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On Generalized q-Close-to-Convexity

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Abstract: A new class of generalized q-close-to-convex functions is defined and investigated using the quantum calculus. Distortion theorems, the rate of growth of coefficients and some other interesting properties of this class are studied. Relevant connections to various known results are also pointed out. The technique of this paper may motivate further research.

Keywords: q-Difference operator, q-starlike, bounded radius rotation, q-close-to-convex, distortion result.
2010 AMS Subject Classification: 30C45

1 Introduction

Let A denote the class of all functions $f(z)$ which are analytic in the open unit disk $E=\{z:|z|<1\}$ and are of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

Let $S\subset A$ be the class of univalent functions in E. A functions $f\in S$ is said to be starlike of order α if and only if

$$\Re\frac{zf'(z)}{f(z)}>\alpha,\quad 0\le\alpha<1,\quad z\in E.$$

The class consisting of such functions is denoted by $S^*(\alpha)$. For $\alpha=0$, we have the well-known class S^* of starlike functions, see [3].

Let $f_i\in A$, $i=1,2$. Then we say that $f_1(z)$ is subordinate to $f_2(z)$, and write $f_1(z)\prec f_2(z)$, if there exists a Schwartz function w, analytic in E with $w(0)=0$ and $|w(z)|<1$, $(z\in E)$, such that

$$f_1(z)=f_2(w(z)).$$

If $f_2\in S$, it is known that the above subordination is equivalent to $f_1(0)=f_2(0)$ and $f_1(E)\subset f_2(E)$.

The q-difference Calculus or quantum Calculus was initiated at the beginning of 19^{th} century and was developed by Jackson [6,7]. Recently the area of q-Calculus has attracted the serious attention of researchers due to its applications in various branches of mathematics and physics. See also [8,11,12,15,16].
The q-difference operator D_q acting on functions $f\in A$, given by (1) and $0<q<1$, is defined as

$$D_qf(z)=\frac{f(z)-f(q(z))}{(1-q)f(z)},\quad (z\neq 0), \tag{2}$$

$$D_qf(0)=f'(0)\quad\text{and}\quad D_q^2f(z)=D_q(D_qf(z)).$$

From (2), we deduce that

$$D_qf(z)=1+\sum_{n=2}^{\infty}[n]_qz^{n-1}, \tag{3}$$

where

$$[n]_q=\frac{1-q^n}{1-q} \tag{4}$$

As $q\to 1^-$, $[n]_q\to n$.
As a right inverse, the q-integral is defined in [7] as

$$\int_0^z f(t)d_q(t)=z(1-q)\sum_{n=0}^{\infty}q^nf(zq^n). \tag{5}$$

In [5], the class S^* is generalized by replacing the derivative with q-difference operator D_q and the right half plane by a suitable domain. This generalized class is denoted by S_q^*. An analytic function $f\in S_q^*\subset A$ is called q-starlike and is defined as follows, see [5].

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Let $f \in A$. Then, $f \in S_q^*$, if

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad 0<q<1, \quad z \in E.$$

We note that, as $q \leftrightarrow 1^-$, the closed dick $|w - \frac{1}{1-q}| \leq \frac{1}{1-q}$ becomes the right half plane and the class S_q^* reduces to S^*.

It has been proved in [5] that $S^* = \cap_{0<q<1} S_q^*$. The class $S_q^*(\alpha)$ of q-starlike functions of order α is defined in [1] as follows.

Definition 1. *Let $f \in A$. Then $f \in S_q^*(\alpha)$, if*

$$\left| \frac{\frac{z(D_q f)(z)}{f(z)} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| < \frac{1}{1-q},$$

where $0<q<1$, $0 \leq \alpha < 1$ and $z \in E$.

When $\alpha = 0$, the class $S_q^*(\alpha)$ coincides with the class S_q^*.

