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Integral Inequalities via α-Prinvex Functions

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Abstract: Hermite-Hadamard's inequality is considered as one of the most interesting result in theory of inequalities. The main objective of this article is to derive several new Hermite-Hadamard type of integral inequalities via α-preinvex and $\log-\alpha$-preinvex functions. Some special cases are also discussed.

Keywords: Convex functions, preinvex functions, α-preinvex functions, Hermite-Hadamard inequalities.
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1 Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Inequality (1) can be considered as necessary and sufficient condition for a function to be convex. For useful details on Hermite-Hadamard type of integral inequalities, see [1,2,3,4,6,7,16,17,20,22,23].
In recent years, several extensions and generalizations have been considered for classical convexity using novel and innovative techniques, see [5,8,10,11,18,21,22,24]. A significant generalization of convex functions was that of invex functions which was introduced by Hanson [9]. Weir et al. [24] introduced the class of convex functions, which is called preinvex functions. It is known that a preinvex function may not be a convex function see [24]. Different properties and the role of preinvex functions in optimization, variational inequalities, equilibrium problems and integral inequalities have been studied and investigated, see [14,15,16,17,18,21]. In [11] the concept of α-invex function was introduced. It has been shown [11] that α-preinvex (α-invex) have useful and important applications in generalized convex programming and multiobjective optimization. Inspired by this ongoing research Noor et al. [21] have introduced the class of logarithmic α-preinvex function.
Motivated by this we in this paper, consider the classes of α-preinvex and logarithmic α-preinvex functions. We prove several Hermite-Hadamard type inequalities for these classes of nonconvex functions. The ideas and techniques used in this paper may be useful for further research.

2 Preliminaries

In this section, we recall some previously known basic results. Let $K_{\alpha\eta}$ be a nonempty closed set in $\mathbb{R}$. Also assume that $f: K_{\alpha\eta} \to \mathbb{R}$, $\eta(.,.): K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}$ and $\alpha(.,.): K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}\setminus\{0\}$ be a bifunction.

Definition 1([11]). *A set $K_{\alpha\eta}$ is said to be a α-invex set, if there exist an arbitrary functions $\alpha, \eta: K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}$, such that*

$$u + t\alpha(v,u)\eta(v,u) \in K_{\alpha\eta}, \qquad \forall u, v \in K_{\alpha\eta}, t \in [0,1].$$

$K_{\alpha\eta}$ is said to be an α-invex set with respect to η and α, if K is α-invex at each $u \in K_{\alpha\eta}$. The α-invex set $K_{\alpha\eta}$ is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(v,u) = 1$ and $\eta(v,u) = v - u$ is an invex set, but the converse is not true.

Remark. If $\alpha(v,u) = 1$, then Definition 1 reduces to the definition for invex set.

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Definition 2([24]). *A set K_η is said to be a invex set, if there exists an arbitrary function $\eta : K_\eta \times K_\eta \to \mathbb{R}$, such that*

$$u + t\eta(v,u) \in K_\eta, \quad \forall u, v \in K_\eta, t \in [0,1].$$

Definition 3([11]). *A function f on $K_{\alpha\eta}$ is said to be α-preinvex function, if there exist arbitrary functions $\alpha, \eta : K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}$, such that*

$$f(u + t\alpha(v,u)\eta(v,u)) \le (1-t)f(u) + tf(v), \quad \forall u, v \in K_{\alpha\eta}, t \in [0,1].$$

An α-preinvex function may not be convex function, see [8].

Remark. If $\alpha(v,u) = 1$, then Definition 3 reduces to the definition for preinvex functions.

Definition 4([24]). *A function f on K_η is said to be preinvex function, if there exists an arbitrary function $\eta : K_\eta \times K_\eta \to \mathbb{R}$, such that*

$$f(u + t\eta(v,u)) \le (1-t)f(u) + tf(v), \quad \forall u, v \in K_\eta, t \in [0,1].$$

Definition 5([21]). *A function f on $K_{\alpha\eta}$ is said to be logarithmic α-preinvex function, if there exists an arbitrary function $\alpha : K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}\, \eta : K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}$, such that*

$$f(u + t\alpha(v,u)\eta(v,u)) \le (f(u))^{(1-t)}(f(v))^t, \quad \forall u, v \in K_{\alpha\eta}, t \in [0,1].$$

Definition 6([7]). *Two functions f and g are said to be similarly ordered (f is g-monotone) on $I \subseteq \mathbb{R}$, if*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \ge 0, \quad \forall x, y \in I.$$

Definition 7([12]). *Let $f \in L_1[a,b]$, the Riemann-Liouville integrals $J^\alpha_{a^+} f$ and $J^\alpha_{b^-} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^\alpha_{a^+}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a,$$

and

$$J^\alpha_{b^-}(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} x^{\alpha-1}\mathrm{d}x,$$

is the Gamma function.

