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A New Mathematical Model and its Application in the Pollution of Air and Water: An Application of Virtual Experience in Qassim Province in Kingdom of Saudi Arabia

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Abstract: In this paper, we present a numerical model of one-dimensional equations of transport equation, where we apply the model in a hypothetical example and compare the results of our model with a virtual experience which deals with the concentration of certain pollutants and their speed diffusion in water in Qassim province in Kingdom of Saudi Arabia.

1 Introduction

In the last four decades, the used of mathematical methods for iterative numerical modeling of large-scale air monitoring in the area, which is slow in the convergence of physical and mathematical equations consisting climatic phenomena solutions. In recent years, the use of alternative methods is fast to solve these equations, one of these methods is Galerkin method. In this project, we will use the finite element method which belongs to the family in general methods of Galerkin. These are used instead of the style differences limited (considered simplistic) to resolve the horizontal and vertical fields in numerical models. Galerkin method can be used in solving systems of equations and partial differential equations and the problems of border free evolutionary (see reference [6], [1], [2],[7], [8] [9], [5]) do not use this method directly in the field of values at grid points after the partition of scale, so as to improve the accuracy of ordinary differential equations systems solutions. There are two methods in this process: the specific elements of the functions of the zero-way, and the spectral way. Galerkin method is one of the ways of Applied Mathematics. And put it in space by using the principles inherited from the transformative formulation, separate mathematical algorithm to find approximate solution in the free boundary problem (see [3]). Galerkin methods differ from the spectral methods because it is not exhaustive, but rather than to determine local values. However, it is a roughly distinct function defined on the whole region and not just separate points (see [2]).

This paper is organized as follows: In the first part, we lay down some assumptions and notations needed throughout the paper. In addition, we study the estimates behavior of the semi discrete problem and prove the stability analysis in Sobolev spaces of Galerkin method for transport equation. In the second section, we apply this model to the hypothetical example and compare the results of this model with a virtual experience for which we will deal with the concentration of certain pollutant and their speed diffusion in air and water.

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2 Finite element methods for hyperbolic equations

We consider the following the advection diffusion equation: find $u(x,t)$ such that $u \in L^2\left(0,T,H^1\right), u_t \in L^2\left(0,T,L^2(\Omega)\right)$

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \left(D\Delta u + \overrightarrow{v}\nabla u\right) = f & \text{on } \Sigma, \\[2mm]
u = 0 \text{ in } \Gamma \times [0,T], \\[2mm]
u(.,0) = u_0 \text{ in } \Omega,
\end{cases}
\tag{1}
$$

where D is a diffusion coefficient satisfies and $\overrightarrow{v}$ is the average velocity satisfies

$$
\bar{v} \in L^2\left(0,T,L^\infty(\Omega)\right) \cap C^0\left(0,T,H^{-1}(\Omega)\right).
$$

Σ is a set in $\mathbb{R}^N \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, and Ω is a smooth bounded domain of $\mathbb{R}^N$ with boundary Γ and the right hand side f is a regular function satisfies

$$
f \in L^2\left(0,T,L^\infty(\Omega)\right) \cap C^1\left(0,T,H^{-1}(\Omega)\right).
\tag{2}
$$

It can be reformulated the equation (1) by the following weak formulation: find $u \in L^2\left(0,T,H^1(\Omega)\right), u_t \in L^2\left(0,T,L^2(\Omega)\right)$

$$
\begin{cases}
(u_t,v) + a(u,v)_\Omega = (f,v)_\Omega, \\[2mm]
u(.,0) = u_0,\ v \in H^1(\Omega), \\[2mm]
u = 0 \text{ in } \Gamma \times [0,T],
\end{cases}
\tag{3}
$$

where

$$
a(u,v) = D(\nabla u, \nabla v)_\Omega + \frac{1}{2}\left(\left(\overrightarrow{v}\nabla u, v\right)_\Omega - \left(\overrightarrow{v}\nabla v, u\right)_\Omega\right).
\tag{4}
$$

The symbol $(.,.)_\Omega$ signifies the inner product in $L^2(\Omega)$ and $\langle .,.\rangle_\Gamma$ indicate the inner product of $L^2(\Gamma_i)$.

