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Multi-Soliton Solutions to Nonlinear Hirota-Ramani Equation

*Harun-Or-Roshid and Md. Nur Alam**

Department of Mathematics, Pabna University of Science and Technology, Bangladesh

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Abstract: A direct rational exponential scheme is proposed to construct exact multi-soliton nonlinear partial differential equations. As an example we consider the well-known nonlinear Hirota-Ramani equation to investigate one-soliton, two-soliton and three-soliton solutions. This work is motivated by the fact that the direct rational exponential method provides completely non-elastic multi-soliton solution although soliton should remain their shape and size unchanged after and before collision. Furthermore, the properties of the acquired multiple soliton solutions are shown by three-dimensional profiles. All solutions are stable and might have applications in physics.

Keywords: Direct rational exponential scheme, Hirota-Ramani equation, Multi-soliton

1 Introduction

Nonlinear wave equations have a significant role in some technical and engineering fields. These equations appear in population models, propagation of fluxons in Josephson junctions, fluid mechanics, solid-state physics, plasma physics, plasma waves and biology etc. However, in recent years, A variety of numerical and analytical methods have been developed to obtain accurate analytic solutions for problems, such as, the Sumudu transform method [1,2,3], the $exp(-\Phi(\eta))$ -expansion method [4,5,6], the (G'/G) -expansion method [7,8,9,10,11,12,13], inverse scattering transform [14], Backlund transformation [15], Darboux transformation [16], analytical methods [17], the exp-function method [18], the Wronskian technique [19], the multiple exp-function method [20], the Hirota's bilinear method [21], the Jacobi elliptic function expansion method [22], the symmetry algebra method [23], etc.
We know that the mainly notable property of faithfully integrable equations is the occurrence of exact solitonic solutions and the existence of one-soliton solution is not itself a precise property of integrable nonlinear partial differential equations, many non-integrable equations also possess simple localized solutions that may be called one-solitonic. On the other hand,there are integrable equations only, which posses exact multi-soliton solutions which describe purely elastic interactions between individual solitons and the KdV equation is one of these integrable equations. Furthermore, some models exist in the literature are completely nonelastic, depending conditions between the wave vectors and velocities. Wazwaz [24,25,26,27] investigated multiple soliton solutions such type of elastic and non-elastic phenomena.

Our aim in this paper is to present an application of the direct rational exponential scheme to non-linear Hirota-Ramani equations to be solved one-soliton, two-soliton and three-soliton solutions by this method for the first time.

2 Multi-Soliton Solution of Hirota-Ramani Equation

In this section, we bring to bear the proposed approach to explain the both elastic and nonelastic interaction clearly to the simplest non-linear Hirota-Ramani equation [28,29,30,31,32],

$$u_t - u_{xxt} + au_x(1-u_t) = 0, \quad (1)$$

where $u(x,t)$ is the amplitude of the relevant wave mode and $a \neq 0$ is a real constant. This equation was first

* Corresponding author e-mail: nuralam.pstu23@gmail.com

introduced by Hirota and Ramani in [29]. Ji established some soliton solutions of this equation by Exp-function method [30]. This equation is completely integrable by using the inverse scattering method. Equation (1) is considered in [28,29,30,31,32], where new kind of solutions were obtained. This equation is commonly used in different brushwood of physics such as plasma physics, fluid physics, and quantum field theory. It also describes a range of wave phenomena in plasma and solid state [29].
For single soliton solution we first consider solution as

$$u(x,t)=r\frac{k_1c_1exp(k_1x+w_1t)}{a_o+c_1exp(k_1x+w_1t)}. \qquad (2)$$

Inserting (2) and (1), and then equating the coefficients of different powers of $(exp(k_1x+w_1t))^i,(i=\cdots\cdot,2,1,0,1,2,\cdots)$ to zero, yields a system of algebraic equations about a_0,c_1,w_1 and k_1 as follows:

$a_0^2(w_1k_1^2-ak_1,w_1)=0,$
$a_0(ark_1^2c_1w_1-2ak_1c_1-2w_1c_1-4k_1^2w_1c_1)=0,$
$c_1^2(k_1^2w_1-ak_1-w_1)=0.$
Solving the above system of algebraic equations for a ,w ,r 0 1 with the aid Maple 13, we achieve the following solution set:
$a_0=const.,w_1=\frac{ak_1}{k_1^2-1},r=6/a,$ and thus the solution is

$$u(x,t)=\frac{6k_1c_1exp(k_1(x+\frac{a}{k_1^2-1}t))}{a[a_0+c_1exp(k_1(x+\frac{a}{k_1^2-1}t))]}. \qquad (3)$$

and corresponding potential function is read as

$$v(x,t)=\frac{6k_1^2c_1^2exp(2k_1(x+\frac{a}{k_1^2-1}t))}{a[a_0+c_1exp(k_1(x+\frac{a}{k_1^2-1}t))]^2}-\frac{6k_1^2c_1exp(k_1(x+\frac{a}{k_1^2-1}t))}{a[a_0+c_1exp(k_1(x+\frac{a}{k_1^2-1}t))]}. \qquad (4)$$

To obtain interaction of two soliton solutions we just suppose

$$u(x,t)=r\frac{\Upsilon_1}{\Upsilon_2}, \qquad (5)$$

where $\Upsilon_1=k_1c_1exp(\xi_1)+k_2c_2exp(\xi_2)+a_{12}(k_1+k_2)c_1c_2exp(\xi_1+\xi_2),\Upsilon_2=a_o+c_1exp(\xi_1)+c_2exp(\xi_2)+a_{12}c_1c_2exp(\xi_1+\xi_2),\xi_1=k_1x+w_1t,\xi_2=k_2x+w_2t$ and the corresponding potential field reads $v=-u_x$.
Inserting Eq.(5) in the equation Eq. (1) via commercial software Maple-13, and setting the coefficients of different power of exponential to zero, we achieve a system of algebraic equation in terms of $r,k_1,k_2,w_1,w_2,c_1,c_2$ and a_{12}. Solving this system of algebraic equations for r,k_1,k_2,w_1,w_2 and a_{12} with the software, we achieve the following solution of the unknown parameters.

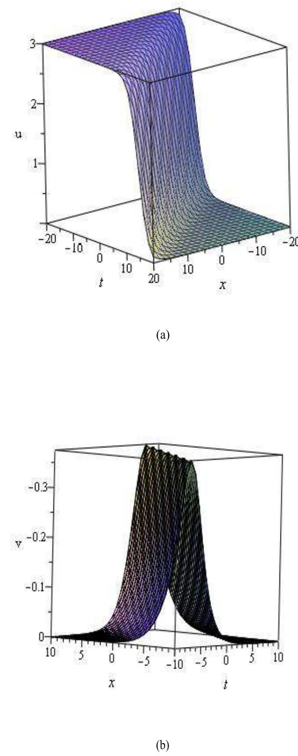


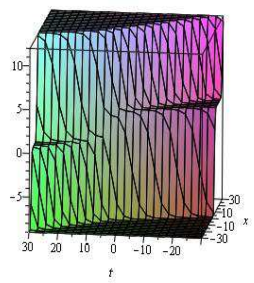
Fig. 1: (a) Profile of the single solitary wave solution Eq. (3) of H-R equation, (b) Potential field Eq. (4) with $k_1=0.5,a=a_0=c_1=1$.