In order to develop our some of our main results, we need following assumption on the bifunctions $\eta(.,.)$ and $\alpha(.,.)$. For more details, see [21].

Condition C. Let $\eta(.,.) : K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R}$ and $\alpha(.,.) : K_{\alpha\eta} \times K_{\alpha\eta} \to \mathbb{R} \setminus \{0\}$ satisfy the assumptions:
I. $\eta(u, u + t\alpha(v,u)\eta(v,u)) = -t\eta(v,u)$;
II. $\eta(v, u + t\alpha(v,u)\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u, v \in K, t \in [0,1]$.

It is worth to mention here that for $\alpha(v,u) = 1$, Condition C collapses to the condition, which is due to Mohan et al. [13].
Throughout the sequel of the paper, it is assumed that the bifunctions $\eta(.,.)$ and $\alpha(.,.)$ satisfy the Condition C, unless otherwise specified.

3 Main Results

In this section, we prove our main results.

Theorem 1. *The product of two α-preinvex functions f and w is α-preinvex if f and w are similarly ordered.*

Proof. Since f and w are α-preinvex functions. Then

$$\begin{aligned}
& f(a + t\alpha(b,a)\eta(b,a))w(a + t\alpha(b,a)\eta(b,a)) \\
& \le [(1-t)f(a) + tf(b)][(1-t)w(a) + tw(b)] \\
& = [1-t]^2 f(a)w(a) + t(1-t)f(a)w(b) \\
& \quad + t(1-t)f(b)w(a) + [t]^2 f(b)w(b) \\
& = (1-t)f(a)w(a) + tf(b)w(b) \\
& \quad -(1-t)f(a)w(a) - tf(b)w(b) + [1-t]^2 f(a)w(a) \\
& \quad + t(1-t)f(a)w(b) + t(1-t)f(b)w(a) + [t]^2 f(b)w(b) \\
& = (1-t)f(a)w(a) + tf(b)w(b) \\
& \quad -t(1-t)[f(a)w(a) + f(b)w(b) \\
& \quad - f(b)w(a) - f(a)w(b)] \\
& \le (1-t)f(a)w(a) + tf(b)w(b).
\end{aligned}$$

This completes the proof. □

Theorem 2. *Let f be α-preinvex function on $K_{\alpha,\eta} = [a, a + \alpha(b,a)\eta(b,a)]$. Then for all $t \in [0,1]$, we have*

$$f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right) \le \frac{1}{\alpha(b,a)\eta(b,a)} \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)\mathrm{d}x \le \frac{f(a) + f(b)}{2}.$$

Proof. Since f is α-preinvex function and the bifunctions $\eta(.,.)$ and $\alpha(.,.)$ satisfy Condition C. Thus

$$f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$

$$= \int_0^1 f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) dt$$
$$\leq \frac{1}{2}\int_0^1 [f(a+t\alpha(b,a)\eta(b,a)) + f(a+(1-t)\alpha(b,a)\eta(b,a))] dt$$
$$= \frac{1}{\alpha(b,a)\eta(b,a)} \int_a^{a+\alpha(b,a)\eta(b,a)} f(x) dx$$
$$= \int_0^1 f(a+t\alpha(b,a)\eta(b,a)) dt \leq \frac{f(a)+f(b)}{2}.$$

This completes the proof. □

Essentially using the technique of [23], we prove a lemma which will be useful in proving our next result of Fejer type inequality for α-preinvex functions.