We discretize the problem (3) with respect to the time using Euler scheme, then we have

$$
\begin{cases}
\left(\dfrac{u^k - u^{k-1}}{\Delta t}, v\right) + a\left(u^k, v\right)_\Omega = \left(f^k, v\right)_\Omega \text{ in } \Omega, \\[2mm]
u^0(x) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega
\end{cases}
\tag{5}
$$

implies

$$
\begin{cases}
\left(\dfrac{u^k}{\Delta t}, v\right) + a\left(u^k, v\right)_\Omega = \left(f^k + \dfrac{u^{k-1}}{\Delta t}, v\right)_\Omega \text{ in } \Omega, \\[2mm]
u^0(x) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.
\end{cases}
\tag{6}
$$

It can be reformulated (5) to the following coercive system of elliptic variational equation

$$
\begin{cases}
b\left(u^k, v\right) = \left(f^k + \lambda u^{k-1}, v\right) = \left(F\left(u^{k-1}\right), v\right), \\[2mm]
u^0(x) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{7}
$$

such that

$$
\begin{cases}
b\left(u^k, v\right) = \lambda\left(u^k, v\right) + a\left(u^k, v\right),\ u^k \in H_0^1(\Omega), \\[2mm]
\lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n},\ k = 1,...,n.
\end{cases}
\tag{8}
$$

In our paper we will interest by the experiment side of the one-dimensional of transport equations

2.1 One-dimensional of transport equations

Consider the following one-dimensional of transport equations

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} + a_0 u = f \text{ in } Qt =]\alpha, \beta[\times]0,T[\\[2mm]
u(\alpha,t) = \varphi(t),\ t \in]0,T[, \\[2mm]
u(x,0) = u(x) \quad x \in \Omega.
\end{cases}
\tag{9}
$$

We multiply (9) by $\upsilon \in H_0^1(\Omega)$ and integrate on $]\alpha, \beta[\subset \mathbb{R}$, then we have for all $x \in H_0^1(\Omega)$:

$$
\int_\Omega \frac{\partial u}{\partial t}\upsilon dx + \int_\Omega a\frac{\partial u}{\partial x}\upsilon dx + \int_\Omega a_0 u\upsilon dx = \int_\Omega f\upsilon dx.
\tag{10}
$$

Using the Green formula we have

$$
\int_\Omega \left(\frac{\partial u}{\partial t}v - a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + a_0 uv\right)dx + \int_\Gamma \frac{\partial u}{\partial \eta}vd\sigma = \int_\Omega f\upsilon dx.
\tag{11}
$$

Problem (11) becomes:

$$
b(u,v) = l(v), \text{ for all } v \in H_0^1(\Omega),
\tag{12}
$$

where

$$
b(u,\upsilon) = \int_\Omega \left(\frac{\partial u}{\partial t}v - a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + a_0 uv\right)dx \text{ and } l(\upsilon) = \int_\Omega f\upsilon dx
$$

By using theorem of Lax-Milgram, it can be easily to prove the problem (3) admits a unique solution.

2.1.1 The discrete problem

We introduce a subspace V^h of finite dimensional space V, then we define the approximate solution u_h of the solution u as the solution of the following problem:: find $u_h \in V^h$ solution of

$$b(u_h, v_h) = l(v_h), \text{for all } v_h \in V^h \qquad (13)$$

where

$$V^h = \left\{ \begin{array}{l} v_h \in C^2(\Omega) \cap H^1(\Omega), \\ v_h \mid_{I=[a,b]} \in P_1 \text{ in } \Omega, \ u(\alpha) = 0 \end{array} \right\} \qquad (14)$$

2.1.2 The space discretization

We defined the following space:

$$V_h^1 = \left\{ v_h \in c^0 [\alpha, \beta] : v_h \setminus I_j \in P_1(I_j), \forall I_j \in \tau_h \right\}. \qquad (15)$$

We have

$$\left\{ \begin{array}{l} \int_\alpha^\beta \frac{\partial u_h}{\partial t} v_h dx + \int_\alpha^\beta \left(a \frac{\partial u_h}{\partial x} + a_0 u_h \right) v_h dx = \int_\alpha^\beta f \, v_h dx \quad \forall v_h \in V_h^1, \\ u_h(t) = \varphi_h(t) \ , x = \alpha, \\ u_h(0) = u_{0,h}, \end{array} \right. \qquad (16)$$

2.2 Existence and uniqueness of the solution

We have for all $v_h \in V_h^1$,

$$\left\{ \begin{array}{l} \int_\alpha^\beta \frac{\partial u_h}{\partial t} v_h dx + \int_\alpha^\beta \left(a \frac{\partial u_h}{\partial x} + a_0(t) \right) u_h v_h dx = \int_\alpha^\beta f(t) v_h dx \\ u_{h(t)} = \varphi_h(t) \qquad en \ x = \alpha, \\ u_h(0) = u_0. \end{array} \right. \qquad (17)$$

