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Differential Operators Homotopy Perturbation Method (DOHPM): an automated selection procedure for Adjustment Parameters.

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Abstract: This paper proposes a modified version of HPM method (DOHPM method), introducing in a systematic fashion, adjustment parameters in order to obtain analytical approximate solutions for nonlinear differential equations by using certain differential operators to turn a nonlinear differential equation into other of higher order. Comparison with exact solution shows that DOHPM method is highly efficient if the initial guess is chosen adequately.

Keywords: Nonlinear differential equation, Homotopy perturbation method, approximative methods.

1 Introduction

The importance of research in nonlinear differential equations is that many phenomena, practical or theoretical, are of nonlinear nature. On the engineering and science fields, physical phenomena are frequently modelled using nonlinear differential equations. Scientists who work in such disciplines constantly face the problems of solving linear and nonlinear ordinary differential equations, partial differential equations, and systems of nonlinear ordinary differential equations. Recently a wide variety of methods focused to find approximate solutions to nonlinear differential equations, as an alternative to classical methods, have been reported. Such as those based on: variational approaches [1,2,3,4,5], tanh method [6], exp-function [7,8], Adomian's decomposition method [9,10,11,12,13,14,15], parameter expansion [16], homotopy perturbation method [17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,15,36,37,38,39,40,41,42], homotopy analysis method [43,44], perturbation method [45,46], modified Taylor series method [47], optimal homotopy analysis method [48,49,50,51], differential transform method [48,49], Lie group analysis [51] among others. Also, a few exact solutions to nonlinear differential equations have been reported occasionally [52].

From all the above methods, the HPM method is one of the most employed because has been successfully used in many nonlinear problems, and its practical application is simpler than other techniques. However, HPM method often requires adjustment parameters in order to obtain better results [26,30,31,34,35,36], therefore this study proposes a systematic fashion to introduce adjustment parameters in order to enhance the HPM method. Finally, as we will see, DOHPM method is inspired in the method of coefficients undetermined, therefore will be instructive establish an analogy between them.

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This paper is organized as follows. In Section 2, a brief review of the basic idea for the HPM method is provided. Section 3, will introduce the basic idea of method of undetermined coefficients. In Section 4, we will present DOHPM method, as a modified version of HPM method. Additionally, Section 5 present two cases study. Besides a discussion on the results is presented in Section 6. Finally, a brief conclusion is given in Section 7.

2 Standard HPM

The standard Homotopy Perturbation Method (HPM) was proposed by Ji Huan He [17,18], it was introduced like a powerful tool to approach various kinds of nonlinear problems. The homotopy perturbation method can be considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted a small parameter as traditional perturbation methods. For example, HPM method requires neither small parameter nor linearization, but only few iterations to obtain accurate solutions [17, 18].

To figure out how HPM method works, consider a general nonlinear equation in the form

$$A(u) - f(r) = 0 \qquad r \in \Omega, \tag{1}$$

with the following boundary conditions

$$B(u, \frac{\partial u}{\partial \eta}) = 0, \qquad r \in \Gamma, \tag{2}$$

where$A$ is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the domain boundary for Ω. A can be divided into two operators L and N, where L is linear and N nonlinear; from this last statement, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Generally, a homotopy can be constructed in the form [17,18]

$$\begin{aligned} H(v,p) = (1-p)[L(v) - L(u_0)] + \\ p[L(v) + N(v) - f(r)] = 0, \qquad p \in [0,1], r \in \Omega. \end{aligned} \tag{4}$$

or

$$\begin{aligned} H(v,p) = L(v) - L(u_0) + p[L(u_0) + N(v) - f(r)] = 0, \\ p \in [0,1], r \in \Omega, \end{aligned} \tag{5}$$

where p is a homotopy a parameter, whose values are within range of 0 and 1, u_0 is the first approximation for the solution of (3) that satisfies the boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p.

$$v = v_0 + v_1 p + v_2 p^2 + \cdots \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms, there can be found values for the sequence u_0, $u_1, u_2, \ldots$

When $p \to 1$, it yields in the approximate solution for (1) in the form

$$v = v_0 + v_1 + v_2 + v_3 \ldots \tag{7}$$

Another way to build a homotopy, which is relevant for this paper, it is by considering

the following general equation

$$L(v) + N(v) = 0, \tag{8}$$

where $L(v)$ and $N(v)$ are the linear and no linear operators respectively. It is desired that solution for $L(v) = 0$ describes, accurately, the original nonlinear system.

