \tfrac{4!(1-\lambda)}{(n+1)g_4^F(\lambda)}\Big(3zn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda) \\ &+2yn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x \\ &+2yn(n-1)\ {}_{\mathscr{H}}\mathscr{G}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)+\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big) \\ &\times\Big(n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!} \\ &+n(n-1)\ {}_{\mathscr{H}}\mathscr{G}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x+n\ {}_{\mathscr{H}}\mathscr{G}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big) \\ &-n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^3}{2!\,3!}-n(n-1) \\ &\ {}_{\mathscr{H}}\mathscr{G}_{n-2}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\tfrac{x^2}{2!}-n\ {}_{\mathscr{H}}\mathscr{G}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)x \\ &-{}_{\mathscr{H}}\mathscr{G}_{n}^{\mathscr{F}}(\mathscr{X},\mathscr{Y},\lambda)\Big)+\int_0^x \tfrac{4!(1-\lambda)}{(n+1)g_4^F(\lambda)}\Big(3z+2y \\ &(x-\xi)+\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)\tfrac{(x-\xi)^2}{2!}-n\tfrac{(x-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.\end{aligned} \tag{97}$$

Proof. We first consider the fourth order differential equation for the 3VHFGP ${}_HG_n^F(x,y,z;\lambda)$ of the following form:

$$\Bigg(D_x^4-\tfrac{4!(1-\lambda)}{(n+1)g_4^F(\lambda)}\Big(3zD_x^3+2yD_x^2+\Big(x-\tfrac{n+1}{2(1-\lambda)}\Big)D_x -n\Big)\Bigg){}_HG_n^F(x,y,z;\lambda)=0. \tag{98}$$

Using the similar approach as in Theorem 4.1, we get assertion (97). Thus we omit it.

Remark 4.1. Taking $\lambda=-1$ and using relations ${}_{\mathscr{H}}\mathscr{E}_n^{\mathscr{F}}(\mathscr{X},\mathscr{Y},-1)={}_{\mathscr{H}}\mathscr{E}_n(\mathscr{X},\mathscr{Y})$ and $e_3^F(-\lambda)=e_3$ in equation (4.1), we the following homogeneous volterra integral equation for the 3-variable Hermite-Euler polynomials ${}_HE_n(x,y,z)$:

$$\begin{aligned}\Psi(x) &= -\tfrac{12}{e_3}\Big(3zn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}(\mathscr{X},\mathscr{Y}) \\ &+2yn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}(\mathscr{X},\mathscr{Y})x \\ &+2yn(n-1)\ {}_{\mathscr{H}}\mathscr{E}_{n-2}(\mathscr{X},\mathscr{Y})+\Big(x-\tfrac{1}{2}\Big) \\ &\times\Big(n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}(\mathscr{X},\mathscr{Y})\tfrac{x^2}{2!}+n(n-1) \\ &\ {}_{\mathscr{H}}\mathscr{E}_{n-2}(\mathscr{X},\mathscr{Y})x+n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}(\mathscr{X},\mathscr{Y})\Big) \\ &-n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{E}_{n-3}(\mathscr{X},\mathscr{Y})\tfrac{x^3}{2!\,3!}) \\ &-n(n-1\ {}_{\mathscr{H}}\mathscr{E}_{n-2}(\mathscr{X},\mathscr{Y})\tfrac{x^2}{2!}-n\ {}_{\mathscr{H}}\mathscr{E}_{n-1}(\mathscr{X},\mathscr{Y})x \\ &-{}_{\mathscr{H}}\mathscr{E}_{n}(\mathscr{X},\mathscr{Y})\Big)+\int_0^x\Big(-\tfrac{12}{e_3}\Big(3z+2y(x-\xi) \\ &+\Big(x-\tfrac{1}{2}\Big)\tfrac{(x-\xi)^2}{2!}\Big)-n\tfrac{(x-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.\end{aligned} \tag{99}$$

Remark 4.2 Taking $\lambda=-1$ and using relations ${}_{\mathscr{H}}\mathscr{G}_n^{\mathscr{F}}(\mathscr{X},\mathscr{Y},-1)={}_{\mathscr{H}}\mathscr{G}_n(\mathscr{X},\mathscr{Y})$ and $g_4^F(-\lambda)=g_4$ in equation (4.9), we the following homogeneous volterra integral equation for the 3-variable Hermite-Genocchi polynomials ${}_HG_n(x,y,z)$:

$$\begin{aligned}\Psi(x) &= \tfrac{48}{(n+1)g_4}\Big(3zn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}(\mathscr{X},\mathscr{Y}) \\ &+2yn(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}(\mathscr{X},\mathscr{Y})x \\ &+2yn(n-1)\ {}_{\mathscr{H}}\mathscr{G}_{n-2}(\mathscr{X},\mathscr{Y})+\Big(x-\tfrac{n+1}{4}\Big) \\ &\times\Big(n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}(\mathscr{X},\mathscr{Y})\tfrac{x^2}{2!} \\ &+n(n-1)\ {}_{\mathscr{H}}\mathscr{G}_{n-2}(\mathscr{X},\mathscr{Y})x+n\ {}_{\mathscr{H}}\mathscr{G}_{n-1}(\mathscr{X},\mathscr{Y})\Big) \\ &-n(n-1)(n-2)\ {}_{\mathscr{H}}\mathscr{G}_{n-3}(\mathscr{X},\mathscr{Y})\tfrac{x^3}{2!\,3!}-n(n-1) \\ &\ {}_{\mathscr{H}}\mathscr{G}_{n-2}(\mathscr{X},\mathscr{Y})\tfrac{x^2}{2!}-n\ {}_{\mathscr{H}}\mathscr{G}_{n-1}^{\mathscr{F}}(\mathscr{X},\mathscr{Y})x \\ &-{}_{\mathscr{H}}\mathscr{G}_{n}(\mathscr{X},\mathscr{Y})\Big)+\int_0^x \tfrac{48}{(n+1)g_4}\Big(3z+2y(x-\xi) \\ &+\Big(x-\tfrac{n+1}{4}\Big)\tfrac{(x-\xi)^2}{2!}-n\tfrac{(x-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.\end{aligned} \tag{100}$$

Further, we remark that the corresponding results for the other special cases of polynomials given in Table 1 can be obtained by substituting suitable values of λ, x, y, z.

Acknowledgement

We would like to thank the reviewers for their careful reading, valuable suggestions and comments that have improved the manuscript substantially. The first author is supported by the Research Fund of Hasan Kalyoncu University in 2017.

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