Lemma 1. *Let f be α-preinvex function then we have*

$$f(2a+\alpha(b,a)\eta(b,a)-x) \leq f(a)+f(b)-f(x).$$

Proof. Let $x \in [a, a+\alpha(b,a)\eta(b,a)]$. Then we have

$$\begin{aligned} & f(2a+\alpha(b,a)\eta(b,a)-x) \\ &= f(2a+\alpha(b,a)\eta(b,a)-a-t\alpha(b,a)\eta(b,a)) \\ &= f(a+(1-t)\alpha(b,a)\eta(b,a)) \\ &\leq t f(a)+(1-t)f(b) \\ &= [f(a)+f(b)]-[(1-t)f(a)+t f(b)] \\ &\leq [f(a)+f(b)]-f(a+t\alpha(b,a)\eta(b,a)) \\ &= f(a)+f(b)-f(x). \end{aligned}$$

This proof is complete. □

Theorem 3. *Let $f: K=[a, a+\alpha(b,a)\eta(b,a)] \to (0,\infty)$ be a α-preinvex function with $a < a+\alpha(b,a)\eta(b,a)$ and $w: [a, a+\alpha(b,a)\eta(b,a)] \to \mathbb{R}$ is a non-negative and integrable function, then we have*

$$f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) \int_a^{a+\alpha(b,a)\eta(b,a)} w(x) dx$$
$$\leq \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x) dx$$
$$\leq \frac{f(a)+f(b)}{2} \int_a^{a+\alpha(b,a)\eta(b,a)} w(x) dx.$$

Proof. Since f is α-preinvex function and the bifunctions $\eta(.,.)$ and $\alpha(.,.)$ satisfy the Condition C, we have

$$f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) \int_a^{a+\alpha(b,a)\eta(b,a)} w(x) dx$$
$$= \int_a^{a+\alpha(b,a)\eta(b,a)} f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) w(x) dx$$
$$= \int_a^{a+\alpha(b,a)\eta(b,a)} f\left(\frac{2a+\alpha(b,a)\eta(b,a)-x+x}{2}\right) w(x) dx$$
$$\leq \frac{1}{2}\int_a^{a+\alpha(b,a)\eta(b,a)} \{f(2a+\alpha(b,a)\eta(b,a)-x)+f(x)\} w(x) dx$$
$$= \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x) dx$$
$$= \frac{1}{2}\int_a^{a+\alpha(b,a)\eta(b,a)} f(2a+\alpha(b,a)\eta(b,a)-x) w(x) dx + \frac{1}{2}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x) dx$$
$$\leq \frac{1}{2}\int_a^{a+\alpha(b,a)\eta(b,a)} [f(a)+f(b)]-f(x)] w(x) dx + \frac{1}{2}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x) dx$$
$$\leq \frac{f(a)+f(b)}{2} \int_a^{a+\alpha(b,a)\eta(b,a)} w(x) dx.$$

This completes the proof. □

Theorem 4. *Let $f: K=[a, a+\alpha(b,a)\eta(b,a)] \to (0,\infty)$ and $w: K=[a, a+\alpha(b,a)\eta(b,a)] \to (0,\infty)$ be α-preinvex functions with $a < a+\alpha(b,a)\eta(b,a)$, then we have*

$$2f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) w\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) - \left[\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\right]$$
$$\leq \frac{1}{\alpha(b,a)\eta(b,a)} \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x) dx$$
$$\leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

where

$$M(a,b) = f(a)w(a)+f(b)w(b) \tag{2}$$

and

$$N(a,b) = f(a)w(b)+f(b)w(a). \tag{3}$$

Proof. Since f and w are α-preinvex functions and the bifunctions $\eta(.,.)$ and $\alpha(.,.)$ satisfy Condition C, we have

$$f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right) w\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right)$$

$$
\begin{aligned}
&= f\left(\frac{a+t\alpha(b,a)\eta(b,a)}{2}+\frac{a+(1-t)\alpha(b,a)\eta(b,a)}{2}\right)\\
&\times w\left(\frac{a+t\alpha(b,a)\eta(b,a)}{2}+\frac{a+(1-t)\alpha(b,a)\eta(b,a)}{2}\right)\\
&\leq \frac{1}{4}[\{f(a+t\alpha(b,a)\eta(b,a))\\
&\quad +f(a+(1-t)\alpha(b,a)\eta(b,a))\}\\
&\quad \times\{w(a+t\alpha(b,a)\eta(b,a))\\
&\quad +w(a+(1-t)\alpha(b,a)\eta(b,a))\}]\\
&= \frac{1}{4}[f(a+t\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\\
&\quad +f(a+t\alpha(b,a)\eta(b,a))w(a+(1-t)\eta(b,a))\\
&\quad +f(a+(1-t)\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\\
&\quad +f(a+(1-t)\alpha(b,a)\eta(b,a))\\
&\quad \times w(a+(1-t)\alpha(b,a)\eta(b,a))]\\
&\leq \frac{1}{4}[f(a+t\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\\
&\quad +f(a+(1-t)\alpha(b,a)\eta(b,a))\\
&\quad \times w(a+(1-t)\alpha(b,a)\eta(b,a))\\
&\quad +\{(1-t)f(a)+tf(b)\}\{tw(a)+(1-t)w(b)\}\\
&\quad +\{tf(a)+(1-t)f(b)\}\{(1-t)w(a)+tw(b)\}]\\
&= \frac{1}{4}[f(a+t\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\\
&\quad +f(a+(1-t)\alpha(b,a)\eta(b,a))\\
&\quad \times w(a+(1-t)\alpha(b,a)\eta(b,a))\\
&\quad +[2t(1-t)M(a,b)+2(t^2+(1-t)^2)N(a,b)]].
\end{aligned}
$$