By the homotopy technique, a homotopy is constructed as follows

$$(1-p)L(v) + p[L(v) + N(v)] = 0. \tag{9}$$

Again, it is assumed that solution for (9) can be written in the form (6); thus, taking the limit when $p \to 1$ results in the approximate solution of (8).

3 Basic Idea of Method of Undetermined Coefficients

As it is well known, a linear differential equation of constant coefficients

$$\begin{aligned} a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_2 y''(x) + a_1 y'(x) \\ + a_0 y(x) = f(x), \end{aligned} \tag{10}$$

can be written as [53,54]

$$(a_n D^{(n)} + a_{n-1} D^{(n-1)} + \ldots + a_2 D^2 + a_1 D + a_0) y(x) = f(x). \tag{11}$$

Expression $P(D) = a_n D^{(n)} + a_{n-1} D^{(n-1)} + \ldots + a_2 D^2 + a_1 D + a_0$ where, $D^n y = d^n y / dx^n$, is called linear differential operator of order n.

For this case, where a_i (i=0,1,2,..n) are constants, $P(D)$ can be possibly factorized, in terms of differential operators of lower order, and therefore it can be handled as an ordinary polynomial. Also it can be shown that the factors of $P(D)$ commute.

Suppose that $f(x)$ is a function which has derivatives up to order n, then if

$$(a_n D^{(n)} + a_{n-1} D^{(n-1)} + \ldots + a_2 D^2 + a_1 D + a_0) f(x) = 0, \quad (12)$$

is said that $a_n D^{(n)} + a_{n-1} D^{(n-1)} + \ldots + a_2 D^2 + a_1 D + a_0$, annihilate to $f(x)$.

For example

1.D^nannihilate the functions $1, x, x^2, x^3, \ldots, x^{n-1}$.
2.$(D-a)^n$annihilate the functions e^{ax}, xe^{ax}, x^2e^{ax}, $x^{n-1}e^{ax}$.
3.$D^2+\beta^2$ annihilate to $cos(\beta x)$ and $sen(\beta x)$.

In [53,54] can be find additional examples of annihilators operators.

Since a linear non homogeneous ordinary differential equation of constant coefficients,

can be written as

$$P(D)y = g(x), \quad (13)$$

if$P_1(D)$ is an annihilator operator of $g(x)$, then

$$P_1(D)P(D)y = 0. \quad (14)$$

General solution of linear equations is expressed as the sum of the complementary and particular solutions. Complementary solution results of solving

$$P(D)y = 0, \quad (15)$$

particular solution is finding of solving homogeneous equation (14) [53,54].

4 DOHPM method

The method of undetermined coefficients to solve nonhomogeneous differential equations converts the original nonhomogeneous equation (13) to one higher order homogeneous (14). The process for obtaining the particular solution, leads to calculate values for certain coefficients (constants), in order that the solution of (14) corresponds to the solution of (13).

In this section, we will propose by analogy how obtaining analytical approximate solutions for nonlinear differential equations, using a modified version of HPM.

Consider for instance, the homotopy given by (9), this equation can be simplified as

$$L(v) + pN(v) = 0. \quad (16)$$

It is important to notice, that in order to obtain a good approximation, DOHPM requires that the solution of $L(v) = 0$ sketches the main qualitative characteristics of (8).

The application of an adequate differential annihilator operator $P(D)$ to (16), results in the following differential equation of higher order than (16)

$$P(D)P_1(D)(v) + pP(D)N(v) = 0, \quad (17)$$

where linear operator $L(v)$ has been expressed as $P_1(D)(v) = L(v)$.

For obtaining the equation (17), we used the obvious fact that

$$P(D)(0) = 0. \quad (18)$$

Operator $P(D)$ is selected such that the solution of the equation $P(D)P_1(D)(v) = 0$, describes better the nonlinear equation (8) than $L(v) = 0$ ($P_1(D)(v) = 0$).