Now according to the cases in the method we have
Set-1: $r=\frac{6}{a},a_0=const.,a_{12}=\frac{(k_1^2+k_2^2-k_1k_2-3)(k_1-k_2)^2}{(k_1^2+k_2^2+k_1k_2-3)(k_1+k_2)^2},$ $w_1=\frac{ak_1}{k_1^2-1},w_2=\frac{ak_2}{k_2^2-1}$ then

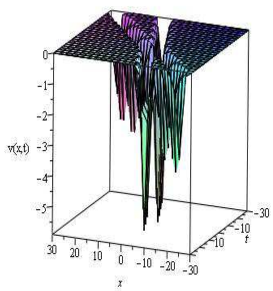
$$u(x,t)=\frac{6}{a}\cdot\frac{\Upsilon_1}{\Upsilon_2}, \qquad (6)$$

where $\Upsilon_1=k_1c_1exp(\xi_1)+k_2c_2exp(\xi_2)+a_{12}(k_1+k_2)c_1c_2exp(\xi_1+\xi_2),\Upsilon_2=a_o+c_1exp(\xi_1)+c_2exp(\xi_2)+a_{12}c_1c_2exp(\xi_1+\xi_2),a_{12}=\frac{(k_1^2+k_2^2-k_1k_2-3)(k_1-k_2)^2}{(k_1^2+k_2^2+k_1k_2-3)(k_1+k_2)^2},\xi_1=k_1(x+\frac{ak_1}{k_1^2-1}t),\xi_2=k_2(x+\frac{ak_2}{k_2^2-1}t)$ and $a,a_0,c_1,c_2,k_1\neq\pm1,k_2\neq\pm1$ are arbitrary constants.
The corresponding potential field reads $v=-u_x$.

From careful analyses of Eq. (6) as Fig. 2 and corresponding potential energy shows that two soliton with different wave height (before collision i.e., $t<0$), interact at ($t=0$) and scatter (after collosion i.e., $t>0$) with different wave height. It is conclude that for all the ranges of two arbitrary parameters k_1,k_2 , soliton changes their shape and size and a non-elastic scatter occurs. **Set-2:** $r=\frac{6}{a},a_0=a_{12}=0,w_2=const.,w_1=$

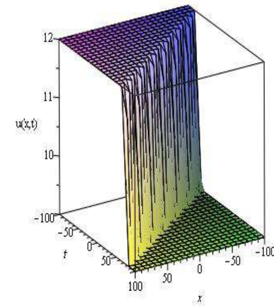


(a)

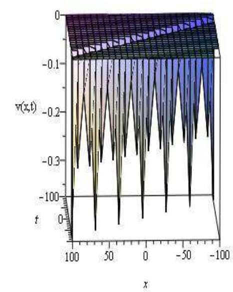


(b)

Fig. 2: (a) Profile of two non-elastic soliton solution Eq. (6) of H-R equation, (b) Potential field with $a = c_1 = c_2 = 1, a_0 = k_1 = 2, k_2 = -1.5$.



(a)



(b)

Fig. 3: (a) Profile of interaction of two soliton solution Eq. (7) of H-R equation, (b) Corresponding potential field $v(x,t)$ with $k_1 = 2, k_2 = 1.5, c_1 = c_2 = a = 1, w_2 = 8$.

$\frac{w_2(k_1-k_2)^2-w_2-a(k_1-k_2)}{(k_1-k_2)^2-1}$ then

$$u(x,t) = \frac{6}{a}\cdot\frac{k_1c_1exp(\xi_1)+k_2c_2exp(\xi_2)}{c_1exp(\xi_1)+c_2exp(\xi_2)}, \quad (7)$$

where $\xi_1 = k_1x + \frac{w_2(k_1-k_2)^2-w_2-a(k_1-k_2)}{(k_1-k_2)^2-1}t, \xi_2 = k_2x + w_2t$ and $a, w_2, c_1, c_2, k_1, k_2$ are arbitrary constants.
The corresponding potential field reads $v = -u_x$.