Integrating both sides with respect to t on $[0,1]$ and using change of variable technique, we have

$$
\begin{aligned}
&2f\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right)w\left(\frac{2a+\alpha(b,a)\eta(b,a)}{2}\right)\\
&\quad -\left[\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\right]\\
&\leq \frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x)\mathrm{d}x. \qquad (4)
\end{aligned}
$$

Now we prove right side of the inequality

$$
\begin{aligned}
&f(a+t\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\\
&\leq [(1-t)f(a)+tf(b)][(1-t)w(a)+tw(b)]\\
&= (1-t)^2f(a)w(a)+t(1-t)f(b)w(a)\\
&\qquad +t(1-t)f(a)w(b)+t^2f(b)w(b).
\end{aligned}
$$

Integrating both sides with respect to t on $[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x)\mathrm{d}x\\
&\leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b). \qquad (5)
\end{aligned}
$$

Combining (4) and (5) completes the proof. □

Theorem 5. *Let $f: K=[a,a+\alpha(b,a)\eta(b,a)]\to(0,\infty)$ and $w: K=[a,a+\alpha(b,a)\eta(b,a)]\to(0,\infty)$ be α-preinvex functions with $a<a+\alpha(b,a)\eta(b,a)$, then we have*

$$
\begin{aligned}
&\frac{1}{\alpha(b,a)\eta(b,a)}\\
&\times\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x\\
&\leq \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are given by (2) and (3) respectively.

Proof. Proof directly follows from the definition of α-preinvex functions.

Theorem 6. *Let $f: K=[a,a+\alpha(b,a)\eta(b,a)]\to(0,\infty)$ and $w: K=[a,a+\alpha(b,a)\eta(b,a)]\to(0,\infty)$ be α-preinvex functions with $a<a+\alpha(b,a)\eta(b,a)$, then we have*

$$
\begin{aligned}
&\frac{1}{\alpha(b,a)\eta(b,a)}\\
&\times\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x\\
&\leq \frac{1}{4}\Theta(a,b),
\end{aligned}
$$

where $\Theta(a,b)=[f(a)]^2+[f(b)]^2+[w(a)]^2+[w(b)]^2$.

Proof. Since f and w are α-preinvex functions. Thus

$$
\begin{aligned}
&\frac{1}{\alpha(b,a)\eta(b,a)}\\
&\times\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x\\
&= \int_0^1 f(a+t\alpha(b,a)\eta(b,a))\\
&\quad \times w(a+(1-t)\alpha(b,a)\eta(b,a))\mathrm{d}t\\
&\leq \frac{1}{2}\int_0^1\{f(a+t\alpha(b,a)\eta(b,a))\}^2\mathrm{d}t\\
&\quad +\frac{1}{2}\int_0^1\{w(a+(1-t)\alpha(b,a)\eta(b,a))\}^2\mathrm{d}t\\
&\leq \frac{1}{2}\int_0^1\{(1-t)f(a)+t(f(b))\}^2\mathrm{d}t\\
&\quad +\frac{1}{2}\int_0^1\{tw(a)+(1-t)w(b)\}^2\mathrm{d}t
\end{aligned}
$$

$$= \frac{1}{6}[\{f(a)\}^2 + \{f(b)\}^2 + \{f(a)\}\{f(b)\}]$$
$$+ \frac{1}{6}[\{w(a)\}^2 + \{w(b)\}^2 + \{w(a)\}\{w(b)\}]$$
$$\leq \frac{1}{4}\Theta(a,b).$$

This completes the proof. □

Our next result is Hermite-Hadamard inequality for α-preinvex functions via fractional integrals.