Next, we apply the HPM method to (17), assuming that

$$v = v_0 + v_1 p + v_2 p^2 + \ldots \quad (19)$$

Substituting (19) into (17) and equating identical powers of p terms, there can be found values for the sequence $v_0, v_1, v_2, \ldots$

When $p \to 1$, it yields in the approximate solution for (17) in the form

$$v = v_0 + v_1 + v_2 + v_3 \ldots \quad (20)$$

As occur with the method of undetermined coefficients, (17) contains additional parameters to be determined (as it will be seen, these parameters could be inserted from the beginning as part of the operator $P(D)$) . The method DOHPM consists in adjusting them, so that an approximate solution of (17) corresponds to an approximate analytic solution of (8).

In order that additional constants, resulting from the solution of (17), can be employed as adjustment parameters, it is proposed that the boundary conditions of the differential equation of lowest order resulting from (17), are the same as those of (16). Equations for the other orders are solved applying the usual procedure of HPM to (17).

5 Cases Study

5.1 Quadratic Riccati Equation

Riccati equation is an important case of nonlinear differential equation, because its applications in engineering sciences, such as stochastic realization theory, optimal control, and robust stabilization. Recent applications of this equation, includes such areas as financial mathematics [5] among others. We will consider the quadratic Ricatti equation [5,15]

$$\frac{dy}{dx} = 2y(x) - y^2(x) + 1, \qquad y(0) = 0. \quad (21)$$

It should be noticed that (21) has exact solution [15] (see Figure 1 and discussion section).

The above equation can be expressed in terms of operators as

$$(D-2+y)y=1, \tag{22}$$

An adequate homotopy, which takes into account that the first approximation for the solution of (21) satisfies the initial condition $y(0)=0$, and adopts the correct asymptotic behavior (see Figure 1) is given in accordance with (16) by

$$(D+\alpha)y+p(-\alpha-2+y)y=1, \tag{23}$$

where α is an adjustment parameter.

By substituting (6) into (23), leads to first order approximation

$$(D+\alpha)v_0=1, \qquad v_0(0)=0 \tag{24}$$

with solution $v_0(x)=\frac{1}{\alpha}(1-e^{-\alpha x})$.

Although this function satisfies the conditions $v_0(0)=0$ and $v_0(\infty)=\frac{1}{\alpha}=constant$, is too restrictive because it only contains one adjust parameter.

To improve it, we apply the annihilator operator D to (23), to obtain (see (17))

$$D(D+\alpha)y+p(-(\alpha+2)Dy+Dy^2)=0, \tag{25}$$

Substituting (19) into (25), and arranging coefficients with p powers we construct the following equations

$$p^0:(D+\alpha)Dv_0=0, \qquad v_0=0, \tag{26}$$

$$p^1:(D+\alpha)Dv_1-(\alpha+2)Dv_0+D^2v_0=0,$$
$$v_1=0, \qquad v_1'=0, \tag{27}$$

$$p^2:(D+\alpha)Dv_2-(\alpha+2)Dv_1+2v_0Dv_1+2v_1Dv_0=0,$$
$$v_2=0, \qquad v_2'=0, \tag{28}$$

$$p^3:(D+\alpha)Dv_3-(\alpha+2)Dv_2+D(v_1^2+2v_0v_2)=0,$$
$$v_3(0)=0, \qquad v_2'(0)=0 \tag{29}$$

...

Note that the initial conditions from (26) and (24) are chosen equal, in order to dispose of one additional parameter.

After solving (26), we obtain

$$p^0:v_0(x)=c_0(1-e^{-\alpha x}), \tag{30}$$

which satisfies the condition $v_0(0)=0$ and has the correct asymptotic behavior (see Figure 1).

Instead of adjusting any of the constants c_0 and α, to satisfy the initial condition $y'(0)$ of second order equation

$$D^2y-2Dy+Dy^2=0 \tag{31}$$

where (31) results from application of D to (22). We will use c_0 and α, in order that the approximate solutions of (31), correspond to approximate solutions for (21) (or (22)).

