Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110313

Inference on the Stress-Strength Model from Weibull Gamma Distribution

Mohamed A. W. Mahmoud[1], *Rashad M. EL-Sagheer*[1,*] *and Mahmoud M. M. Mansour*[2]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2] Department of Basic Science, Faculty of Engineering, The British University in Egypt, El Sherouk City, Cairo, Egypt

Received: 2 Nov. 2016, Revised: 25 Dec. 2016, Accepted: 29 Dec. 2016
Published online: 1 May 2017

Abstract: The point at issue of this paper is to deliberate point and interval estimations of the stress - strength function, R. The maximum likelihood, Bayes, and parametric bootstrap estimators are obtained as point estimations of R. Based on the maximum likelihood estimate (MLE) of R, the distribution of R is determined and hence its confidence interval (CI) is estimated. The variance of $\hat{R}$ has been got in a closed form. Furthermore, four bootstrap CIs of R have been obtained. The results of Bayes estimation are computed under the squared error loss (SEL) and the LINEX loss functions. The acceptance rejection principle algorithm is applied to obtain the credible CI of R. Finally, two explanatory examples are introduced to explicate the precision of the obtained estimators .

Keywords: Stress-strength model, progressive type II censoring, bootstrap bias corrected confidence interval (Boot-BC), Bootstrap accelerated bias corrected confidence interval (Boot-BCa), Bayesian estimation.

1 Introduction

The stress strength model has been known in the mechanical as follows, the stress is the mechanical loads and forces, while the strength is the physical effort that can resist the loads to perform its required function. When the stress exceeds the strength, the failure occurs. If X represents the strength and Y represents the stress, the main theme of statisticians is to estimate the probability of failure or reliability of this model. Since the reliability concept is general, so the stress strength model can be applied in different fields outside of the scope of mechanics, for more details see [1]. The WG distribution is appropriate for phenomenon of loss of signals in telecommunications which is called fading when multipath is superimposed on shadowing.

A random variable T is said to have WG distribution, with scale parameter α and two shape parameters θ and β, if its probability density function (pdf) is given by:

$$f(t;\alpha,\theta,\beta) = \frac{\theta\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\theta-1}\left(1+\left(\frac{t}{\alpha}\right)^{\theta}\right)^{-(\beta+1)}, \quad t>0;\alpha,\theta,\beta>0, \tag{1}$$

and the cumulative distribution function (CDF) is

$$F(t) = 1-\left(1+\left(\frac{t}{\alpha}\right)^{\theta}\right)^{-\beta} \quad ,t>0;\alpha,\theta,\beta>0. \tag{2}$$

For more detials about WG distribution and its properties see, [2,3,4]. Let $X\sim \text{WG}(\alpha,\theta,\beta_1)$ and $Y\sim \text{WG}(\alpha,\theta,\beta_2)$ be independent random variables and

$$R = P(Y<X) = \int_0^{\infty}\int_0^{x} f(x)\,g(y)\,dydx = \frac{\beta_2}{\beta_1+\beta_2}, \tag{3}$$

where $f(x)$ and $g(y)$ are the pdf of the strenth variable X and the stress variable Y respectively. The data or observations are assumed to be progressively Type-II censored (PROG-II-C) from the two WG distributions with two commonly known parameters α,θ and different shape parameters β_1 and β_2. The PROG-II-C scheme can be described as follows. First, the experimenter places n independent and identical units on the life test. When the first failure occurs, say at time $t_{(1)}, r_1$ of the surviving units are randomly selected and removed from the test. When the second failure occurs at time $t_{(2)}, r_2$ of the surviving units are randomly selected and removed from the test. The test is continued and when the $(m-1)$th failure occurs at time $t_{(m-1)}, r_{m-1}$ of the surviving units

* Corresponding author e-mail: rashadmath@azhar.edu.eg

are randomly selected and removed from the test. This experiment terminates when the m th failure occurs at time t_m, and the remaining surviving units $r_m = n-(r_1+r_2+...+r_{m-1})-m$ are all removed from the test. For more information on progressive censoring, we refer the reader to [5,6,7,8,9]. The estimation of R has been studied by several authors, based on different populations and various observations, see [1,10,11,12]. The remainder of this paper is organized as follows: Section 2 gives the MLE of R in addition to the corresponding CI. Section 3 concerns with four types of bootstrap confidence intervals. In Section 4 the Bayesian estimation of R is computed under the SEL and the LINEX loss functions and the credible CI of R is also obtained, using the acceptance rejection principle algorithm. Two illustrative examples, one of them is simulated and the other represents a real life data, are developed to explain the theoretical results in Section 5. Eventually, conclusion is inserted in Section 6.

2 Maximum Likelihood Estimation

Suppose $\underline{X} = (X_{1:M}, X_{2:M}, ..., X_{m:M})$ is a progressively Type II censored sample from $\mathrm{WG}(\alpha,\theta,\beta_1)$ with censored scheme $\underline{r} = (r_1, r_2, ..., r_m)$ and $\underline{Y} = (Y_{1:N}, Y_{2:N}, ..., Y_{n:N})$ is a progressively Type II censored sample from $\mathrm{WG}(\alpha,\theta,\beta_2)$ with censored scheme $\underline{\acute{r}} = (\acute{r}_1, \acute{r}_2, ..., \acute{r}_n)$. Hence, the likelihood function of $\beta_1$and β_2 is given by

$$L(\beta_1,\beta_2|\alpha,\theta,\underline{x},\underline{y}) = c_1\prod_{i=1}^{m}\{f(x_i)[1-F(x_i)]^{r_i}\}\times c_2\prod_{j=1}^{n}\left\{g(y_j)[1-G(y_j)]^{\acute{r}_j}\right\}, \quad (4)$$

where

$$c_1 = M(M-1-r_1)(M-2-r_1-r_2)...\times (M-m+1-r_1...-r_{m-1}),$$
$$c_2 = N(N-1-\acute{r}_1)(N-2-\acute{r}_1-\acute{r}_2)...\times (N-n+1-\acute{r}_1...-\acute{r}_{n-1}),$$

for more detials, see [5].

