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Linear Combinations of Generalized q-Starlike Functions

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Abstract: In this paper, a new concept of bounded radius rotation is introduced to define a new class of generalized q-starlike functions using the quantum calculus. Some geometric properties of linear combinations of such functions are studied in this paper. The techniques of this paper may motivate further research activities.

Keywords: Convex, q-starlike, bounded radius rotation, linear combination.
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1 Introduction

Let A denote the class of analytic functions $f(z)$ defined in the open unit disc $E=\{z:|z|<1\}$ with the normalization $f(0)=0$, $f'(0)=1$. One-one analytic functions in this class are usually called univalent analytic functions. A function $f\in A$ is called starlike $(f\in S^*)$ if $\Re\{\frac{zf'(z)}{f(z)}\}>0$, $z\in E$. Also f is convex $(f\in C)$, if and only if, $zf'\in S^*$.
In 1990, a q-analogue of starlike functions was introduced by Ismail et. al. [2] by using q-difference operator D_qf, $0<q<1$. This operator is defined by the equation

$$(D_qf)(z)=\frac{f(z)-f(qz)}{z(1-q)},\quad z\neq 0,\quad (D_qf)(0)=f'(0)\tag{1}$$

From (1.1), we can deduce that

$$D_qf(z)=1+\sum_{n=2}^{\infty}[n]_qa_nz^n,\tag{2}$$

where

$$[n]_q=\frac{(1-q^n)}{1-q}\tag{3}$$

As $q\to 1^-$, $[n]_q\to n$.

The set $B\subset\mathbb{C}$ is defined as μ-geometric, if it contains all geometric sequences $\{z\mu\}_{n=0}^{\infty}$ for $z\in B$, $\mu\in E$.

A function $f\in A$ is said to belong to the class S_q^* of q-starlike functions, if

$$\left|\frac{z}{f(z)}(D_qf(z))-\frac{1}{1-q}\right|\leq\frac{1}{1-q},\quad z\in E,\tag{4}$$

where $D_qf(z)$ is defined by (1.1) on a q-geometric set with $q\in(0,1)$.
As $q\to 1^-$, the closed dics $|w-(1-q)^{-1}|\leq(1-q)^{-1}$ becomes the right half plane and the class S_q^* reduces to class S^* of starlike function.

It is known [3,6] that (1.4) holds if and only if

$$\frac{zD_qf(z)}{f(z)}\prec\frac{1+z}{1-qz},\tag{5}$$

where $\prec$ denotes subordination.
From (1.5), it can be seen that the linear transformation $\frac{1+z}{1-qz}$ maps $|z|=r$ onto the circle with center $C(r)=\frac{1+qr^2}{1-q^2r^2}$ and the radius $\sigma(r)=\frac{(1+q)r}{1-q^2r^2}$. Thus, using Subordination principle, we can write

$$\left|\frac{zD_qf(z)}{f(z)}-\frac{1+qr^2}{1-q^2r^2}\right|\leq\frac{(1+q)r}{1-q^2r^2}.\tag{6}$$

We can define a related class C^* as follows.

Let $f=zf_1'$, $f_1\in A$. Then $f_1\in C_q^*$, if and only if, $f\in S^*$.

When $q\to 1^-$, C_q^* reduces to class C of convex univalent

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functions.

For the following definitions, we refer to [3].

Definition 1. *Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p \in P_m(q)$, if and only if,*

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z),$$

where

$$p_i(z) \prec \frac{1+z}{1-qz}, \quad i = 1,2, \quad q \in (0,1), \quad m \geq 2.$$

For $m = 2$, $P_2(q) = P(q)$ consists of all functions subordinate to $\frac{1+z}{1-qz}$, $z \in E$. Also, $\lim_{q\to 1^-} P(q) = P$, the class of functions with positive real part.

Definition 2. *Let $f \in A$. Then $f \in R_q^*(m)$, if and only if, $\frac{zD_q f}{f} \in P_m(q)$, $z \in E$.*

f, in this case, is called a function of q-bounded radius rotation.

We note that $R_q^*(2) = S_q^*$ and as $q \to 1^-$, $R_q^*(m) = R_m$, the class of functions with bounded radius rotation.

Following the technique of Robertson [4], the inequality (1.6) can easily be generalized for the class $R_q^*(m)$ of function of q-bounded radius rotation as follows.

Lemma 1. *Let $f \in R_q^*(m)$. Then, for $m \geq 2$, $q \in (0,1)$,*

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1+qr^2}{1-q^2r^2} \right| \leq \frac{\frac{m}{2}(1+q)r}{1-q^2r^2}. \tag{7}$$

We shall need the following lemmas to prove our main results.