From careful analyses of Eq. (7) as Fig. 3 and corresponding potential energy shows that two soliton with different wave height (before collision i.e., $t < 0$), interact at ($t = 0$) and elastic scatter (after collosion i.e., $t > 0$) with same shape, size of wave. It is conclude that for all the ranges of two arbitrary parameters k_1, k_2 , soliton remain unchanges their shape and size and a elastic scatter occurs.

To obtain interaction of three soliton solutions we just suppose

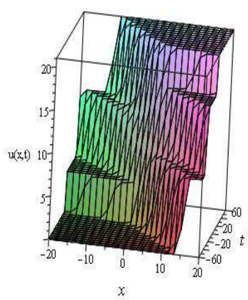
$$u(x,t) = r\frac{\Upsilon_1}{\Upsilon_2}, \quad (8)$$

where $\Upsilon_1 = k_1c_1exp(\xi_1) + k_2c_2exp(\xi_2) + k_3c_3exp(\xi_3) + a_{12}(k_1+k_2)c_1c_2exp(\xi_1+\xi_2) + a_{23}(k_2+k_3)c_2c_3exp(\xi_2+\xi_3) + a_{13}(k_1+k_3)c_1c_3exp(\xi_1+\xi_3) + a_{123}(k_1+k_2+k_3)c_1c_2c_3exp(\xi_1+\xi_2+\xi_3), \Upsilon_2 = a_0 + c_1exp(\xi_1) + c_2exp(\xi_2) + c_3exp(\xi_3) + a_{12}c_1c_2exp(\xi_1+\xi_2) + a_{23}c_2c_3exp(\xi_2+\xi_3) + a_{13}c_1c_3exp(\xi_1+\xi_3) + a_{123}c_1c_2c_3exp(\xi_1+\xi_2+\xi_3), \xi_1 = k_1x + w_1t, \xi_2 = k_2x + w_2t, \xi_3 = k_3x + w_3t$ and and the corresponding potential field reads $v = -u_x$. Inserting (8) in the equation (1) via commercial software Maple-13, and setting the coefficients of different power of exponential to zero, we achieve a system of algebraic and solving the system of algebraic equations via software, we achieve the following solution of the unknown parameters. **Set-1:** $r = \frac{6}{a}, a_0 = const., a_{12} = \frac{(k_1^2+k_2^2-k_1k_2-3)(k_1-k_2)^2}{(k_1^2+k_2^2+k_1k_2-3)(k_1+k_2)^2}, a_{23} = \frac{(k_2^2+k_3^2-k_2k_3-3)(k_2-k_3)^2}{(k_2^2+k_3^2+k_2k_3-3)(k_2+k_3)^2}, a_{13} = \frac{(k_1^2+k_3^2-k_1k_3-3)(k_1-k_3)^2}{(k_1^2+k_3^2+k_1k_3-3)(k_1+k_3)^2}, a_{123} = a_{12}a_{23}a_{13}$,
$w_1 = \frac{ak_1}{k_1^2-1}, w_2 = \frac{ak_2}{k_2^2-1}, w_3 = \frac{ak_3}{k_3^2-1}$ then

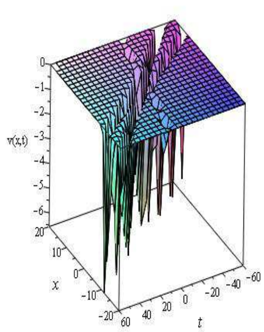
$$u(x,t) = \frac{6}{a}\cdot\frac{\Upsilon_1}{\Upsilon_2}, \quad (9)$$

where
$\Upsilon_1 = k_1c_1exp(\xi_1) + k_2c_2exp(\xi_2) + k_3c_3exp(\xi_3) + a_{12}(k_1+k_2)c_1c_2exp(\xi_1+\xi_2) + a_{23}(k_2+k_3)c_2c_3exp(\xi_2+\xi_3) +$