Then $L(\beta_1,\beta_2|\alpha,\theta,\underline{x},\underline{y})$ or $L(\beta_1,\beta_2)$, for notation simplicity, can be written as follows:

$$L(\beta_1,\beta_2) = c_1c_2\beta_1^m\beta_2^n\left(\frac{\theta}{\alpha}\right)^{m+n}\times \prod_{i=1}^{m}\left\{\left(\tfrac{x_i}{\alpha}\right)^{\theta-1}\left(1+\left(\tfrac{x_i}{\alpha}\right)^{\theta}\right)^{-(\beta_1 r_i+\beta_1+1)}\right\}\times \prod_{j=1}^{n}\left\{\left(\tfrac{y_j}{\alpha}\right)^{\theta-1}\left(1+\left(\tfrac{y_j}{\alpha}\right)^{\theta}\right)^{-\left(\beta_2\acute{r}_j+\beta_2+1\right)}\right\}. \quad (5)$$

The log-likelihood function may then be written as

$$\ln L(\beta_1,\beta_2) = \ln c_1+\ln c_2+(m+n)\ln\left(\frac{\theta}{\alpha}\right)+m\ln\beta_1+n\ln\beta_2 +(\theta-1)\sum_{i=1}^{m}\ln\left(\frac{x_i}{\alpha}\right)+(\theta-1)\sum_{j=1}^{n}\ln\left(\frac{y_j}{\alpha}\right) -\sum_{i=1}^{m}(\beta_1 r_i+\beta_1+1)\ln\left(1+\left(\frac{x_i}{\alpha}\right)^{\theta}\right)- \sum_{j=1}^{n}(\beta_2\acute{r}_j+\beta_2+1)\ln\left(1+\left(\frac{y_j}{\alpha}\right)^{\theta}\right),$$

thus we have the likelihood equations for β_1 and β_2 respectively, as

$$\frac{\partial \ln L}{\partial \beta_1} = \frac{m}{\beta_1}-\sum_{i=1}^{m}(r_i+1)\ln\left(1+\left(\frac{x_i}{\alpha}\right)^{\theta}\right) = 0,$$

$$\frac{\partial \ln L}{\partial \beta_2} = \frac{n}{\beta_2}-\sum_{j=1}^{n}(\acute{r}_j+1)\ln\left(1+\left(\frac{y_j}{\alpha}\right)^{\theta}\right) = 0.$$

Then

$$\hat{\beta}_1 = \frac{m}{\sum_{i=1}^{m}(r_i+1)\ln\left(1+\left(\frac{x_i}{\alpha}\right)^{\theta}\right)},$$
$$\hat{\beta}_2 = \frac{n}{\sum_{j=1}^{n}(\acute{r}_j+1)\ln\left(1+\left(\frac{y_j}{\alpha}\right)^{\theta}\right)},$$

and the MLE of R, say$\hat{R}$, can be written as

$$\hat{R} = \frac{\hat{\beta}_2}{\hat{\beta}_1+\hat{\beta}_2}. \quad (6)$$

To find the PDF of $\hat{R}$, the following lemma is needed

Lemma 2.1. If the random variable $X\sim WG(\alpha,\theta,\beta_1)$,then $T = Ln\left(1+\left(\frac{X}{\alpha}\right)^{\theta}\right)\sim Exp(\beta_1)$.

Proof. The proof is omitted.

Also, the following transformation can be considered:

$$S_1 = MT_1,$$
$$S_2 = (M-R_1-1)(T_2-T_1),$$
$$\vdots$$
$$S_m = (M-R_1...-R_{m-1}-(m-1))(T_m-T_{m-1}).$$

[5] has proved that S_i 's are independent and identically distributed exponential random variables, i.e.$S_i\sim Exp(\beta_1), i = 1,...,m$. Furthermore,

$$\sum_{i=1}^{m}S_i = \sum_{i=1}^{m}(R_i+1)T_i = \sum_{i=1}^{m}(R_i+1)Ln\left(1+\left(\frac{X_i}{\alpha}\right)^{\theta}\right) = U.$$

Accordingly, U has a gamma distribution with the shape parameter m and the scale parameter β_1, then

$$\hat{\beta}_1 = \frac{m}{U}\text{ and }\hat{\beta}_2 = \frac{n}{V},$$

where V has a gamma distribution with the shape parameter n and the scale parameter β_2.

Hence,

$$\hat{R} = \frac{1}{1+(m/n)(V/U)} = \frac{1}{1+(\beta_2/\beta_1)Z},$$

wher $Z = m\beta_1 V/n\beta_2 U$ has a F distribution with degrees of freedom $2n$ and $2m$,, taking into account the independence of the two gamma random variables U and V.