We can set $v_h = u_h(t)$, we get

$$\int_\alpha^\beta \frac{\partial u_h}{\partial t} u_h + \int_\alpha^\beta \left(a \underbrace{\frac{\partial u_h}{\partial x} u_h}_{\zeta} + a_0 u_h^2 \right) dx = \int_\alpha^\beta f(t) u_h dx. \qquad (18)$$

Then

$$\int_\alpha^\beta a(x) \frac{\partial u_h}{\partial x} u_h dx = \left[\frac{1}{2} a(x) u_h^2 \right]_\alpha^\beta - \frac{1}{2} \int_\alpha^\beta a\prime(x) u_h^2 dx.$$

so

$$\int_\alpha^\beta a(x) \frac{\partial u_h}{\partial x} u_h dx = \left[\frac{1}{2} a(x) u_h^2 \right]_\alpha^\beta - \frac{1}{2} \int_\alpha^\beta a\prime(x) u_h^2 dx.$$

Then

$$\int_\alpha^\beta a(x) \frac{\partial u_h}{\partial x} u_h dx = \frac{1}{2} a(\beta) u_h^2(\beta)$$

$$- \frac{1}{2} a(\alpha) \underbrace{u_h^2(\alpha)}_{=0} - \frac{1}{2} \int_\alpha^\beta a\prime(x) u_h^2 dx$$

Therefore,

$$\int_\alpha^\beta a(x) \frac{\partial u_h}{\partial x} u_h dx = \frac{1}{2} a(\beta) u_h^2(\beta) - \frac{1}{2} \int_\alpha^\beta a\prime(x) u_h^2 dx \qquad (19)$$

Equation (18) becomes

$$\frac{1}{2} \frac{\partial}{\partial t} \int_\alpha^\beta u_h^2 dx - \frac{1}{2} a(\beta) u_h^2(\beta) \\ + \int_\alpha^\beta \left(a_0(t) u_h^2 - \frac{1}{2} a\prime(x) u_h^2 \right) dx = \int_\alpha^\beta f(t) u_h dx. \qquad (20)$$

We set

$$I = \frac{1}{2} \frac{\partial}{\partial t} \int_\alpha^\beta u_h^2 dx - \frac{1}{2} a(\beta) u_h^2(\beta)$$

$$+ \int_\alpha^\beta \left(a_0(t) - \frac{1}{2} a\prime(x) \right) u_h^2 dx$$

and

$$II = \int_\alpha^\beta f(t) u_h dx.$$

Suppose

$$0 \le \mu_0 \le \left(a_0(t) - \frac{1}{2} a\prime(x) \right). \qquad (21)$$

By using Young inequality

$$\int_\alpha^\beta f(t) u_h dx \le \frac{\|f(t)\|_{l^2}}{2} + \frac{\|u_h\|_{l^2}}{2}. \qquad (22)$$

We can write

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_h\|_{l^2}^2 + \frac{1}{2}a\left(\beta\right)u_h^2\left(\beta\right) + \int_\alpha^\beta \mu_0 u_h^2 dx \leq I$$

and

$$II \leq \frac{\|f\left(t\right)\|_{l^2}^2}{2} + \frac{\|u_h\|_{l^2}^2}{2}.$$

Using (20) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_h\|_{l^2}^2 + \frac{1}{2}a\left(\beta\right)u_h^2\left(\beta\right) + \mu_0\|u_h\|_{l^2}^2$$

$$\leq I$$

and

$$II \leq \frac{\|f\left(t\right)\|_{l^2}}{2} + \frac{\|f\left(t\right)\|_{l^2}}{2}$$

Integrating on $[0,t]$

$$\frac{1}{2}\left[\|u_h\left(t\right)\|_{l^2}^2 - \|u_{h0}\|_{l^2}^2\right] - \frac{1}{2}a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau$$

$$+\int_0^t \mu_0\|u_h\left(t\right)\|_{l^2}^2 d\tau \leq \int_0^t f\left(\tau\right)u\left(\tau\right)d\tau dx.$$

Then

$$\|u_h\left(t\right)\|_{l^2}^2 - \|u_{h0}\|_{l^2}^2 - a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau$$

$$+2\int_0^t \mu_0\|u_h\|_{l^2}^2 \leq 2\int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau,$$

thus

$$\|u_h\left(t\right)\|_{l^2}^2 + \int_0^t 2\mu_0\|u_h\left(\tau\right)\|_{l^2}^2 d\tau$$

$$-a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau \leq 2\int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau + \|u_{h0}\|_{l^2}^2,$$

or

$$\|u_h\left(t\right)\|_{l^2}^2 + 2\int_0^t \mu_0\|u_h\left(\tau\right)\|_{l^2}^2 d\left(t\right)$$

$$-a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau \leq 2\int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau + \|u_{h0}\|_{l^2}^2$$

implies

$$\|u_h\|_{l^2}^2 + 2\int_0^t \mu_0\|u_h\left(\tau\right)\|_{l^2}^2 d\left(\tau\right)$$

$$-a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau \leq 2\int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau + \|u_{h0}\|_{l^2}^2.$$