In the same way, we obtain the solutions for equations (27)-(29).

$$p^1:v_1(x)=\frac{c_0^2e^{-2\alpha x}}{\alpha}+2c_0^2e^{-\alpha x}x-e^{-\alpha x}c_0\alpha x-e^{-\alpha x}c_0$$
$$-2e^{-\alpha x}c_0x-\frac{2c_0e^{-\alpha x}}{\alpha}+\frac{c_0(-c_0+\alpha+2)}{\alpha} \tag{32}$$

$$p^2:v_2(x)=-e^{-\alpha x}c_0\alpha x-\frac{2c_0^3e^{-\alpha x}}{\alpha^2}-\frac{4c_0e^{-\alpha x}}{\alpha}$$
$$-e^{-\alpha x}c_0+\frac{3c_0^2e^{-2\alpha x}}{\alpha}+4c_0^2e^{-\alpha x}x+\frac{4c_0^2e^{-\alpha x}}{\alpha}$$
$$-4e^{-\alpha x}c_0x+\frac{c_0(\alpha^2-6c_0-3c_0\alpha+4\alpha+2c_0^2+4)}{\alpha^2}$$
$$+2c_0^2e^{-2\alpha x}x+2c_0^2\alpha e^{-\alpha x}x^2-\frac{1}{2}c_0\alpha^2e^{-\alpha x}x^2$$
$$-2c_0\alpha e^{-\alpha x}x^2+\frac{8c_0^2xe^{-\alpha x}}{\alpha}-\frac{4c_0xe^{-\alpha x}}{\alpha}$$
$$-\frac{2c_0^3xe^{-\alpha x}}{\alpha}+\frac{4c_0^2xe^{-2\alpha x}}{\alpha}-\frac{4c_0^3xe^{-2\alpha x}}{\alpha}$$
$$+4c_0^2e^{-\alpha x}x^2-2c_0e^{-\alpha x}x^2-2c_0^3e^{-\alpha x}x^2$$
$$-\frac{c_0^3e^{-3\alpha x}}{\alpha^2}+\frac{6c_0^3e^{-2\alpha x}}{\alpha^2}+\frac{8c_0^2e^{-\alpha x}}{\alpha^2}$$
$$-\frac{4c_0e^{-\alpha x}}{\alpha^2}-\frac{2c_0^3e^{-2\alpha x}}{\alpha^2}-\frac{c_0^2(-3c_0+4\alpha+8)e^{-\alpha x}}{\alpha^2}. \tag{33}$$

$$
\begin{aligned}
p^3 : v_3(x) = & -e^{-\alpha x}c_0\alpha x + \frac{c_0^3 e^{-\alpha x}}{\alpha^2} - \frac{4c_0 e^{-\alpha x}x^3}{3} - \frac{6c_0 e^{-\alpha x}}{\alpha} \\
& - c_0 e^{-\alpha x} - \frac{5c_0^2 e^{-2\alpha x}}{\alpha} - 6e^{-\alpha x}c_0 x - 2c_0^2 e^{-2\alpha x}x \\
& + 2c_0^2 e^{-\alpha x}x^2 - \tfrac{1}{2}c_0\alpha^2 e^{-\alpha x}x^2 - 3c_0\alpha e^{-\alpha x}x^2 \\
& - \frac{12c_0 x e^{-\alpha x}}{\alpha} + \frac{c_0^3 x e^{-\alpha x}}{\alpha} - \frac{8c_0^2 x e^{-2\alpha x}}{\alpha} + \frac{4c_0^3 x e^{-2\alpha x}}{\alpha} \\
& + 8c_0^2 e^{-\alpha x}x^2 - 6c_0 e^{-\alpha x}x^2 - c_0^3 e^{-\alpha x}x^2 + \frac{c_0^3 e^{-3\alpha x}}{2\alpha^2} \\
& - \frac{20c_0^2 e^{-2\alpha x}}{\alpha^2} - \frac{12c_0 e^{-\alpha x}}{\alpha^2} + \frac{6c_0^3 e^{-2\alpha x}}{\alpha^2} + \frac{2c_0^2 e^{-\alpha x}\alpha^2 x^3}{3} \\
& - \frac{2\alpha c_0^3 x^3 e^{-\alpha x}}{3} + \frac{8c_0^2 e^{-\alpha x}x^3}{3} - \tfrac{1}{6}c_0 e^{-\alpha x}\alpha^3 x^3 \\
& - c_0 e^{-\alpha x}\alpha^2 x^3 - 2\alpha c_0 e^{-\alpha x}x^3 + \frac{2c_0^3 x e^{-\alpha x}}{\alpha^2} \\
& - \frac{8c_0^2 x e^{-2\alpha x}}{\alpha^2} - \frac{2c_0^3 x^2 e^{-\alpha x}}{\alpha} + \frac{8c_0^3 x e^{-2\alpha x}}{\alpha^2} - \frac{4c_0 x^2 e^{-\alpha x}}{\alpha} \\
& - \frac{8c_0 x e^{-\alpha x}}{\alpha^2} + \frac{8c_0^2 x^2 e^{-\alpha x}}{\alpha} - \frac{4c_0^3 x^3 e^{-\alpha x}}{3} + \frac{8c_0^2 x^3 e^{-\alpha x}}{3} \\
& - \frac{20c_0^2 e^{-2\alpha x}}{\alpha^3} + \frac{c_0^3 e^{-3\alpha x}}{\alpha^3} + \frac{12c_0^3 e^{-2\alpha x}}{\alpha^3} + \frac{2c_0^3 e^{-\alpha x}}{\alpha^3} \\
& - \frac{8c_0 e^{-\alpha x}}{\alpha^3} + \frac{c_0^2(16\alpha^2+64\alpha-19c_0\alpha-38c_0+64)e^{-\alpha x}}{2\alpha^3} \\
& + \frac{c_0(\alpha^3-12c_0\alpha-3c_0\alpha^2+6\alpha^2+2c_0^2\alpha+8+12\alpha-12c_0+4c_0^2)}{\alpha^3}.
\end{aligned} \tag{34}
$$