Theorem 7. *Let f be a α-preinvex function and $f \in L[a, a + \alpha(b,a)\eta(b,a)]$. Then, we have*

$$f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)}{(2\alpha(b,a)\eta(b,a))^{\alpha}}$$
$$\times \left[J^{\alpha}_{[a+\alpha(b,a)\eta(b,a)]^-} f(a) + J^{\alpha}_{a^+} f(a + \alpha(b,a)\eta(b,a))\right]$$
$$\leq \frac{f(a) + f(b)}{2}.$$

Proof. Since f is a α-preinvex function and the bifunctions $\eta(.,.)$ and $\alpha(.,.)$ satisfy the Condition C. Then

$$f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$
$$= f\left(\frac{a + t\alpha(b,a)\eta(b,a) + a + (1-t)\alpha(b,a)\eta(b,a)}{2}\right)$$
$$\leq \frac{1}{2} f(a + t\alpha(b,a)\eta(b,a))$$
$$+ \frac{1}{2} f(a + (1-t)\alpha(b,a)\eta(b,a)).$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\frac{2}{\alpha} f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)$$
$$\leq \int_0^1 t^{\alpha-1} f(a + t\alpha(b,a)\eta(b,a))\mathrm{d}t$$
$$+ \int_0^1 (1-t)^{\alpha-1} f(a + (1-t)\alpha(b,a)\eta(b,a))\mathrm{d}t.$$

Let $u = a + t\alpha(b,a)\eta(b,a)$ and $v = a + (1-t)\alpha(b,a)\eta(b,a)$ and by change of variable technique, we have

$$\frac{2}{\alpha} f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$
$$\leq \int_a^{a+\alpha(b,a)\eta(b,a)} \left(\frac{u-a}{\alpha(b,a)\eta(b,a)}\right)^{\alpha-1} \frac{f(u)}{\alpha(b,a)\eta(b,a)}\mathrm{d}u$$
$$+ \int_a^{\alpha(b,a)\eta(b,a)} \left(\frac{a + \alpha(b,a)\eta(b,a) - v}{\alpha(b,a)\eta(b,a)}\right)^{\alpha-1}$$
$$\times \frac{f(v)}{\alpha(b,a)\eta(b,a)} dv.$$

This implies that

$$2f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)}{\alpha(b,a)\eta(b,a)^{\alpha}} \left[J^{\alpha}_{[\alpha(b,a)\eta(b,a)]^-} f(a)\right.$$
$$\left. + J^{\alpha}_{a^+} f(\alpha(b,a)\eta(b,a))\right].$$

Also

$$f(a + t\alpha(b,a)\eta(b,a)) + f(a + (1-t)\alpha(b,a)\eta(b,a))$$
$$\leq f(a) + f(b).$$

Multiplying above inequality by $t^{\alpha-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\frac{\Gamma(\alpha+1)}{\alpha(b,a)\eta(b,a)^{\alpha}}$$
$$\times \left[J^{\alpha}_{[\alpha(b,a)\eta(b,a)]^-} f(a) + J^{\alpha}_{a^+} f(\alpha(b,a)\eta(b,a))\right]$$
$$\leq f(a) + f(b).$$

After suitable rearrangements the proof is complete. □

Now, we prove some results for logarithmic α-preinvex functions.

Theorem 8. *Let f be a differentiable logarithmic α-preinvex function on $K_{\alpha\eta}$. Then $u \in K_{\alpha\eta}$ is the minimum of f on $K_{\alpha\eta}$ if and only if $u \in K_{\alpha\eta}$ satisfies the inequality*

$$\left\langle \alpha(v,u)\frac{f'(u)}{f(u)}, \eta(v,u) \right\rangle \geq 0, \quad \forall u, v \in K_{\alpha\eta}, \tag{6}$$

where f' is the differential of f.

Proof. The proof is left for the interested readers. □

Before proceeding further one may consult [7] for the details on arithmetic, geometric, logarithmic and extended logarithmic means. Our next result is Hermite-Hadamard's inequality via logarithmic α-preinvex functions. The proof of the result is obvious using the definition of logarithmic α-preinvex functions.