The pdf of $\hat{R}$ can be obtained as

$$f_{\hat{R}}(r) = \frac{\left(\frac{n}{m}\right)^n}{\beta(m,n)}\left(\frac{\beta_1}{\beta_2}\right)^n \times \frac{(1-r)^{n-1}}{r^{n+1}\left(1+\frac{n\beta_1}{m\beta_2}\left(\frac{1-r}{r}\right)\right)^{m+n}}, 0<r<1. \quad (7)$$

To calculate the variance of $\hat{R}$, The expectation and the second moment of $\hat{R}$ can be obtained, respectively, as follows:

$$E\left[\hat{R}\right] = \frac{m\Gamma(m+n)}{\Gamma(m+n+1)}\left(\frac{m}{n}\right)^m\left(\frac{\beta_2}{\beta_1}\right)^m \times {}_2F_1\left(m+n, m+1; m+n+1; 1-\frac{m\beta_2}{n\beta_1}\right), \quad (8)$$

$$E\left[\hat{R}^2\right] = \frac{m(m+1)\Gamma(m+n)}{\Gamma(m+n+2)}\left(\frac{m}{n}\right)^m\left(\frac{\beta_2}{\beta_1}\right)^m \times {}_2F_1\left(m+n, m+2; m+n+2; 1-\frac{m\beta_2}{n\beta_1}\right), \quad (9)$$

where ${}_2F_1$ is the hypergeometric function given by,

$${}_2F_1(a,b,c,w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tw)^{-a}\,dt,$$

cf. [13], p.110.

Hence, the variance of $\hat{R}$ can be calculated.

Since

$$\frac{1-R}{R} \times \frac{\hat{R}}{1-\hat{R}} = \frac{\beta_1}{\beta_2} \times \frac{mV}{nU} = Z \sim F(2n, 2m), \quad (10)$$

the $100(1-\gamma)\%$ confidence interval of R is

$$\left[\frac{1-\hat{R}}{(1-\hat{R})+\hat{R}\,F_{\frac{\gamma}{2}}(2n,2m)}, \frac{1-\hat{R}}{(1-\hat{R})+\hat{R}\,F_{1-\frac{\gamma}{2}}(2n,2m)}\right]. \quad (11)$$

The following is the interval estimation for R based on bootstrap confidence intervals.

3 Bootstrap confidence intervals

The confidence intervals are proposed based on the parametric bootstrap methods where the parametric model for the data is known as $f(\underline{x};\alpha,\theta,\beta_1)$ and $g(\underline{y};\alpha,\theta,\beta_2)$ up to the unknown parameters (β_1,β_2). The bootstraping data are sampled from $f\left(\underline{x};\alpha,\theta,\hat{\beta}_1\right)$ and $g\left(\underline{y};\alpha,\theta,\hat{\beta}_2\right)$, where $\left(\hat{\beta}_1,\hat{\beta}_2\right)$ the MLEs from the original data. A lot of papers dealt only with percentile bootstrap method (Boot-p) based on the idea of Efron [14] and bootstrap-t method (Boot-t) based on the idea of Hall [15], such as, [7,16] and among others. In this article, additional two types of Bootstrap CIs, Boot-BC and Boot-BCa based on the idea of DiCiccio and Efron [17], are discussed. The following algorithm is followed to obtain bootstraping samples for the four methods:

(1)Based on the original PROG-II-C samples, $X_{1:M} < X_{2:M} < \ldots < X_{m:M}$ and $Y_{1:N} < Y_{2:N} < \ldots < Y_{n:N}$,compute$\hat{\beta}_1, \hat{\beta}_2$ and $\hat{R}$ from (6).

(2)Use $\hat{\beta}_1$and $\hat{\beta}_2$ to generate a bootstrap samples, $\underline{X}^* \equiv X^*_{1:M} < X^*_{2:M} < \ldots < X^*_{m:M}$ and $\underline{Y}^* \equiv Y^*_{1:N} < Y^*_{2:N} < \ldots < Y^*_{n:N}$,respectively, with the same values of r_i, $i = 1,2,\ldots,m$ and $\acute{r}_j, j = 1,2,\ldots,n$ using the algorithm presented in [18].

(3)As in Step1 based on $\underline{X}^*$and $\underline{Y}^*$ compute the bootstrap data estimates of $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{R}$ say $\hat{\beta}_1^*, \hat{\beta}_2^*$ and $\hat{R}^*$.

(4)Repeat the previous steps 2 and 3 B times and arrange all $\hat{R}^*$ in ascending order to obtain the bootstrap sample $\left(\hat{R}^{*[1]}, \hat{R}^{*[2]}, \ldots, \hat{R}^{*[B]}\right)$.

3.1 Bootstrap-p confidence interval

Let $\Phi(z) = P(\hat{R}^* \leq z)$ be the cumulative distribution function of $\hat{R}^*$. Define $\hat{R}^*_{Boot} = \Phi^{-1}(z)$ for given z. The approximate bootstrap-p $100(1-\zeta)\%$ confidence interval of $\hat{R}^*$ is given by

$$\left[\hat{R}^*_{Boot}\left(\tfrac{\zeta}{2}\right), \hat{R}^*_{Boot}\left(1-\tfrac{\zeta}{2}\right)\right].$$