...

and so on.

By substituting solutions (30), (32)-(34) into (19) and calculating the limit when $p \to 1$, results in a third order approximation

$$
y(x) = \lim_{p\to 1}\left(\sum_{i=0}^{3} v_i p^i\right). \tag{35}
$$

Constants c_0 and α, are calculated using the Nonlinear Fit build-in command from Maple 15, obtaining

$$
\begin{aligned}
y(x) = & \, 2.4252 - 8.0294xe^{-3.6046x} - 7.329x^3 e^{-3.6046x} \\
& - 9.9728x^2 e^{-3.6046x} - 0.15446xe^{-7.2092x} \\
& - 2.26285e^{-3.6046x} - 0.1618e^{-7.2092x} \\
& - 0.00054e^{-10.814x}.
\end{aligned} \tag{36}
$$

5.2 Approximate Solution of Gelfand's Equation.

As it is known, Gelfand's equation [44] (also known as Bratu's problem in 1D) models the chaotic dynamics in combustible gas thermal ignition. Therefore it is important to search for accurate solutions for this equation.

The equation to solve is

$$
\frac{d^2y(x)}{dx^2} + \varepsilon e^{y(x)} = 0, \qquad 0 \le x \le 1, \qquad y(0)=0, \qquad y(1)=0, \tag{37}
$$

where ε is a positive parameter, which value we choose as $\varepsilon = 3$.

It is possible to find a handy solution for (37) by applying the DOHPM method, and identifying terms:

$$
L(y) = y''(x), \tag{38}
$$

$$
N(y) = e^{y(x)}, \tag{39}
$$

where prime denotes differentiation respect to x.

To solve (37), first we expand the exponential term of Gelfand,s problem, resulting

$$
y'' + \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \ldots\right) = 0, \qquad 0 \le x \le 1, \qquad y(0)=0, \qquad y(1)=0, \tag{40}
$$

in terms of differential operators

$$
\left(D^2 + \varepsilon\left(1 + \frac{1}{2}y + \frac{1}{6}y^2\right)\right)y + \varepsilon = 0. \tag{41}
$$

In order to obtain an analytical solution we construct a homotopy in accordance with (16)

$$
\left(D^2 + p\varepsilon\left(1 + \frac{1}{2}y + \frac{1}{6}y^2\right)\right)y + \varepsilon = 0, \tag{42}
$$

applying the annihilator operator D^2 to (42), we obtain

$$
\left(D^4 + p\varepsilon\left(D^2 + \frac{1}{2}D^2 y + \frac{1}{6}D^2 y^2\right)\right)y = 0. \tag{43}
$$