Lemma 2. [5]. *Let a, d, k, ρ be reals with $a > d \geq 0, k > 0$ and $\rho \in (0,\pi)$. Suppose $|u-a| \leq d$ and $|v-a| \leq d$ and set*

$$w = \frac{u}{1+ke^{i\rho}} + \frac{v}{1+k^{-1}e^{-i\rho}}.$$

Then

$$\Re(w) \geq a - d(\sec\frac{\rho}{2}).$$

Lemma 3. [3]. *Let $f \in R_q^*(m)$. Then $f \in S_q^*$ in $|z| < r_q^*$, where*

$$r_q^* = \frac{4}{m(1+q) + \sqrt{m^2(1+q)^2 - 16q}}. \tag{8}$$

From Lemma 1.1 and a modified version of well known result due to Brannan [1], we have:

Lemma 4. *Let $f \in R_q^*(m)$. Then*

$$|\arg f(z)| \leq \frac{m}{2}(1+q)\sin^{-1} r.$$

In our discussion, throughout this paper, we will take $m \geq 2$ and $q \in (0,1)$ unless otherwise stated.

2 Main Results

Theorem 1. *Let $f, g \in R_q^*(m)$ and let*

$$F(z) = \gamma f(z) + (1-\gamma) g(z),$$

where $0 \leq \arg\frac{\gamma}{1-\gamma} \leq \sigma < \pi$. Then $F \in S_q^*$ in $|z| < r_{q,m}$ where $r_{q,m}$ is the smallest positive value of r satisfying the equation

$$T(r) = (1+qr^2)\cos\left(\frac{\sigma}{2} + \frac{m}{2}(1+q)\sin^{-1}\right) - \frac{m}{2}(1+q)r = 0.$$

Proof. q-difference operator of F gives us

$$D_q F(z) = \gamma D_q f(z) + (1-\gamma) D_q g(z),$$

and therefore

$$\begin{aligned} \frac{zD_q F(z)}{F(z)} &= \frac{\gamma z D_q f(z) + (1-\gamma) D_q g(z)}{\gamma f(z) + (1-\gamma) g(z)} \\ &= \frac{zD_q f(z)}{f(z)} \left[1 + \left(\frac{\gamma}{1-\gamma} \cdot \frac{f(z)}{g(z)}\right)^{-1}\right]^{-1} \\ &\quad + \frac{zD_q g(z)}{g(z)} \left[1 + \left(\frac{\gamma}{1-\gamma} \cdot \frac{f(z)}{g(z)}\right)\right]^{-1}. \end{aligned} \tag{9}$$

Let

$$u = \frac{zD_q g(z)}{g(z)}, \quad v = \frac{zD_q f(z)}{f(z)}, \quad k = \left|\frac{\gamma}{1-\gamma} \cdot \frac{f(z)}{g(z)}\right|. \tag{10}$$

From (2.1) and (2.2), we have

$$w(z) = \frac{zD_q F(z)}{F(z)} = \frac{u}{1+ke^{i\rho}} + \frac{v}{1+k^{-1}e^{-i\rho}}. \tag{11}$$

We now apply Lemma 1.1 and Lemma 1.2 to (2.3) and have

$$\Re\left\{\frac{zD_q F(z)}{F(z)}\right\} \geq \frac{1+qr^2}{1-q^2r^2} - \frac{\frac{m}{2}(1+q)r}{1-q^2r^2} \sec\left(\frac{\rho}{2}\right), \tag{12}$$

where

$$\begin{aligned} \rho &= \arg\left(\frac{\gamma}{1-\gamma} \cdot \frac{f(z)}{g(z)}\right) \\ &= 2n\pi + \arg\left(\frac{\gamma}{1-\gamma}\right) + \arg f(z) - \arg g(z). \end{aligned}$$

This gives us

$$|\rho| = \sigma + m(1+q)\sin^{-1} r.$$

Therefore

$$\Re\left\{\frac{zD_q F(z)}{F(z)}\right\} > 0,$$

if

$$T(r) = (1+qr^2)\cos\left(\frac{\sigma}{2} + \frac{m}{2}(1+q)\sin^{-1} r\right) - \frac{m}{2}(1+q)r > 0.$$

We note that

$$T(r) = \cos\frac{\sigma}{2}, \quad for \quad r = 0,$$

and

$$T(r) = -\frac{m}{2}(1+q)\sin\frac{\pi-\sigma}{m(1+q)} < 0$$
$$when \quad r = \sin\left(\frac{\pi-\sigma}{m(1+q)}\right).$$

This implies that $T(r) = 0$ has a root in the interval $(0, \sin(\frac{\pi-\sigma}{m(1+q)}))$ and right hand side of (2.4) is positive in the disc $|z| < r_{q,m}$, where $r_{q,m}$ is the least positive value of r satisfying $T(r) = 0$. This completes the proof.

We note the following special cases as:

Corollary 1. *For $q \to 1^-$, $f, g \in R_m$ are the functions of bounded radius rotation and it follows, from Theorem 2.1, that $F = \gamma f + (1-\gamma)g$ is starlike in $|z| < r_m^*$, where r_m^* is the least positive root of*

$$T_m^*(r) = A(1+r^2) - mr = 0, \quad A = \cos\left(\frac{\sigma}{2} + m\sin^{-1} r\right).$$

This gives us

$$r_m^* = \frac{m + \sqrt{m^2 - 4A^2}}{2A}.$$

As a special case of Corollary 2.1, we take $m = 2$. Then

$$A = A_2 = \cos(\frac{\sigma}{2} + 2\sin^{-1} r), \quad \text{and} \quad \lim_{q\to 1^-} R_q^*(2) = S^*.$$

From these observations, we deduce the radius of starlikeness of linear combination of two starlike functions is given by $r_2^* = \frac{1-\sqrt{1-A_2^2}}{A^2}$.