Following the similar method used in [10], we note that

$$\frac{\frac{z(D_q f)(z)}{f(z)} - \alpha}{1-\alpha} \prec \frac{1+z}{1-qz}.$$

That is, $f \in S_q^*(\alpha)$, if and only if

$$\frac{z(D_q f)(z)}{f(z)} \prec \frac{1+\{1-\alpha(1+q)\}z}{1-qz}. \tag{6}$$

Definition 2. *Let $p(z)$ be analytic in E with $p(0)=1$. Then $p \in P_q(m,\alpha)$, if and only if*

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \tag{7}$$

where

$$p_i(z) \prec \frac{1+\{1-\alpha(1+q)z\}}{1-qz},$$
$$i=1,2,\ m \geq 2,\ q \in (0,1),\ \alpha \in [0,1), z \in E.$$

For $m=2$, $P_q(2,\alpha) = P_q(\alpha)$, and $P_q(2,0) = P_q$ consists of analytic functions subordinate to $\frac{1+z}{1-qz}$ and $\lim_{q\to 1^-} P_q = P$ is the well known class of functions with positive real part. Also $\lim_{q\to 1^-} P_q(m,0) = P_m$ is the class introduced and studied in [17].

Definition 3. *Let $f \in A$. Then $f \in R_q^*(m,\alpha)$, if and only if*

$$\frac{z(D_q f)}{f} \in P_q(m,\alpha), \quad z \in E.$$

We note that, for $q \to 1^-$, $R_q^*(m,0) = R_q^*(m)$ reduces to the class R_m which consists of functions of bounded radius rotation, see [3,14].

We now introduce the concept of q-close-to-convexity in generalized form as follows.

Definition 4. *Let $f \in A$. Then $f \in K_q^*(m,\alpha)$, if and only if there exists $g \in R_q^*(m,\alpha)$ such that*

$$\frac{zf'(z)}{g(z)} \prec \frac{1+qz}{1-qz}, \quad z \in E$$

For $\alpha = 0$, $m=2$, we have the class K_q^* of q-close-to-convex functions and

$$\lim_{q\to 1^-} K_q^*(m,0) = T_m,$$

where T_m is the class of generalized close-to-convex functions introduced and discussed in [1].
Also, for $m=2$, $\alpha=0$ and $q \to 1^-$, we obtain the class k of close-to-convex univalent functions, see [9].

Definition 5. *Let $f \in A$. Then, for $a \geq 0$, $0 < r\gamma \leq 1$, $f(z)$ is said to belong to the class $T_q^a(m,\alpha,\gamma)$, if and only if there exists $g \in K_q^*(m,\alpha)$ such that*

$$zf'(z) + af(z) = (a+1)z(g'(z))^\gamma. \tag{8}$$

We note that

$$T_q^0(m,\alpha,1) = K_q^*(m,\alpha)$$

Also

$$f \in T_q^\infty(m,\alpha,1) = Q_q(m,\alpha) \quad \Longrightarrow \quad f(z) = zg'(z),\ g \in K_q^*(m,\alpha).$$

Let

$$G(a,b;c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\mathrm{d}u, \tag{9}$$

where $\Re(a)>0$, $\Re(c-a)>0$, Γ denotes the Gamma function and $G(a,b;c,z)$ represents the hypergeometric function.

Unless otherwise stated, throughout this paper we take $0 \leq \alpha < 1$, $0 < r \leq 1$, $m \geq 2$, $a \geq 0$ and $z \in E$.

2 Preliminary Results

Lemma 1. [1] *Let $g \in A$. Then $g \in S_q^*(\alpha)$, if and only if, there exists a probability measure μ supported on the unit circle such that*

$$\frac{zg'(z)}{g(z)} = 1 + \sum_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z)\mathrm{d}\mu(\sigma),$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} (1-\alpha)\frac{2\ln q}{q^n - 1} z^n, \quad z \in E.$$

Lemma 2. *Let $g \in S_q^*(\alpha)$. Then there exists $g_1 \in S_q^*$ such that*

$$g(z) = z\left(\frac{g_1(z)}{z}\right)^{1-\alpha}. \tag{10}$$

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The proof follows easily from the definition and simple computations.