By substituting (19) into (43), and equating identical powers of p, we obtain the equations

$$
p^0 : D^4 v_0 = 0, \qquad v_0(0) = 0, \qquad v_0(1) = 0, \tag{44}
$$

$$p^1 : D^4 v_1 + \varepsilon\left(D^2 v_0 + \frac{1}{2}D^2 v_0^2 + \frac{1}{6}D^2 v_0^3\right) = 0,$$

$$v_1(0) = 0, \quad v_1'(0) = 0, \quad v_1(1) = 0, \quad v_1'(1) = 0. \tag{45}$$

...

After solving the above equations, we obtain

$$p^0 : v_0(x) = A(x^3 - x) + B(x^2 - x), \tag{46}$$

$$p^1 : v_1(x) = \frac{-A^3x^{11}}{660} - \frac{A^2Bx^{10}}{180} + \left(\frac{A^2B}{144} + \frac{A^3}{144} - \frac{AB^2}{144}\right)x^9$$
$$+ \left(\frac{-B^3}{336} - \frac{A^2}{112} + \frac{AB^2}{56} + \frac{A^2B}{56}\right)x^8$$
$$+ \left(\frac{-A^3}{84} - \frac{A^2B}{42} - \frac{AB}{42} + \frac{B^3}{84}\right)x^7$$
$$+ \left(\frac{-BA^2}{60} + \frac{A^2}{30} - \frac{B^2}{60} - \frac{B^3}{60} + \frac{AB}{30} - \frac{AB^2}{30}\right)x^6$$
$$+ \left(\frac{AB}{20} + \frac{A^3}{120} + \frac{AB^2}{40} + \frac{B^2}{20} + \frac{A^2B}{40} + \frac{B^3}{120} - \frac{3A}{20}\right)x^5$$
$$+ \left(\frac{-B}{4} - \frac{B^2}{24} - \frac{A^2}{24} - \frac{AB}{12}\right)x^4$$
$$+ \left(\frac{9A}{20} + \frac{B}{2} - \frac{A^2B}{1680} - \frac{B^2A}{5040} + \frac{A^2}{280} + \frac{AB}{420} - \frac{5A^3}{11088}\right)x^3$$
$$+ \left(\frac{-3A}{10} - \frac{B}{4} - \frac{A^2B}{315} - \frac{B^2A}{420} + \frac{23A^2}{1680} + \frac{3AB}{140} + \frac{B^2}{120} - \frac{13A^3}{9240} - \frac{B^3}{1680}\right)x^2 \tag{47}$$

...

and so on.

By substituting solutions (46) and (47) into (19) and calculating the limit when $p \to 1$, results in a first order approximation

$$y(x) = \lim_{p\to 1}\left(\sum_{i=0}^{1} v_i p^i\right). \tag{48}$$

Constants A and B, are calculated using the Nonlinear Fit build-in command from Maple 15, which results in

$$\begin{aligned} y(x) = {} & 0.358856x^4 + 2.377556x + 0.04x^8 - 1.188778x^3 \\ & -1.728x^2 + 0.13x^6 + 0.17064x^5 - 0.16x^7 \\ & -1.13431514x10^{-10}x^9 + 1.102877x10^{-19}x^{10} \\ & -3.6556x10^{-29}x^{11}, \end{aligned} \tag{49}$$

where $A = -3.76891x10^{-9}$ and $B = -2.377557$.

6 Discussion

This paper proposes a modified version of HPM method, introducing in a systematic way, adjustment parameters in order to obtain analytical approximate solutions for nonlinear differential equations. DOHPM method is inspired in the method of undetermined coefficients to solve linear non homogeneous ordinary differential equations of constant coefficients, which employ the concept of annihilator operator to convert the original nonhomogeneous equation (13) into one higher order homogeneous (14). The process of obtaining the particular solution, leads to calculate values for certain coefficients (constants), in order that the solution of (14) corresponds to the solution of original equation (13). In the same fashion, DOHPM method uses differential operators to turn a nonlinear differential equation into other of higher order, however is not required that these operators, apply to solve non homogeneous equations as in the case of undetermined coefficients. The criteria for their use is rather, that the new linear part describes better the nonlinear equation to be solved than the original one, although in our examples D and D^2 are indeed annihilator operators when acts on (23) and (42) respectively.