3.2 Bootstrap-t confidence inteval

Consider the order statistics $\mu^{*[1]} < \mu^{*[2]} < \ldots < \mu^{*[B]}$ where

$$\mu^{*[p]} = \frac{\sqrt{B}(\hat{R}^{*[j]}-\hat{R})}{\sqrt{Var\left(\hat{R}^{*[j]}\right)}}, \; j = 1,2,\ldots,B.$$

where $Var\left(\hat{R}^{*[j]}\right)$ is obtained using (8) and (9). Let $W(z) = P(\mu^* < z)$, be the cumulative distribution

function of μ^*.
For a given z, define

$$\hat{R}^*_{Boot\ -t} = \hat{R} + B^{\frac{-1}{2}} \sqrt{Var\left(\hat{R}^*\right)} W^{-1}(z).$$

Thus, the approximate bootstrap-t $100(1-\zeta)\%$ confidence interval of $\hat{R}^*$ is given by

$$\left[\hat{R}^*_{Boot\ -t}\left(\tfrac{\zeta}{2}\right),\ \hat{R}^*_{Boot\ -t}\left(1-\tfrac{\zeta}{2}\right)\right].$$

3.3 Bootstrap bias corrected confidence interval

Let $\Phi(z) = \zeta$ be the standard normal cumulative distribution function, with $z_\zeta = \Phi^{-1}(\zeta)$. Define the bias-correction constant $z_\circ$ from the following probability $P(\hat{R}^* \le \hat{R}) = G(z_\circ)$ where $G(.)$ is the CDF of the bootstrap distribution and

$$P(\hat{R}^* \le \hat{R}) = \frac{\#\left\{\hat{R}^{*[j]} < \hat{R}\right\}}{B},\ j = 1,2,...,B.$$

thus

$$z_\circ = \Phi^{-1}\left(\frac{\#\left\{\hat{R}^{*[j]} < \hat{R}\right\}}{B}\right),\ j = 1,2,...,B. \quad (12)$$

For a given ζ, and the bias-correction constant $z_\circ$,then

$$\hat{R}^*_{Boot\ -BC} = G^{-1}\left[\Phi\left(2z_\circ + z_\zeta\right)\right]. \quad (13)$$

Thus, the approximate bootstrap-BC $100(1-\zeta)\%$ confidence interval of $\hat{R}^*_{Boot\ -BC}$ is given by

$$\left[\hat{R}^*_{Boot\ -BC}\left(\tfrac{\zeta}{2}\right),\ \hat{R}^*_{Boot\ -BC}\left(1-\tfrac{\zeta}{2}\right)\right].$$

3.4 Bootstrap bias corrected accelerated confidence interval

Let $\Phi(z) = \zeta$ be the standard normal cumulative distribution function, with $z_\zeta = \Phi^{-1}(\zeta)$ and the bias-correction constant $z_\circ$ which is defined in (12). Then

$$\hat{R}^*_{Boot\ -BCa} = G^{-1}\left[\Phi\left(z_\circ + \frac{z_\circ + z_\zeta}{1 - a(z_\circ + z_\zeta)}\right)\right], \quad (14)$$

where a is called the acceleration factor which is estimated by a simple jack-knife method. Let $\underline{x}_i$ and $\underline{y}_j$ represent the original data with the ith point omitted and the jth point omitted, say $\underline{x}_2 = x_{1;M} < x_{3;M} < ... < x_{m:M}$, ,$\underline{y}_2 = y_{1;N} < y_{3;N} < ... < y_{n:N}$. Assume that $\hat{\Omega}^i_1 = \hat{\Omega}_1(\underline{x}_i)$ $\hat{\Omega}^j_2 = \hat{\Omega}_2(\underline{y}_j)$ be the MLE estimate of, $\Omega_1 \equiv \beta_1$ and $\Omega_2 \equiv \beta_2$,constructed from this data. Let $\bar{\Omega}_1$ and $\bar{\Omega}_2$ be the mean of the $\hat{\Omega}^i_1$'s and $\hat{\Omega}^j_2$'s, respectively . Then $a = a_1 + a_2$ is estimated by

$$a_1 = \frac{\sum_{i=1}^m \left(\bar{\Omega}_1 - \hat{\Omega}^i_1\right)^3}{6\left[\sum_{i=1}^m \left(\bar{\Omega}_1 - \hat{\Omega}^i_1\right)^2\right]^{\frac{3}{2}}} \text{ and } a_2 = \frac{\sum_{i=1}^n \left(\bar{\Omega}_2 - \hat{\Omega}^j_2\right)^3}{6\left[\sum_{j=1}^n \left(\bar{\Omega}_2 - \hat{\Omega}^j_2\right)^2\right]^{\frac{3}{2}}}.$$

For more details see [19] and [20]. If $a_1 = a_2 = 0$, Equation (14) reduces to equation (13). Then, the approximate bootstrap-BCa $100(1-\zeta)\%$ confidence interval of $\hat{R}^*_{Boot\ -BCa}$ is given by

$$\left[\hat{R}^*_{Boot\ -BCa}\left(\tfrac{\zeta}{2}\right),\ \hat{R}^*_{Boot\ -BCa}\left(1-\tfrac{\zeta}{2}\right)\right].$$

4 Bayesian Estimation of R

The Bayesian approach deals with the parameters as random, and uncertainties on the parameters are described by a joint prior distribution, which is developed before the failure data are collected, and is based on historical data, experience with similar products, design specifications, and experts' opinions. The ability of incorporating prior knowledge in the analysis makes the Bayesian approach very helpful in the reliability analysis because one of the main challenges associated with the reliability analysis is the limited availability of data.