Using a result proved in [10] for $R_q^*(m)$ and Lemma 2.2, we have

Lemma 3. *Let $g \in R_q^*(m,\alpha)$. Then there exist $s_i \in S_q^*(\alpha)$, $i=1,2$ such that*

$$\frac{g(z)}{z} = \left[\frac{\left(\frac{S_1(z)}{z}\right)^{\frac{m}{4}+\frac{1}{2}}}{\left(\frac{S_2(z)}{z}\right)^{\frac{m}{4}-\frac{1}{2}}}\right].$$

When $\alpha = 0$ and $q \to 1^-$, we obtain a well known result for functions of bounded radius rotation, see [3].

Using (3) and a result proved in [10] for the class $R_q^*(m)$, we have the following result.

Lemma 4. *Let $f \in R_q^*(m,\alpha)$. Then, for $|z| = r < 1$,*

$$\frac{(1-qr)^{q_3(1-\alpha)(\frac{m}{4}-\frac{1}{2})}}{(1+qr)^{q_3(1-\alpha)(\frac{m}{4}+\frac{1}{2})}} \leq \left|\frac{f(z)}{z}\right| \leq \frac{(1+qr)^{q_3(1-\alpha)(\frac{m}{4}-\frac{1}{2})}}{(1+qr)^{q_3(1-\alpha)(\frac{m}{4}+\frac{1}{2})}}, \tag{11}$$

where

$$q_3 = q_1 q_2,\ q_1 = \frac{q+1}{q},\ q_2 = \frac{1-q}{\log q^{-1}},\ q \in (0,1). \tag{12}$$

Lemma 5.[4] *Let $h(z)$ be analytic in E, $h(0) = 1$ and $\Re h(z) > 0$, $z = re^{i\theta}$. Then*

$$\int_0^{2\pi} |h(re^{i\theta})|^\lambda d\theta < c_1(\lambda)\frac{1}{(1-\lambda)^{\lambda-1}},$$

where $c_1(\lambda)$ is a constant and $\lambda > 1$.

3 Main Results

Theorem 1. *Let $f \in T_q^0(m,\alpha,\gamma) = T_q(m,\alpha,\gamma)$. Then*

$$\frac{2^{1-Q}}{q(M+2)}\{G(a,b;c,-1) - r_1^{Q+1}G(a,b;c,-r_1)\}$$
$$\leq \left|\frac{f(z)}{z}\right|$$
$$\leq \frac{2^{1-Q}}{q(M+2)}\{G(a,b;c,-1) - r_1^{-(Q+1)}G(a,b;c,-r_1^{-1})\},$$

where

$$M = \gamma(1-\alpha)q_3\left(\frac{m}{4}-\frac{1}{2}\right), \quad Q = \gamma(1-\alpha)q_3,$$
$$a = M+1, \quad b = 2-Q,\ c = M+2, \quad r_1 = \frac{1-r}{1+r},$$
$$|z| = r, \quad q_3 = q_1 q_2. \tag{13}$$

Proof. Since $f \in T_q(m,\alpha,\gamma)$, there exists $g \in K_q^*(m,\alpha)$ such that

$$f'(z) = (g'(z))^\gamma = \left(\frac{g_1(z)}{z}\cdot p(z)\right)^\gamma,$$
$$g_1 \in R_q^*(m,\alpha),\ p(z) \prec \frac{1+qz}{1-qz}. \tag{14}$$

Let d_r denote the radius of the largest Schlicht disk centered at the origin and contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_0, $|z_0| = r$ such that $|f(z_0)| = d_r$ and we have

$$d_r = |f(z_0)| = \int_c |f'(z)||dz|$$
$$\geq \int_c \frac{(1-q|z|)^M}{(1+q|z|)^{M+Q}}\cdot\left(\frac{1-q|z|}{1+q|z|}\right)^\gamma |dz|$$
$$\geq \int_0^{|z|} \frac{(1-qs)^{M+r}}{(1+qs)^{M+\gamma+Q}} ds,$$

where we have used (14) and Lemma 2.4. Thus, we have

$$|f(z_o)| \geq \int_0^{|z|}\left(\frac{1-qs}{1+qs}\right)^{M+\gamma}\frac{ds}{(1+qs)^Q}. \tag{15}$$