As aforementioned, the strategy in this paper, to systematically obtain adjustment parameters, is that the boundary conditions of the differential equation of lowest order resulting from (17), are the same as those of (16), while equations for the other orders are solved applying the usual procedure of HPM to (17). Thus, instead of adjusting these parameters to the resulting higher order equation, we used them to obtain a good approximate solution of the original equation.

The above procedure proves to be a tool with great potential, especially if the solution of equation $L = 0$, describes adequately the original nonlinear equation (see (8)), because the lower-order equation resulting from the method DOHPM, directly depends on this linear equation, whereby if solution $v_0(x)$ resembles in its general characteristics to the exact solution for (8), then an appropriate adjustment of the above parameters can result in a good approximate analytical solution to (8) of the form (20). Although our examples used D and D^2 to obtain equations of higher order, other differential operators can be used, inclusive can be applied consecutively (as in the Method of Undetermined Coefficients) to obtain solutions with more parameters, and ease of adjustment. As first case study we chose the case of Riccati equation (21), which has the exact solution $y(x) = 1 + \sqrt{2}Tanh\left(\sqrt{2}x + \frac{1}{2}Ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$ [15]. Figure 2 and Figure 3 show that third order approximation (36) is a good approximation, with a maximum absolute error less than 0.07, it is expected that if more terms are considered of (20), a better approximation will be obtained, however, if we operate on (23) with $D(D+\beta)$ instead of D, (where β is other adjustment parameter) the lowest-order equation (26) would have been $(D+\beta)(D+\alpha)Dv_0 = 0$, and the solution satisfying $v_0 = 0$, would be $v_0(X) = B(e^{-\alpha x} - 1) + C(e^{-\beta x} - 1)$, thus we would have four adjustment parameters B, C, α and β instead of two. For instance, if we require of just three parameters, then we would make the above solution also satisfies $v_0'(0) = 1$, deduced from (21), incidentally it gives greater accuracy to the initial part of the approach. It is clear that applying more operators of the form $(D+\gamma)$ to (23) is allowed, and thus more parameters are obtained, although, we may face computational problems by the requirement to adjust many parameters. In particular Figure 2 compares (36) with the HPM third order approximation obtained following the standard procedure explained in section 2. The above mentioned figure shows the accuracy of DOHPM for the whole domain $x \geq 0$, while HPM rapidly diverges from the exact solution. On that matter, unlike of HPM, DOHPM zero approximation employs exponential terms with negative exponents to model the correct asymptotic behavior. In this case study, we guarantee that the linear part produce negative exponential terms; improving notoriously the convergence [26]. In our second case study, we solved approximately Gelfand,s equation. Figure 4 shows the comparison between (order 1) approximation (49) for $\varepsilon = 3$ with the four order Runge Kutta (RK4) numerical solution. It can be noticed that figures are very similar showing the accuracy of (49). This is confirmed by Figure 5, which shows that the maximum absolute error is about 0.0035, this proves the efficiency of DOHPM method, especially because only was considered the first-order approximation. This is a consequence of applying D^2 to (42), introducing the adjustment parameters A and B. Is important to note that the lowest order approximation (46) resembles in its general characteristics to the exact solution for (37) (see Figure 4). Finally in order to compare the accuracy of our results, the same Figure 4 also shows the HPM third order approximation for Gelfand's problem. We can notice that DOHPM approximations are more precise (zero and first order), although we consider the third-order approximation of HPM. This indicates that the proposed methodology adequately accelerates the convergence of the proposed problem.