Then

$$\|u_h\|_{l^2}^2 + 2\int_0^t \mu_0\|u_h\left(\tau\right)\|_{l^2}^2 d\left(\tau\right)$$

$$-a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau \leq 2\underbrace{\int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau}_{A} + \|u_{h0}\|_{l^2}^2.$$

$$(23)$$

By Using Cauchy Schwarz inequality

$$A = \int_0^t\int_\alpha^\beta f\left(\tau\right)u\left(\tau\right)dxd\tau$$

$$= \int_0^t \|f\left(\tau\right)\|_{l^2}\|u_h\left(\tau\right)\|_{l^2} d\left(\tau\right).$$

Using Young inequality, we have for all $n \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+^*$

$$m \times n \leq \varepsilon m^2 + \frac{n^2}{4\varepsilon}$$

$$A = \int_0^t \|f\left(\tau\right)\|_l^2\|u_h\left(\tau\right)\|_{l^2} d\left(\tau\right)$$

$$\leq \int_0^t \left(\varepsilon\|u_h\left(\tau\right)\|_{l^2}^2 + \frac{\|f\left(\tau\right)\|^2}{4\varepsilon}\right)d\tau.$$

We put $\varepsilon = \frac{1}{2}\mu_0$. Then

$$A \leq \int_0^t \left(\frac{1}{2}\mu_0\|u_h\left(\tau\right)\|_{l^2}^2 + \frac{\|f\left(\tau\right)\|_{l^2}^2}{4\mu_0}\right)d\tau$$

Problem (23) equivalent to

$$\|u_h\left(t\right)\|_{l^2}^2 + \mu_0\int_0^t \|u_h\left(\tau\right)\|_{l^2}^2 d\left(\tau\right)$$

$$+a\left(\beta\right)\int_0^t u_h^2\left(\beta,\tau\right)d\tau$$

$$\leq \int_0^t \left(\frac{\|f\left(\tau\right)\|_{l^2}^2}{\mu_0} + \mu_0\|u_h\left(\tau\right)\|_{l^2}^2\right)d\left(\tau\right) + \|u_{h0}\|_{l^2}^2$$

$$\|u_h(t)\|_{l^2}^2 + \int_0^t \left(2\mu_0 \|u_h(\tau)\|_{l^2}^2 - \mu_0 \|u_h(\tau)\|_{l^2}^2\right) d\tau$$

$$+a(\beta)\int_0^t u_h^2(\beta,\tau)\,d\tau$$

$$\leq \frac{1}{\mu_0}\|f(\tau)\|_{l^2}^2\,dx + \|u_{h\,0}\|_{l^2}^2,$$

thus, we have

$$\|u_h(t)\|_{l^2}^2 + \int_0^t \mu_0 \|u_h(\tau)\|_{l^2}^2 \, d(\tau) + \int_0^t a(\beta) u_h^2(\beta,\tau)\,d\tau$$

$$\leq \int_0^t \frac{1}{\mu_0}\|f(\tau)\|_{l^2}^2\,dx + \|u_{h0}\|_{l^2}^2$$

If we have in particular cases of f and a_0 are null, we get

$$\|u_h(t)\|_{l^2(\Omega)}^2 \leq \|u_{0h}\|_{l^2(\Omega)}^2$$

which reflects the stability of the energy of the system.

2.3 The time discretization

By sing Euler time method of the problem (12). The problem can be reformulated, for all $n \geq 0$: find $u_h^n \in V_h$

$$\frac{1}{\Delta t}\int_\alpha^\beta \left(u_h^n - u_h^{n-1}\right) v_h dx + \int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} v_h dx \tag{24}$$

$$+\int_\alpha^\beta a_0^n v_h dx = \int_\alpha^\beta f^n v_h dx$$

with $u_h^n(\alpha)=\varphi^n$ and $u_h^0 = u_{0h}$ if $f=0$ and $\tau=0$, we set $v_h = u_h^n$ in (24) we find

$$\frac{1}{\Delta t}\int_\alpha^\beta \left(u_h^n - u_h^{n-1}\right) u_h^n dx \tag{25}$$

$$+\int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} u_h^n dx + \int_\alpha^\beta a_0^n \left(u_h^n\right)^2 dx = 0$$