Let $\frac{1-qs}{1+qs} = t$. Then $dt = -\frac{2q}{(1+qs)^2}ds$ and we can write (15) as

$$|f(z_o)| \geq \frac{2^{(1-Q)}}{q}\int_{\frac{1-|z|}{1+|z|}}^1 t^M(1+t)^{Q-2}dt$$
$$= -\frac{2^{(1-Q)}}{q}\int_0^{\frac{1-r}{1+r}} t^M(1+t)^{-(2-Q)}dt + \frac{2^{(1-Q)}}{q}\int_0^1 t^M(1+t)^{-(2-Q)}dt$$
$$= I_1 + I_2. \tag{16}$$

To calculate I_1, let $\frac{1-r}{1+r} = r_1$ and $r_1 u = t$. Then

$$I_1 = -\frac{2^{(1-Q)}}{q}\int_0^{r_1}(r_1u)^M r_1(1+r_1u)^{-(2-Q)}du$$
$$= -\frac{2^{(1-Q)}}{q}r_1^{Q+1}\int_0^1 u^M(1+r_1u)^{-(2-Q)}du$$
$$= -\frac{2^{(1-Q)}}{q}r_1^{Q+1}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\cdot G(a,b;c,-r_1), \tag{17}$$

where a, b, c, r_1, M and Q are as given by (13).
We now calculate I_2

$$I_2 = \frac{2^{(1-Q)}}{q}\int_0^1 t^M(1+t)^{-(2-Q)}dt$$
$$= \frac{2^{(1-Q)}}{q}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\cdot G(a,b;c,-1). \tag{18}$$

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Since

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}=\frac{\Gamma(M+1)}{\Gamma(M+2)}=\frac{1}{M+1}=\frac{1}{\{\gamma(1-\alpha)q(\frac{m}{4}-\frac{1}{2})\}+2},$$

it follows from (16), (17) and (18) that

$$|f(z)| \geq \frac{2^{1-Q}}{q(M+2)}\{G(a,b;c,-1)-r_1^{Q+1}G(a,b;c,-r_1)\}. \quad (19)$$

To calculate the upper bound for $|f(z)|$, we proceed in the similar way and note that

$$|f'(z)|\leq \frac{(1+q|z|)^{M+1}}{(1-q|z|)^{M+Q+1}}.$$

That is,

$$|f'(z)|\leq \int_0^{|z|}\frac{(1+qs)^{M+1}}{(1-qs)^{M+Q+1}}ds$$

With $\frac{1+qs}{1-qs}=\xi$ and $-\frac{2q}{(1-qs)^2}ds=d\xi$, we have

$$\begin{aligned}|f(z)| &\leq -\frac{1}{2q}\int_1^{\frac{1-|z|}{1+|z|}}\xi^M\left(\frac{2}{1+\xi}\right)^{2-Q}d\xi\\ &=\frac{2^{1-Q}}{q(M+2)}\{G(a,b;c,-1)-r_1^{-(Q+1)}\\ &\qquad\times G(a,b;c,-r_1^{-1})\}. \quad (20)\end{aligned}$$

Combining (19) and (20), we obtain the desired result.

By letting $r\to 1$ in Theorem 3.1, we have the following covering result.

Theorem 2.*Let $f\in T_q(m,\alpha,\gamma)$. Then $f(E)$ contains the Schlicht disk*

$$|z|<\frac{2^{1-Q}}{q(M+2)},$$

where Q,M are given by (13).

Corollary 1.*Let $f\in T_q(m,\alpha,\gamma)$ and $q\to 1^-$. Then $f(E)$ contains the Schlicht disk*

$$|z|<\frac{2^{(1-2\gamma(1-\alpha))}}{\{\gamma(1-\alpha)(\frac{m}{2}-1)\}+2}.$$

When $\alpha=0$, $r=1$, we have $|z|<\frac{1}{m+2}$ and for $m=2$, this result reduces to a well known covering theorem $|z|<\frac{1}{4}$ for close-to-convex functions.