$a_{13}(k_1+k_3)c_1c_3exp(\xi_1+\xi_3)$ $+ a_{123}(k_1+k_2+k_3)c_1c_2c_3exp(\xi_1+\xi_2+\xi_3)\Upsilon, \Upsilon_2 = a_0 + c_1exp(\xi_1) + c_2exp(\xi_2) + c_3exp(\xi_3) + a_{12}c_1c_2exp(\xi_1+\xi_2) + a_{23}c_2c_3exp(\xi_2+\xi_3) + a_{13}c_1c_3exp(\xi_1+\xi_3) + a_{123}c_1c_2c_3exp(\xi_1+\xi_2+\xi_3), a_{12} = \frac{(k_1^2+k_2^2-k_1k_2-3)(k_1-k_2)^2}{(k_1^2+k_2^2+k_1k_2-3)(k_1+k_2)^2}, a_{23} = \frac{(k_2^2+k_3^2-k_2k_3-3)(k_2-k_3)^2}{(k_2^2+k_3^2+k_2k_3-3)(k_2+k_3)^2}, a_{13} = \frac{(k_1^2+k_3^2-k_1k_3-3)(k_1-k_3)^2}{(k_1^2+k_3^2+k_1k_3-3)(k_1+k_3)^2},$
$\xi_1 = k_1(x+\frac{ak_1}{k_1^2-1}t), \xi_2 = k_2(x+\frac{ak_2}{k_2^2-1}t), \xi_3 = k_3(x+\frac{ak_3}{k_3^2-1}t)$ and $a_0, c_1, c_2, c_3, k_1, k_2, k_3$ are arbitrary constants.
The corresponding potential field reads $v = -u_x \neq$.



(a)



(b)

Fig. 4: (a) Profile of three solitary wave fusion solution Eq. (9) of H-R equation, (b) Corresponding potential field $v(x,t)$ with $k_1 = 1.5, k_2 = 2.5, k_3 = 3, c_1 = c_2 = c_3 = 1, a = a_0 = 2$.

From careful analyses of Eq. (9) as Fig. 4 and corresponding potential energy shows that two soliton with different wave height (before collision i.e., $t < 0$), interact at ($t = 0$) and scatter (after collosion i.e., $t > 0$) with different wave height. It is conclude that for all the ranges of two arbitrary parameters k_1, k_2, k_3 , soliton changes their shape and size and a non-elastic scatter occurs.

3 Conclusion

The direct rational exponential scheme offers a simple and straightforward way to study exact solutions to NLPDEs. The method has been applied to the Hirota-Ramani equation and onewave, two-wave and three-wave solutions have been obtained in this paper. The 3D profiles of obtained solutions are given to visualize the shape, size of wave solutions and both elastic and non-elastic interactions are found. Overcoming the difficulties of calculations by some simple techniques via Maple-13 softwere, we finally construct some new explicit two soliton and three-soliton solutions for the Hirota-Ramani equation. It is point out that the procedure is very easy, any examiner can easily realized the idea of the scheme and can be applied to obtain the multi-soliton solutions of other nonlinear partial differential equations.

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Harun-Or-Roshid is Associate Professor of Mathematics at Pabna University of Science and Technology, Bangladesh. He received the PhD degree in Mathematics for Engineering Science at Non-linear Oscillations. His research interests are in the areas of applied mathematics and mathematical physics including the mathematical methods and models for non-linear complex systems, Soliton theory in case of traveling waves. He has published about 50 research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of mathematical journals.

Md. Nur Alam is Assistant Professor of Mathematics at Pabna University of Science and Technology, Bangladesh. He has received his B. Sc.(Hons) and M. Sc. (Thesis) in Mathematics from University of Rajshahi, Bangladesh in 2008 and 2009. He has received his M. Phil. research Mathematical Physics from Pabna University of Science and Technology, Pabna-6600, Bangladesh in 2015. His fields of interest are Analytical and Numerical Techniques for Differential Equations and Modeling, Computer aided geometric design and modeling, Isogeometric Analysis and Computer graphics.