The left hand of (25) less than

$$\frac{1}{\Delta t}\|u_h^n\|_{l^2(\Omega)}^2 - \frac{1}{\Delta t}\left(\frac{\int_\alpha^\beta (u_h^n)^2}{2} + \frac{\int_\alpha^\beta (u_h^{n-1})^2}{2}\right)$$

$$+\int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} u_h^n dx + \int_\alpha^\beta a_0^n (u_h^n)^2\,dx,$$

$$\leq \frac{1}{\Delta t}\|u_h^n\|_{l^2(\Omega)}^2 - \frac{1}{\Delta t}\left[\frac{\|u_h^n\|_{l^2(\Omega)}^2}{2} + \frac{\|u_h^{n-1}\|_{l^2(\Omega)}^2}{2}\right]$$

$$+\int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} u_h^n dx + \int_\alpha^\beta a_0^n (u_h^n)^2\,dx$$

$$\leq \frac{1}{2\Delta t}\|u_h^n\|_{l^2(\Omega)}^2 - \frac{1}{2\Delta t}\|u_h^{n-1}\|_{l^2(\Omega)}^2$$

$$+\underbrace{\int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} u_h^n dx + \int_\alpha^\beta a_0^n (u_h^n)^2\,dx}_{B''}$$

$$\leq \frac{1}{2\Delta t}\|u_h^n\|_{l^2(\Omega)}^2 - \frac{1}{2\Delta t}\|u_h^{n-1}\|_{l^2(\Omega)}^2 + B'$$

calculation of B''

$$B'' = \int_\alpha^\beta a(x)\frac{\partial u_h^n}{\partial(x)} u_h^n dx + \int_\alpha^\beta a_0^n (u_h^n)^2\,dx$$

thus, we have

$$B'' = \frac{1}{2}a(\beta)(u_h^n)^2(\beta)$$

$$+\int_\alpha^\beta \left(a_0^n - \frac{1}{2}a'(x)\right)(u_h^n)^2\,dx$$

Using (21)

$$\mu_0 \leq a_0(x,t) - \frac{1}{2}a'(x),$$

then

$$B'' = \frac{1}{2}a(\beta)(u_h^n)^2(\beta) + \int_\alpha^\beta \left(a_0^n - \frac{1}{2}a'(x)\right)(u_h^n)^2$$

$$\geq \frac{1}{2}a(\beta)(u_h^n)^2(\beta) + \mu_0 \|u_h^n\|_{l^2}^2.$$

Then, we deduce

$$\frac{1}{2\Delta t}\left(\left\|u^{n+1}\right\|_{l^2}^2 - \left\|u_h^n\right\|_{l^2}^2\right) \tag{26}$$

$$+a(\beta)u^n(\beta) + \mu_0\left\|u^{n+1}\right\|_{l^2}^2 \leq 0.$$

We Sum from 0 to $m-1$, we get, for $m \geq 1$,

$$\left\|u^m\right\|_{l^2}^2$$

$$+2\Delta t \left(\sum_{j=1}^{m}\left\|u_h^j\right\|_{l^2}^2 + \sum_{i=1}^{m} a(\beta)u^{j+1}(\beta)\right)^2$$

$$\leq \left\|u_h^0\right\|_{l^2}^2.$$

In particular, we can conclude that for all $m \geq 0$.

$$\left\|u_h^m\right\|_{l^2} \leq \left\|u_h^0\right\|_{l^2}^2$$

2.4 Matrix form

We have:

$$\begin{cases} \frac{1}{\Delta t}\int_\alpha^\beta \left(u_h^{n+1} - u_h^n\right)v_h dx + \int_\alpha^\beta a\frac{\partial u_h}{\partial x}v_h dx \\ + \int_\alpha^\beta a_0 u^{n+1}v_h dx = \int_\alpha^\beta f^{n+1}v_h dx, \end{cases} \tag{27}$$

$$u_{(\alpha)}^{n+1} = \varphi^{n+1},$$

$$u_h^0 = u_{0h}$$

thus

$$\frac{1}{\Delta t}\int_\alpha^\beta u_h^{n+1}v_h dx - \frac{1}{\Delta t}\int_\alpha^\beta u_h^n v_h dx$$

$$+\int_\alpha^\beta a\frac{\partial u_h^{n+1}}{\partial x}v_h dx + \int_\alpha^\beta a_0 u^{n+1}v_h dx$$