Let the prior knowledge of parameters β_1 and, β_2 be described by the following independent prior distributions:

$$\left.\begin{aligned} \pi(\beta_1) &= \frac{\lambda_1^{\mu_1}}{\Gamma(\mu_1)} \beta_1^{\mu_1 - 1} e^{-\beta_1 \lambda_1},\ \beta_1, \mu_1, \lambda_1 > 0, \\ \pi(\beta_2) &= \frac{\lambda_2^{\mu_2}}{\Gamma(\mu_2)} \beta_2^{\mu_2 - 1} e^{-\beta_2 \lambda_2},\ \beta_2, \mu_2, \lambda_2 > 0 \end{aligned}\right\}. \quad (15)$$

Hence, the joint prior of the parametersβ_1 and, β_2 can be written as follow

$$\pi(\beta_1, \beta_2) = \frac{\lambda_1^{\mu_1}}{\Gamma(\mu_1)} \frac{\lambda_2^{\mu_2}}{\Gamma(\mu_2)} \beta_1^{\mu_1 - 1} \beta_2^{\mu_2 - 1} e^{-(\beta_1 \lambda_1 + \beta_2 \lambda_2)} \quad (16)$$

The joint posterior density function of β_1 and, β_2, denoted by $\pi^*(\beta_1, \beta_2 | \alpha, \theta, \underline{x}, \underline{y})$ can be written as

$$\pi^*(\beta_1, \beta_2 | \alpha, \theta, \underline{x}, \underline{y}) = \frac{L\left(\beta_1, \beta_2 | \alpha, \theta, \underline{x}, \underline{y}\right) \times \pi(\beta_1, \beta_2)}{\int_0^\infty \int_0^\infty L\left(\beta_1, \beta_2 | \alpha, \theta, \underline{x}, \underline{y}\right) \times \pi(\beta_1, \beta_2)\, d\beta_1 d\beta_2}. \quad (17)$$

Then

$$\pi^*(\beta_1, \beta_2 | \alpha, \theta, \underline{x}, \underline{y}) \propto \beta_1^{n + \mu_1 - 1} \beta_2^{m + \mu_2 - 1} e^{-(\beta_1 \lambda_1 + \beta_2 \lambda_2)} \times \prod_{i=1}^m \left(1 + \left(\frac{x_i}{\alpha}\right)^\theta\right)^{-\beta_1 (r_i + 1)} \times \prod_{j=1}^n \left(1 + \left(\frac{y_j}{\alpha}\right)^\theta\right)^{-\beta_2 (\acute{r}_j + 1)} \quad (18)$$

The conditional posterior densities of β_1 and β_2 can be given as

$$\pi_1^*(\beta_1 | \beta_2, \alpha, \theta, \underline{x}, \underline{y}) \equiv gamma\left[m + \mu_1, \lambda_1 + \sum_{i=1}^m \left\{(r_i + 1) \ln\left(1 + \left(\tfrac{x_i}{\alpha}\right)^\theta\right)\right\}\right], \quad (19)$$

$$\pi_2^*(\beta_2|\beta_1,\alpha,\theta,\underline{x},\underline{y}) \equiv$$
$$gamma\left[n+\mu_2,\lambda_2+\sum_{j=1}^{n}\left\{(\acute{r}_j+1)\ln\left(1+(\tfrac{y_j}{\alpha})^{\theta}\right)\right\}\right]. \quad (20)$$

Applying transformation techniques, the posterior PDF of R is

$$f_{R|Data}(r)=K\times\left(\frac{1-r}{r}\right)^{m+\mu_1}\times$$
$$\left(1+\frac{\Phi_1(r_i,\lambda_1,x_i)}{\Phi_2(\acute{r}_j,\lambda_2,y_j)}\left(\frac{1-r}{r}\right)\right)^{-(m+n+\mu_1+\mu_2)}, 0<r<1, \quad (21)$$

where

$$\Phi_1(r_i,\lambda_1,x_i)=\lambda_1+\sum_{i=1}^{m}\left\{(r_i+1)\ln\left(1+(\tfrac{x_i}{\alpha})^{\theta}\right)\right\},$$
$$\Phi_2(\acute{r}_j,\lambda_2,y_j)=\lambda_2+\sum_{j=1}^{n}\left\{(\acute{r}_j+1)\ln\left(1+(\tfrac{y_j}{\alpha})^{\theta}\right)\right\}$$

and

$$K=\frac{1}{\beta\,(m+\mu_1,n+\mu_2)}\left(\frac{\Phi_1(r_i,\lambda_1,x_i)}{\Phi_2(\acute{r}_j,\lambda_2,y_j)}\right)^{m+\mu_1}.$$