Theorem 3.*Let $f\in T_q^a(m,\alpha,\gamma)$ be given by* (1). *Then, for $(M+Q)>\frac{1}{2}$, and $M,\ Q$ as given by* (13), *we have*

$$a_n=O(1)\left(\frac{a+1}{n+a}\right)n^{(M+Q)},\quad (n\to\infty),$$

where $O(1)$ is a constant depending only on $q,\ m,\ \alpha,\ \gamma$ and $M,\ Q$ are as given by (13).

*Proof.*Since $f\in T_q^a(m,a,\gamma)$, we can write

$$\frac{1}{(a+1)}\{zf'(z)+af(z)\}=z\left[\frac{g_1(z)}{z}\cdot p(z)\right]^\gamma,\ g_1\in R_q^*(m,\alpha \quad (21)$$

and

$$p(z)\prec\frac{1+qz}{1-qz}.$$

Now, from Cauchy Theorem, Lemma 2.2, Lemma 2.3 and (21), we have

$$\begin{aligned}&\left(\frac{n+a}{a+1}\right)|a_n|\\ &\leq\frac{1}{2\pi r^{n-1}}\int_0^{2\pi}\left|\frac{\left(\frac{s_1(z)}{z}\right)^{\gamma(1-\alpha)(\frac{m}{4}+\frac{1}{2})}}{\left(\frac{s_2(z)}{z}\right)^{\gamma(1-\alpha)(\frac{m}{4}-\frac{1}{2})}}\right||p(z)|^\gamma d\theta,\\ &\qquad s_1,s_2\in S_q^*. \quad (22)\end{aligned}$$

Using distortion result for $s_2\in S_q^*$ (see [10]) in (22), and applying Schwartz inequality together with subordination, we have

$$\begin{aligned}&\left(\frac{n+a}{a+1}\right)|a_n|\\ &\leq\frac{(1+q)^M}{2\pi r^{n-1}}\int_0^{2\pi}\left|\frac{s_1(z)}{z}\right|^{\gamma(1-\alpha)(\frac{m}{4}+\frac{1}{2})}|p(z)|^\gamma d\theta\\ &\leq\frac{(1+q)^M}{r^{n-1}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{s_1(z)}{z}\right|^{\gamma(1-\alpha)(\frac{m}{2}+1)}d\theta\right)^{\frac{1}{2}}\\ &\quad\times\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\right)^{\frac{1}{2}}\\ &\leq d_1(\alpha,\gamma,m,q)\left(\int_0^{2\pi}\frac{d\theta}{|1-qre^{i\theta}|^{2(M+Q)}}\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_0^{2\pi}\frac{d\theta}{|1-qre^{i\theta}|^2}\right)^{\frac{1}{2}}\\ &\leq d_2(\alpha,\gamma,m,q)\frac{1}{(1-r)^{M+Q}},\end{aligned}$$

where $d_1,\ d_2$ are constants and we have used Lemma 2.5. The proof is complete by choosing $r=1-\frac{1}{n},\ (n\to\infty)$.

We have the following special case.

Corollary 2.*Let $a=0,\ \gamma=1$ and $\alpha=0$. Then $f\in T_q(m,0,1)$ and we have*

$$a_n=O(1)n^{\{q_3(\frac{m}{4}+\frac{1}{2})-1\}}\quad (n\to\infty).$$

When $q\to 1^-$, $a_n=O(1)\cdot n^{\frac{m}{2}}$, and this is a result proved in [13]. For $m=2,\ q\to 1^-,\ f$ is close-to-convex and in this case we obtain a well known result, see [3].

Denote by $L(r,f)$, the length of the image of the circle $|z|=r$ under f. Then, with the similar techniques used in Theorem 3.3, we can easily prove the following.

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Theorem 4. *Let $f \in T_q(m,0,\gamma)$, $\gamma > \frac{1}{2}$. Then*

$$L(r,f) = O(1)\left(\frac{1}{1-qr}\right)^{M_o+Q_o+\gamma-1}, \quad (r \to 1)$$

where $O(1)$ is a constant, $M_o = \gamma q_3(\frac{m}{4}-\frac{1}{2})$ and $Q_o = \gamma q_3$, q_3 is as given by (12).