$$=\int_\alpha^\beta f^{n+1}v_h dx$$

or (Π) implies

$$\upsilon^h = \sum_{j=1}^{N-1} v_j \varphi_j,$$

$$u^h = \sum_{i=1}^{N-1} u_i \varphi_i,$$

We can write

$$u_h^{n+1} = \sum_{i=0}^{n} u_i^{n+1}\varphi_i; \tag{28}$$

$$\varphi_h = \sum_{j=0}^{n} v_j \varphi_i$$

and we have

$$\int_\alpha^\beta u_h^{n+1}v_h dx - \int_\alpha^\beta u_h^n v_h dx$$

$$+\Delta t \int_\alpha^\beta a(x)\frac{\partial u_h^{n+1}}{\partial x}v_h dx$$

$$+\Delta t \int_\alpha^\beta a_0(x)u_h^{n+1}v_h dx$$

$$= \Delta t \int_\alpha^\beta f^{n+1}(x)v_h dx.$$

We set

$$A\prime = \int_\alpha^\beta u_h^{n+1}v_h dx - \int_\alpha^\beta u_h^n v_h dx,$$

$$B\prime = \Delta t \int_\alpha^\beta a(x)\frac{\partial u_h^{n+1}}{\partial x}v_h dx$$

$$+\Delta t \int_\alpha^\beta a_0(x)u_h^{n+1}v_h dx$$

and

$$C\prime = \Delta t \int_\alpha^\beta f^{n+1}(x)v_h dx.$$

Using (28), we have

$$A\prime = \int_{\sup p\varphi_i \cap \sup p\varphi_j} \left(\begin{array}{c}\sum_{i=1}^{n} u^{n+1} \\ \varphi_i \sum_{j=1}^{n} v_j \varphi_i\end{array}\right) dx \tag{29}$$

$$- \int_{\sup p\ \varphi_i \cap\ \sup p\varphi_j} \left(\begin{array}{c}\sum_{i=1}^{n} u^n \varphi_i. \\ \sum_{j=1}^{n} v_j \varphi_i\end{array}\right) dx$$

with

$$B\prime = \Delta t \int_{\sup p\varphi_i \cap \sup port\varphi_j} \left[\begin{array}{c}\left(\sum_{i=1}^{n} \frac{\partial}{\partial x}u_i^{n+1}\varphi_i\right) \\ \left(\sum_{j=1}^{n} v_j \varphi_i\right)a_0(x)\end{array}\right] dx$$

$$+\Delta t \int_{\sup p\varphi_i \cap \sup p\varphi_j} a_0^{n+1}(x) \left[\begin{array}{c}\left(\sum_{i=1}^{n} u_i^n \varphi_i\right) \\ \left(\sum_{i=1}^{n} v_j \varphi_i\right)\end{array}\right] dx$$

and

$$C\prime = \Delta t \sum_{j=1}^{n} \int_{\sup p\varphi_j} f^{n+1}(x) v_j \varphi_j dx$$

if $|i - j| \geq 2$, we have

$$(\sup p\varphi_i \cap \sup p\varphi_j) \neq \Phi$$

We consider

$$\begin{cases} a(x) = \delta, \\ a_0(x,t) = \Psi(t). \end{cases}$$

where δ is a constant.
(29) becomes

$$A\prime = \sum_{j=1}^{n} v_j \left[\begin{array}{c} \sum_{i=1}^{n} u_i^{n+1} \int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i \varphi_j dx \\ - \sum_{i=1}^{n} u_i^{n} \int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i \varphi_j dx \end{array} \right],$$

$$B\prime = \delta\Delta t \left(\sum_{j=1}^{n} v_j \left[\sum_{i=1}^{n} u_i^{n+1} \int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i\prime \varphi_j dx \right] \right)$$

$$+ \Psi^{n+1}\Delta t \left(\sum_{j=1}^{n} v_j \left[\int_{\sup p\varphi_i \cap \sup p\varphi j} \sum_{i=1}^{n} u_i^{n+1} \varphi_i \varphi_j dx \right] \right)$$

and

$$C\prime = \sum_{j=1}^{n} v_j \left[\int_{\sup p\varphi_j} f^{n+1}(x) \varphi_j dx \right],$$

thus, the equation $A\prime + B\prime = C\prime$ equivalent to

$$\sum_{j=1}^{n} v_j \left[\sum_{i=1}^{n} u_i^{n+1} \int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i \varphi_j dx \right] \quad (30)$$

$$- \sum_{j=1}^{n} v_j \left[\sum_{i=1}^{n} u_i^{n} \int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i \varphi_j dx \right]$$

$$+ \delta\Delta t \sum_{j=1}^{n} v_j \left[\sum_{i=1}^{n} u_i^{n+1} \int_{\varphi_{ii} \cap \varphi j} \varphi_i\prime \varphi_j dx \right]$$

$$+ \left(\Psi^{n+1}\Delta t \right) \sum_{j=1}^{n} v_j \left[\sum_{i=1}^{n} u_i^{n+1} \int_{\varphi_{ii} \cap \varphi j} \varphi_i \varphi_j dx \right]$$

$$= \sum_{j=1}^{n} v_j \left[\int_{\varphi_{ii} \cap \varphi j} f^{n+1}(x) \varphi_j dx \right].$$