The Bayes estimate of R using the squared error loss function, say $\hat{R}_{BSEL}$, can be obtained by calculating the posterior mean of R as follows

$$\hat{R}_{BSEL}=\int_0^1 r f_{R|Data}(r)dr \quad (22)$$
$$=K\int_0^1 r\left(\frac{1-r}{r}\right)^{m+\mu_1}$$
$$\times\left(1+\frac{\Phi_1(r_i,\lambda_1,x_i)}{\Phi_2(\acute{r}_j,\lambda_2,y_j)}\left(\frac{1-r}{r}\right)\right)^{-(m+n+\mu_1+\mu_2)}dr$$
$$=\left(\frac{\Phi_2(\acute{r}_j,\lambda_2,y_j)}{\Phi_1(r_i,\lambda_1,x_i)}\right)^{n+\mu_2}\frac{(m+\mu_1)(n+\mu_2)(n+\mu_2+1)}{\lambda(\lambda+1)(\lambda+2)}\times$$
$${}_2F_1\left(\lambda,n+\mu_2+2;\lambda+3;1-\frac{\Phi_2(\acute{r}_j,\lambda_2,y_j)}{\Phi_1(r_i,\lambda_1,x_i)}\right), \quad (23)$$

where $\lambda=m+n+\mu_1+\mu_2$. The Bayes estimate of R using the LINEX loss function, say $\hat{R}_{BLN}$, can be obtained by calculating the posterior median of R,from the following equation

$$\int_0^{\hat{R}_{BLN}} f_{R|Data}(r)dr=0.5. \quad (24)$$

Then, $\hat{R}_{BLN}$ is the solution of $f_{R|Data}(\hat{R}_{BLN})=0$, after differentiation the both sides of Equation(24) with respect to $\hat{R}_{BLN}$. The mode of the posterior distribution can be obtained numerically by maximizing the PDF of R.

In some cases, it is not easy to obtain the estmation of R from (23) and (24), so the acceptance rejection principle can be used to obtain the Bayesian point estimates of R and also to obtain the corresponding credible interval. The acceptance rejection principle is a simulation procedure and used to generate samples from the posterior distribution. The algorithm of this procedure is introduced and proven by [21] and the steps for Bayesian estimation of R are described in [1]

5 Applications

In this section, two examples are introduced one of them is simulated and the other is a real data set. At first the following lemma is introduced to make the corresponding known parameters in two populations are the same, in case of application our model to a real life data.

Lemma 2. If the random variable $T\sim WG(\alpha,\theta,\beta)$,then $Y=\left(\frac{T}{\alpha}\right)^{\theta}\sim WG(1,1,\beta)$.

Proof. The proof is easy to obtain.

Example 1. (Simulated data)

In this example, two PROG-II-C samples from WG distributions are generated. The algorithm of generation is performed according to the algorithm described in Balakrishnan and Sandhu [18] as the following:

(1)Specify the values of M,N,m and n.
(2)Specify the values of $r_i, i=1,2,...,m$ and $\acute{r}_j, j=1,2,...,n$.
(3)Specify the values of the parameters α,θ,β_1 and β_2.
(4)Generate a random sample with size M and censoring size m from the random variable X , the set of data can be considered as:

$$X_{1;m,M}<X_{2;m,M}<...<X_{m;m,M}.$$

(5)Generate a random sample with size N and censoring size n from the random variable Y , the set of data can be considered as:

$$y_{1;n,N}<y_{2;n,N}<...<y_{n;n,N}.$$

(6)Use the two preceding PROG-II-C samples to compute the MLEs of the stress strength parameter.
(6)Compute the 95% bootstrap conidence intervals for R, using the steps described in Section 3.
(7)Compute the Baye estimates of the model parameters based on acceptance rejection principle.

A simulation data for two PROG-II-C samples from WG distributions with true values $\alpha=6$, $\theta=5.85$, $\beta_1=5$ and $\beta_2=3$, so $R=0.375$. Using progressive censoring schemes $M=30$, $m=10$ with $\underline{r}=(1,0,0,1,0,0,1,0,1,16)$ and $N=26, n=10$ with $\underline{\acute{r}}=(0,1,1,0,1,0,1,1,0,11)$, data have been approximated to two decimal places and they have been presented in Table 1 and Table 2.

Table 1: Simulated PROG-II-C strength data

1.6505	2.4991	3.1526	3.6722	3.9882
2.4604	2.8343	3.5303	3.6857	4.0938

Table 2: Simulated PROG-II-C stress data

1.9338	2.9807	3.6396	3.6823	4.0124
2.6914	3.5425	3.6490	3.8957	4.3085

Table 3: Different point estimates for R

$(.)_{ML}$	$(.)_{Boot-p}$	$(.)_{Boot-t}$	$(.)_{BSEL}$	$(.)_{BLN}$
0.494	0.494	0.482	0.502	0.482

Table 4: 95% confidence intervals for R.

Method	R	Length
ACI	$[0.2824, 0.7050]$	0.42258
Boot -p CI	$[0.2679, 0.6891]$	0.42116
Boot -t CI	$[0.4807, 0.4840]$	0.00328
Boot-BC CI	$[0.2679, 0.6886]$	0.42067
Boot-BCa CI	$[0.1888, 0.6660]$	0.47717
CRI	$[0.0297, 0.9788]$	0.94911

The following figure shows the posterior density function of R, where the prior knowledge parameters are $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 0.001$.

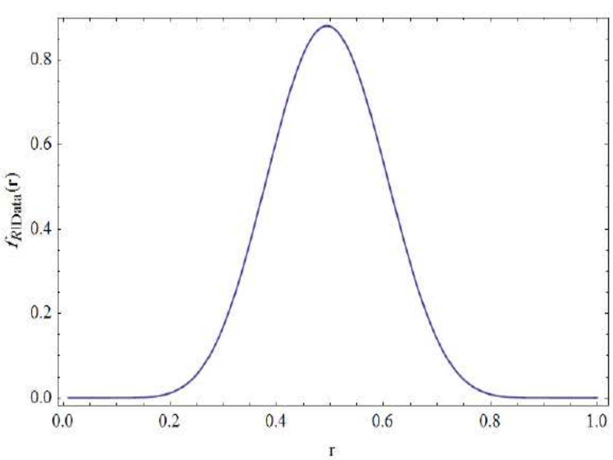

Fig. 1: The posterior density function of R

Example 2. (Real-life data)

The data of Xia et al. [22] can be used as application of stress strength model under PROG-II-C, where these data represent the ordered breaking strengths of jute fibre at gauge lengths 10 mm and 20 mm.