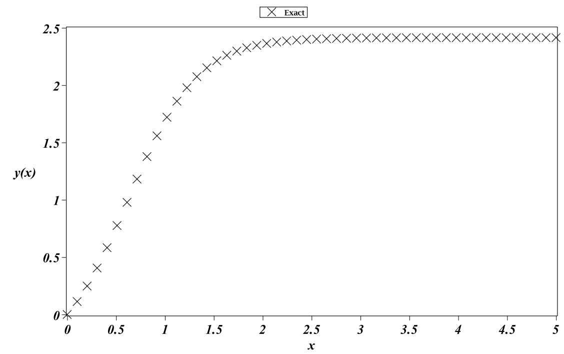

Fig. 1: Exact solution of (21)

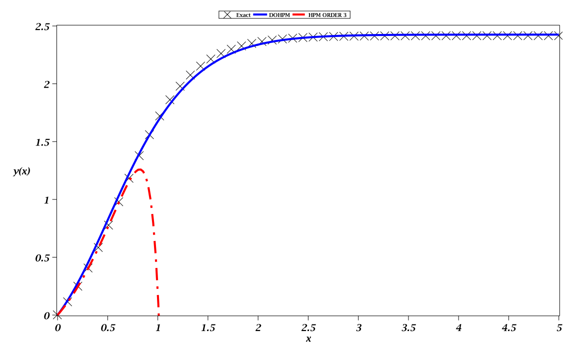

Fig. 2: Exact solution of (21) (diagonal cross), DOHPM approximation (36) (solid line) and HPM third order approximation (dash-dot).

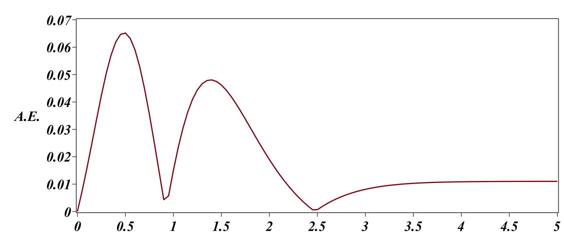

Fig. 3: Absolute error (A.E.) DOHPM approximation (36) for (21).

7 Conclusions

This work presented DOHPM method as a novel modification for HPM method, with high potential to solve no linear differential equations. The method works in a similar way to method of undetermined coefficients to solve linear non homogeneous differential equations, but in this case differential operators are applied on both sides of an homotopy equation, in order to obtain higher-order equations in the successive stages of the method.

As mentioned, the strategy in this paper, was systematically getting adjustment parameters, so that the initial conditions of the differential equation of lowest order resulting from (17), were the same as those of (16), while equations for the other orders were solved applying the usual procedure of HPM to (17). Thus, we disposed of adjustment parameters, which is especially important if solution $v_0(x)$ resembles in its general characteristics to the exact solution for (8). In that case, an appropriate adjustment of the above parameters can result in a good approximate analytical solution to (8) of the form (20).

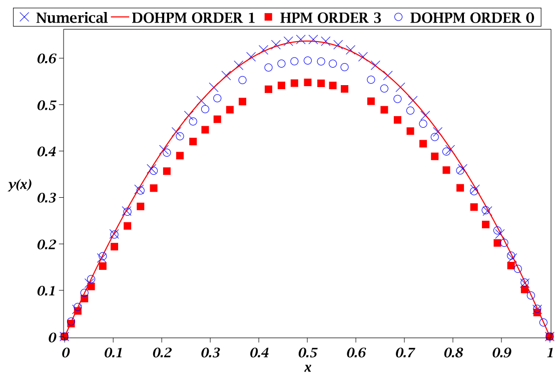

Fig. 4: Order zero (46) (circles) and order one (49) (solid line). DOHPM approximations for (37) (diagonal cross) and HPM third order approximation (squares).

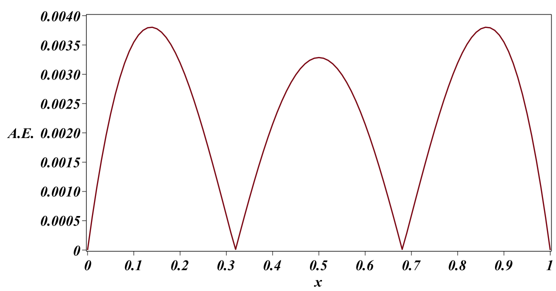

Fig. 5: Absolute error (A.E.) of DOHPM approximation (49) for (37).

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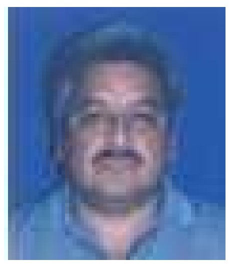

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