(30) gives

$$\sum_{i=1}^{n} u_i^{n+1} \underbrace{\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i \varphi_j dx}_{K}$$

$$- \sum_{i=1}^{n} u_i^{n} \int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i \varphi_j dx$$

$$+ \Delta t \delta \sum_{i=1}^{n} u_i^{n+1} \underbrace{\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi\prime_i \varphi_j dx}_{R}$$

$$+ \Psi^{n+1}\Delta t \sum_{i=1}^{n} u_i^{n+1} \underbrace{\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i \varphi_j dx}_{K}$$

$$= \underbrace{\int_{\sup p\varphi_i \cap \sup p\varphi j} f^{n+1}(x) \ \varphi_j dx}_{F}$$

Therefore, we can write (30) as

$$\left[\mathbf{K} + (\delta\Delta t \cdot \mathbf{R}) + \left(\Psi^{n+1}\Delta t \cdot \mathbf{K} \right) \right] \cdot U^{n+1}$$

$$- \left[\mathbf{K} \cdot U^n \right] = F, \tag{31}$$

where
$K(\varphi_i, \varphi_j)$ and $R(\varphi_i', \varphi_j)$ are symmetric three diagonal matrices of dimension $(N-1) \times (N-1)$

$$K(\varphi_i, \varphi_j) = \int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i(x) \cdot \varphi_j(x) dx$$

and

$$R(\varphi_i', \varphi_j) = \int_{\sup p\varphi_i' \cap \sup p\varphi j} \varphi_i'(x) \cdot \varphi_j(x) \ dx$$

It can be easily calculated $R(\varphi_i', \varphi_j)$, $K(\varphi_i, \varphi_j)$ as follow:
First case: $i = j$

$$\begin{cases} \int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i \varphi_j dx = \int_{\sup p\varphi_i \cap \sup p\varphi_j} (\varphi_i)^2 dx \\ = \int_{x_{i-1}}^{x_i} \left[\frac{x - x_{i-1}}{h} \right]^2 dx + \int_{x_i}^{x_{i+1}} \left[\frac{x_{i+1} - X}{h} \right]^2 dx \\ = h \int_{-1}^{0} [1+s]^2 dx + h \int_0^1 [1-s]^2 dx = \frac{2h}{3} \end{cases}$$

or

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i \varphi_j dx = \frac{2h}{3}, \tag{32}$$

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi'_i\varphi_j dx = \frac{h}{2}\left[(\varphi_0)^2\right]_{-1}^0$$

$$+\frac{h}{2}\left[(\varphi_0)^2\right]_0^1 = 0$$

and

$$\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi'_i\varphi_j dx = 0. \qquad (33)$$

Second case: $j = i-1$
We have

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i\varphi_j dx$$

$$= \int_{\sup p\varphi_i \cap \sup p\varphi_{j-1}} \varphi_i\varphi_{j-1} dx = \varphi_0\varphi_{-1}$$

$$= \frac{h}{6},$$

thus

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i\varphi_j dx = \frac{h}{6}, \qquad (34)$$

$$\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i\varphi'_j dx$$

$$= \int_{\sup p\varphi_i \cap \sup p\varphi_{j-1}} \varphi'_i\varphi_j dx$$

$$= h\int_{\sup p\varphi_0 \cap \sup p\varphi_{-1}} \varphi_0'\varphi_{-1} ds = \frac{h}{2}.$$

Thus, we find

$$\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i\varphi'_j dx = \frac{h}{2}. \qquad (35)$$

Third case: $j = i+1$
We have

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i\varphi_j dx$$

$$= \int_{\sup p\varphi_i \cap \sup p\varphi_{j+1}} \varphi_i\varphi_{j+1} dx$$

$$= \int_{\sup p\varphi_0 \cap \sup p\varphi_1} \varphi_0\varphi_{+1} ds = \frac{h}{6},$$

$$\int_{\sup p\varphi_i \cap \sup p\varphi_j} \varphi_i\varphi_j dx = \frac{h}{6} \qquad (36)$$

$$\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i\varphi'_j dx$$

$$= \int_{\sup p\varphi_i \cap \sup p\varphi_{j-1}} \varphi'_i\varphi_j dx$$

$$= h\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_0'\varphi_{-1} dx = \frac{h}{2}$$

and for $j = i+1$

$$\int_{\sup p\varphi_i \cap \sup p\varphi j} \varphi_i\varphi'_j dx = \frac{h}{2}. \qquad (37)$$

3 Application

We give pollutant transport by the following equation based on simplifying assumptions including neglecting dispersion phenomena vertical transverse and longitudinal dispersion:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0, \qquad (38)$$

where

C pollutant concentration (mg/L); vU: velocity flow (m/s).