Table 5: Values of α, θ, β,K-S and p - values

Data set	1	2
α	53318.4	103809.0
θ	1.62514	1.36076
β	2741.11	2128.26
K-S	0.1058	0.1490
p - value	0.8553	0.4730

Table 6: Transformed Data Set 1.

0.01	0.065	0.126	0.25	0.517	0.86
0.012	0.073	0.172	0.29	0.56	0.88
0.038	0.082	0.18	0.319	0.66	0.88
0.043	0.094	0.21	0.329	0.75	0.93
0.052	0.099	0.23	0.384	0.82	1.04.

Table 7: Transformed Data Set 2.

0.02	0.078	0.16	0.33	0.795	1.031
0.03	0.094	0.18	0.43	0.86	1.085
0.029	0.097	0.19	0.48	0.86	1.13
0.05	0.101	0.20	0.553	0.87	1.24
0.062	0.13	0.27	0.621	0.889	1.25.

Table 8: Values of α, θ, β,K-S and p - values to transformed data Sets

Data set	1	2
α	1.0000	1.0000
θ	1.0000	1.0000
β	2741.11	2128.26
K-S	0.1058	0.1490
p - value	0.8553	0.4730

Table 5. shows the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distribution functions for two data sets separately. Also Mathematica 9 program is used to find the distribution parameters in two cases.

Since the p-value is quite high in two cases, it is evident to accept the null hypothesis that the data is coming from the WG distribution. From Table 5, it is noted that the values of α and θ are different in the data sets, so Lemma 2 can be used.

Where the data in Table 6 and Table 7 are multiplied by 10^{-3}. From the transformed data sets, the following results have been obtained in Table 8.

Using progressive censoring schemes $M = 30$, $m = 15$ with $\underline{r} = (1,0,0,1,0,0,1,0,1,0,0,0,0,0,11)$ and $N = 30, n = 10$ with $\underline{\acute{r}} = (2,2,2,0,1,0,2,1,0,10)$, data have been presented in Table 9.

The results about the statistical inference of R are given in Table 10 and Table 11.

Figure 2 shows the posterior density function of R, where the prior knowledge parameters are
$\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 0$.

Table 9: Generated PROG-II-C data

$x_{1;15,30} < x_{2;15,30} < \ldots < x_{15;15,30} y_{1;10,30} < y_{2;10,30} < \ldots < y_{10;10,30}$				
0.00001	0.000065	0.000178	0.00002	0.000078
0.000012	0.000073	0.00021	0.000027	0.000097
0.000038	0.000082	0.000225	0.000029	0.000101
0.000043	0.000094	0.00025	0.00005	0.000132
0.000052	0.000099	0.000288	0.000062	0.000157

Table 10: Different point estimates for R .

$(.)_{ML}$	$(.)_{Boot-p}$	$(.)_{Boot-t}$	$(.)_{BSEL}$	$(.)_{BLN}$
0.545	0.552	0.534	0.486	0.464

Table 11: 95% confidence intervals for R.

Method	R	Length
ACI	$[0.3470, 0.7438]$	0.396776
Boot -p CI	$[0.3433, 0.7463]$	0.403022
Boot -t CI	$[0.5335, 0.5364]$	0.002880
Boot-BC CI	$[0.3385, 0.7364]$	0.397832
Boot-BCa CI	$[0.2994, 0.7161]$	0.416684
CRI	$[0.0286, 0.9829]$	0.954277

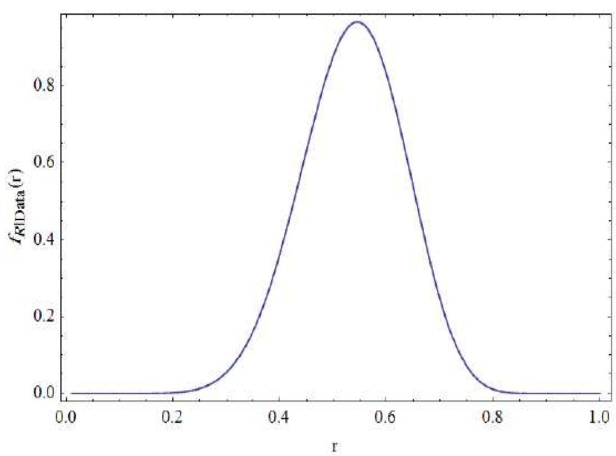

Fig. 2: The posterior density function of R

6 Conclusion

In this paper, the estimation of the stress – strength function for two WG distributions under progressive censoring has been studied. The two WG distributions are assumed to be have commonly known parameters, one of them shape parameter and the other is scale,while the third shape parameter is different in the both two distributions. The MLE of the stress – strength parameter is calculated. Four types of bootstrap CIs are used to obtain 95% CIs for R. The acceptance rejection principle is used to obtain the Bayes estimates of R and the corresponding credible interval. Two applications are given to illustrate the proposed methods.

References

[1] B. Saraçoğlu, I. Kinaci, D. Kundu, On estimation of R=P(Y<X) for exponential distribution under progressive type-II censoring, J. Stat. Comput. Sim. **82,5,** 729-744 (2012).