Corollary 2. *Let $m = 2$. Then, in Theorem 2.1, $f, g \in S_q^*$ and it follows that*

$$\Re\{\frac{zD_qF(z)}{F(z)}\} > 0 \quad in \quad |z| < r_q^*,$$

where r_q^* is the least positive root of

$$T_q^*(r) = B_1 qr^2 - (1+q)r + B_1 = 0,$$

$$B_1 = \cos(\frac{\sigma}{2} + (1+q)\sin^{-1} r).$$

This gives us

$$r_q^* = \frac{(1+q) - \sqrt{(1+q)^2 - 4qB_1^2}}{2qB_1}.$$

Theorem 2. *Let $f, g \in \bigcap_{0<q<1} S_q^*$ and let*

$$F = \gamma g + (1-\gamma)g, \quad with \quad 0 \le \arg(\frac{\gamma}{1-\gamma}) \le \sigma < \pi.$$

Then F maps the disc $|z| < r_\sigma$ onto a convex domain, where r_σ is the least positive value of r that satisfies the equation

$$T_\sigma(r) = Br^2 - 2r_1 r + Br_1^2, r_1 = 2-\sqrt{3},$$
$$B = \cos\left(\frac{\sigma}{2} + 2\sin^{-1}\left(\frac{r}{r_1}\right)\right).$$

Proof. It has been shown in [2] that

$$\bigcap_{0<q<1} S_q^* = S^*.$$

It is well known that $f \in S^*$ is convex in the disc $|z| < r_1 = 2-\sqrt{3}$.
With these facts, we proceed to find the radius of convexity for the function F following the technique used in Theorem 2.1.
We can write

$$\frac{(zF'(z))'}{F'(z)} = \frac{(zf'(z))'}{f'(z)}\left[1 + \left(\frac{\gamma}{1-\gamma}\cdot\frac{f'(z)}{g'(z)}\right)^{-1}\right]^{-1} + \frac{(zg'(z))'}{g'(z)}\left[1 + \left(\frac{\gamma}{1-\gamma}\cdot\frac{f'(z)}{g'(z)}\right)\right]^{-1}.$$

We take

$$k = |\frac{\gamma}{1-\gamma}\cdot\frac{f'(z)}{g'(z)}|, \quad \rho = \arg(\frac{\gamma}{1-\gamma}\cdot\frac{f'(z)}{g'(z)})$$

and

$$v = \frac{(zf'(z))'}{f'(z)}, \quad u = \frac{(zg'(z))'}{g'(z)}.$$

Now, for $r_1 = 2\sqrt{3}$, we have

$$\left|u - \frac{r_1^2 + r^2}{r_1^2 - r^2}\right| \le \frac{2rr_1}{r_1^2 - r^2},$$
$$\left|v - \frac{r_1^2 + r^2}{r_1^2 - r^2}\right| \le \frac{2rr_1}{r_1^2 - r^2}.$$

We formulate

$$w(z) = \frac{(zF'(z))'}{F'(z)} = \frac{u}{1+ke^{i\rho}} + \frac{v}{1+k^{-1}e^{-i\rho}} \qquad (13)$$

with

$$\rho = \arg\left(\frac{\gamma}{1-\gamma}\cdot\frac{f'(z)}{g'(z)}\right)$$
$$= 2n\pi + \arg\left(\frac{\gamma}{1-\gamma}\right) + \arg f'(z) - \arg g'(z).$$

This gives us

$$|\rho| = \sigma + 4\sin^{-1}\left(\frac{r}{r_1}\right).$$

Therefore

$$\Re\left\{\frac{(zF'(z))'}{F'(z)}\right\} > 0,$$

if

$$T_\sigma(r) = \left[(r_1^2 + r^2)\cos\left(\frac{\sigma}{2} + 2\sin^{-1}\left(\frac{r}{r_1}\right) - 2r_1 r\right)\right] = 0,$$

where

$$r_1 = 2 - \sqrt{3}.$$

That is

$$T_\sigma(r) = Br^2 - 2r_1 r + Br_1^2, \quad B = \cos\left(\frac{\sigma}{2} + 2\sin^{-1}\left(\frac{r}{r_1}\right)\right).$$

This gives us

$$r_\sigma = \frac{r_1 - \sqrt{r_1^2 - B^2 r_1^2}}{B}. \tag{14}$$

We note that $r_\sigma \in (0, r_1 \sin(\frac{\pi-\sigma}{4}))$. Hence F maps the dics $|z| < r_\sigma$ onto a convex domain, where r_σ is given by (2.6).

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