For the application of our model, we pick out a virtual experience from Qassim Cities, Qassim city is one of the thirteen administrative regions of Kingdom of Saudi Arabia. Located at the heart of the country, and almost in the center of the Arabian Peninsula, it has a population of $1,370,727$ and an area of $58,046$ km^2. It is known to be the "alimental basket" of the country, for its agricultural assets.

It knows that equation (38) is a hyperbolic equation, here it would be wise to use Euler time scheme discretization combined with Galerkin spatial discretization with respect to the following assumptions: (1)-An nitial conditions knowledge and boundary conditions, (2)-The pollutant concentrations related only to distance time. (3)-The velocity is constant and independent of time.

Considering that the concentrations are zero regardless of the distance downstream of the sampling point such that:

$$C(x,0) = 0 \text{ for } x > 0 \qquad (39)$$

and for boundary conditions as follows:

The pollutant is considered passive conservative, and the principle of mass conservation is taken into account:

$$\int_{-\infty}^{\infty} A(x)C(x,t)dx = M.$$

After an infinite distance downstream from the point of sample collection of concentrations are considered zero:

$$C(\infty, t) = 0 \text{ for } t \geq 0,$$

with

$$\frac{\partial C}{\partial x} \longrightarrow 0 \text{ when } t \text{ tends to } \infty,$$

where:

A is the river section (m^2); C is the pollutant concentration(mg/L); M is the rejection mass (Kg).

It can be calculated velocity folw which given by the following formula:

$$U = Q/S, \tag{40}$$

where:

S : mesh section (m^2); U : flow velocity (m/s); Q is adaily flow of the river average (m^3/s).

Since several phenomena have been neglected, the choice of using one-dimensional models can be detrimental in terms of the accuracy of predictions. Moreover, the importance lies in the small number of data required making a tool in line with the issue of emergency data as follow: Flood discharge (m^3/s) - wet section (m^2) main floo -length (m) - Raw slope (m/m)-high flood.

In our work, the choice fell on 2 pollutants: $NH4$ and $NO2$ and we have a large dilution in water, and its harmfulness and toxicity to living beings. Moreover, a plant that manufactures detergents discharges infected material into a stream. The detergents manufacture is based on products of this type. For this, the laboratories service of any agency of water resources can perform measurements of the chemical composition of the water regularly and in different places. For that our demonstrating problem can enter the input parameters according to the following assumptions:

Insert the U velocity calculated using the formula in section 4 [$u = Q/S$] and we insert the $AT = 10min$ and $h = 2m$ with ($h = \max h_i$). These 2 parameters are chosen arbitrarily. Because it is unconditionally stable according to Euler time discretization. For a maturity of $12h$, we obtain 72 iterations.

Introduction of a number of points: our choice was focused on 5 points due to the low concentration of the pollutant treated to avoid that other phenomenon outweighs the transportation

Read initial data from the file container. (The data are from NET collection of 16 November 2016).

Matrices are calculated according to the previous steps of the later section, and we find the following system:

$$(A + \Delta t U B)C^{n+1} = AC^n,$$

where C^{n+1} is the forecast. We can give our results by the following graph:

Comment The graphs show the temporal evolution of pollutant concentrations of NO_2 at differing distances
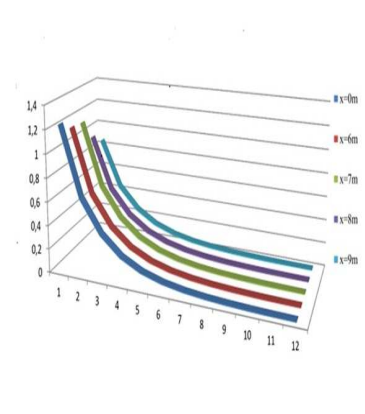

Fig. 1: Concentrations of NO_2 (mg\l).

with respect to the chose a flow velocity of $0.64m/s$. Because higher speeds cause greater dilution, and therefore very low concentrations. In addition the pollutant diffusion has been eliminated and is interested only by the advection.

4 Conclusion

In this paper, a convergence and the stability of discetized transport is established by using Euler time scheme combined with a finite element spatial approximation and we have applied this model to a hypothetical example and compare the calculated results of this model with a virtual experience taken from NET which deals with the concentration of certain pollutants and their speed diffusion in the water.

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