[2] P.S. Bithas, Weibull-gamma composite distribution: An alternative multipath/shadowing fading model, Electron. Lett. **45**,749-751(2009).

[3] G. Molenberghs and G. Verbeke, On the Weibull-gamma frailty model, its infinite moments, and its connection to generalized log-logistic, logistic, Cauchy, and extreme-value distributions, J. Stat. Plan. Infer. **141**, 861-868 (2011).

[4] M.A.W. Mahmoud, Y. Abdel-Aty, N. M. Mohamed, G. G. Hamedani, Recurrence relations for moments of dual generalized order statistics from Weibull-Gamma distribution and its characterizations, Journal of Statistics Applications & Probability **3**, 189-199 (2014).

[5] N. Balakrishnan and R. Aggarwala, Progressive Censoring: Theory, Methods and Applications, Birkhäuser, Boston 2000.

[6] N. Balakrishnan, Progressive censoring methodology: An appraisal, Test **16**, 211-296 (2007) (with discussions).

[7] R. M. EL-Sagheer and Ahsanullah M, Statistical inference for A step - stress partially accelerated life test model based on progressively type - II censored data from Lomax distribution, Journal of Applied Statistical Science **21**, 307-323 (2015).

[8] R. M. EL-Sagheer, Inferences in constant-partially accelerated life tests based on progressive type-II censoring, Bulletin of the Malaysian Mathematical Sciences Society (2016), doi:10.1007/s40840-016-0311-9.

[9] R. M. EL-Sagheer, Estimation of parameters of Weibull-gamma distribution based on progressively censored data, Stat. Pap. (2016), doi:10.1007/s00362-016-0787-2.

[10] M.A.W. Mahmoud, On stress-strength model in Weibull case, The Egyptian Statistical Journal **40**, 119-126 (1996).

[11] M.A.W. Mahmoud, R. M. EL-Sagheer, A. A. Soliman, A. H. Abd Ellah, Bayesian estimation of P[Y ¡ X] Based on Record Values from the Lomax Distribution and MCMC Technique, Journal of Modern Applied Statistical Methods **15**, 488-510 (2016).

[12] R. Valiollahi, A. Asgharzadeh, M. Z. Raqab, Estimation of P(Y ¡ X) for Weibull Distribution Under Progressive Type-II Censoring, Communications in Statistics - Theory and Methods **42,24**, 4476-4498 (2013).

[13] N. M. Temme, Special Functions: An Introduction to the Classical Functions of Mathematical Physics, John Wiley and Sons, New York 1996.

[14] B. Efron, The jackknife, the bootstrap, and other resampling plans, Society for Industrial and Applied Mathematics, Philadelphia 1982.

[15] P. Hall, Theoretical comparison of bootstrap confidence intervals, Ann. Stat. **16**, 927–953 (1988).

[16] A. A. Soliman, A. H. Abd-Ellah, N. A. Abou-Elheggag, E. A. Ahmed, Modified Weibull model: A Bayes study using MCMC approach based on progressive censoring data, Reliability Engineering and System Safety **100**, 48–57 (2012).

[17] T. J. DiCiccio and B. Efron, Bootstrap confidence intervals, Stat. Sci. **11**, 189–228 (1996).

[18] N. Balakrishnan and R. A. Sandhu, A simple simulation algorithm for generating progressively type-II censored samples, The American Statistician **49**, 229–230 (1995).
[19] Efron B, Tibshirani RJ. An Introduction to the Bootstrap. London: Chapman and Hall; 1993.
[20] Davison AC, Hinkley DV. Bootstrap methods and their application. Cambridge: Cambridge university; 1997.
[21] L. Devroye, A simple algorithm for generating random variates with a log-concave density function, Computing **33**, 247–257 (1984).
[22] Z.P. Xia, J.Y. Yu, L.D. Cheng, L.F. Liu, W.M. Wang, Study on the breaking strength of jute fibers using modified Weibull distribution, Composites Part A: Applied Science and Manufacturing **40**, 54–59 (2009).

Mohamed A. W. Mahmoud is presently employed as a professor of Mathematical statistics in Department of Mathematics and Dean of Faculty of Science, Al-Azhar University, Cairo, Egypt. He received his PhD in Mathematical statistics in 1984 from Assiut University, Egypt. His research interests include: Theory of reliability, ordered data, characterization, statistical inference, distribution theory, discriminant analysisand classes of life distributions. He published and Co-authored more than 100 papers in reputed international journals. He supervised more than 62 M. Sc. thesis and more than 75 Ph. D. thesis.

Rashad M. EL-Sagheer is a lecturer of Mathematical Statistics at Mathematics Department Faculty of Science AL -Azhar University Cairo Egypt. He received Ph. D. from Faculty of Science Sohag University Egypt in 2014. His areas of research where he has several publications in the international journals and conferences include: Statistical inference,Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability,censored data, Life testing and Distribution theory. He published and Co-authored more than 25 papers in reputed international journals. He supervised for M. Sc. and Ph. D. students.

Mahmoud M.M. Mansour is an assistant lecturer of Mathematical Statistics at Basic Science Department, Faculty of Engineering, The Britsh University in Egypt, Cairo, Egypt. He received MSc from Faculty of Science AL -Azhar University Egypt in 2014. His research interests include: Theory of reliability, Censored data, Life testing, distribution theory.