As a special case, we note that $f \in T_m$ when we take $\gamma = 1$ and $q \to 1^- \Rightarrow q_3 \to 2$ and in this case we have a result proved in [13] that

$$L(r,f) = O(1)\left(\frac{1}{1-r}\right)^{\frac{m}{2}+1}, \quad (r \to 1)$$

Theorem 5. *Let $G \in A$ and let $\frac{zG'(z)}{g(z)} \prec \frac{1+qz}{1-qz} = p_q(z)$, $g \in \cap_{0<q<1} R_q^*(m,\alpha)$. Let*

$$zf'(z) + af(z) = (a+1)zG'(z). \tag{23}$$

Then f is univalent in $|z| < r_m$, where

$$r_m = \frac{1}{2}\{m - \sqrt{m^2-4}\}. \tag{24}$$

Proof. First we note that $\cap_{0<q<1} S_q^*(\alpha) = S^*(\alpha)$, see [1] and therefore it follows from Lemma 2.3 that $\cap_{0<q<1} R_q^*(m,\alpha) = R_m(\alpha) \subset R_m$, the class of bounded radius rotation.
From (23), we can write

$$f(z) = \frac{a+1}{z^a}\int_0^z t^{a-1}G'(t)\mathrm{d}t, \quad G \in T_q^\infty(m,\alpha,1) \tag{25}$$

In (23), we define $G \in T_q^\infty(m,\alpha,1)$ as

$$zG'(z) = g(z)p(z), \quad g \in \cap_{0<q<1} R_q^*(m,\alpha) = R_m(\alpha),$$

With $a = c+id$, $c > 0$, we can write (23) as

$$f(z) = \frac{(c+1)+id}{z^{c+id}}\int_0^z t^c g(t)p(t)t^{id-1}\mathrm{d}t \tag{26}$$

For $g \in R_m(\alpha)$, there exists $g_1 \in R_m$ such that

$$\frac{g(z)}{z} = \left(\frac{g_1(z)}{z}\right)^{1-\alpha}$$

we define

$$G_1(z) = z\left(\frac{g_1(z)}{z}\right)^{\frac{(1-\alpha)}{c+1}}, \quad g_1 \in R_m \tag{27}$$

It is well known, see [17] that $g_1 \in R_m$ is starlike in $|z| < r_m$, where r_m is given by (24).
From (27), we have

$$\frac{zG_1'(z)}{G_1(z)} = \left(1-\frac{1}{c_1}\right) + \frac{1}{c_1}\frac{zg_1'(z)}{g_1(z)}, \quad c_1 = \frac{1-\alpha}{c+1}.$$

Since $g_1 \in R_m$, $\frac{zg_1'(z)}{g_1(z)} \in P_m$ and P_m is a convex set, it follows $\frac{zG_1'(z)}{G_1(z)} \in P_m$ which implies $G_1 \in R_m$ in E. Therefore $G_1 \in S^*$ in $|z| < r_m$.
Now, let $f_1 \in A$ be defined as

$$f_1(z) = [(c_1+1+id)\int_0^z \{G_1(t)\}^{c_1} p(t)t^{id-1}\mathrm{d}t]^{\frac{1}{c+id}}$$

We note that f_1 is a Bazilevic univalent function in $|z| < r_m$, where r_m is given by (24), see [2].

We observe that $\frac{f_1(z)}{z} \neq 0$, $|z| < r_m$.

$$f_1(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{a+1}}, \quad a = c+id$$

This means that $f(z)$, given by (26), is analytic and, for $(\frac{f(z)}{z})^{\frac{1}{a+1}}$, it is possible to select uniform branch which takes the value one for $z = 0$ and which is analytic for $|z| < r_m$ and also allows us to compute the derivatives in $|z| < r_m$.

Thus we conclude that $f(z)$ is univalent in $|z| < r_m$ where r_m is given by (23) and the proof is complete.

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