Theorem 9. *Let f be a logarithmic α-preinvex function, then for all $t \in [0,1]$, we have*

$$f\left(\frac{2a + \alpha(b,a)\eta(b,a)}{2}\right)$$
$$\leq \exp\left[\frac{1}{\alpha(b,a)\eta(b,a)} \int_a^{a+\alpha(b,a)\eta(b,a)} \log f(x)\mathrm{d}x\right]$$
$$\leq \sqrt{f(a)f(b)}.$$

Theorem 10. *Let f be logarithmic α-preinvex function. Then for all $t \in [0,1]$, we have*

$$f\left(\frac{a+\alpha(b,a)\eta(b,a)}{2}\right) \leq \exp\left[\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} \log f(x)\mathrm{d}x\right]$$
$$\leq \frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x))\mathrm{d}x$$
$$\leq \frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)\mathrm{d}x$$
$$\leq L[f(b), f(a)] \leq A[f(a), f(b)].$$

Proof. The proof of first inequality is obvious from previous result. In order to prove second inequality, we proceed as

$$G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x)) = \exp\left[\log G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x))\right].$$

Integrating above inequality with respect to x on $[a, a+\alpha(b,a)\eta(b,a)]$, we have

$$\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x))\mathrm{d}x$$
$$= \frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} \times \exp\left[\log G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x))\right]\mathrm{d}x$$
$$\geq \exp\left[\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} \log G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x\right]$$
$$= \exp\left[\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} \frac{\log f(x)+\log f(2a+\alpha(b,a)\eta(b,a)-x)}{2}\mathrm{d}x\right]$$
$$= \exp\left[\frac{1}{a+\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} \log f(x)\mathrm{d}x\right].$$

Using $AM-GM$ inequality, we have

$$G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x)) \leq \frac{f(x)+f(2a+\alpha(b,a)\eta(b,a)-x)}{2}.$$

Integrating the above inequality with respect to x on $[a, a+\alpha(b,a)\eta(b,a)]$, we have

$$\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} G(f(x), f(2a+\alpha(b,a)\eta(b,a)-x))\mathrm{d}x \leq \frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)\mathrm{d}x.$$

Since f is logarithmic α-preinvex function, so for all $t \in [0,1]$, we have

$$f(a+t\alpha(b,a)\eta(b,a)) \leq [f(a)]^{1-t}[f(b)]^t.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)\mathrm{d}x \leq \int_0^1 [f(a)]^{1-t}[f(b)]^t$$
$$= f(a)\int_0^1 \left[\frac{f(b)}{f(a)}\right]^t dt$$
$$= \frac{f(b)-f(a)}{\log f(b)-\log f(a)}$$
$$= L[f(b), f(a)] \leq \frac{f(a)+f(b)}{2}.$$

This completes the proof. □

Theorem 11. *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be logarithmic α-preinvex functions, then we have*

$$\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x)\mathrm{d}x \leq L[f(a)w(b), f(a)w(a)] \leq \frac{f(a)w(a)+f(b)w(b)}{2} \leq \frac{1}{4}\Theta(a,b),$$

where $\Theta(a,b) = [f(a)]^2+[f(b)]^2+[w(a)]^2+[w(b)]^2$.

Proof. Since f and w are logarithmic α-preinvex functions. Then

$$f(a+t\alpha(b,a)\eta(b,a)) \le [f(a)]^{1-t}[f(b)]^t,$$
$$w(a+t\alpha(b,a)\eta(b,a)) \le [w(a)]^{1-t}[w(b)]^t.$$

Now

$$\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(x)\mathrm{d}x$$
$$= \int_0^1 f(a+t\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))$$
$$\le \int_0^1 [f(a)w(a)]^{1-t}[f(b)w(b)]^t dt$$
$$= f(a)w(a)\int_0^1 \left[\frac{f(b)w(b)}{f(a)w(a)}\right]^t dt$$
$$= f(a)w(a)\left[\frac{\left[\frac{f(b)w(b)}{f(a)w(a)}\right]^t}{\log\left[\frac{f(b)w(b)}{f(a)w(a)}\right]}\right]_0^1$$
$$= \frac{f(b)w(b)-f(a)w(a)}{\log f(b)w(b)-\log f(a)w(a)} = L[f(b)w(b), f(a)w(a)]$$
$$\le \frac{f(a)w(a)+f(b)w(b)}{2} = A[f(a)w(a), f(b)w(b)]$$
$$\le \frac{1}{2}\int_0^1 \left[\{f(a+t\alpha(b,a)\eta(b,a))\}^2 + \{w(a+t\alpha(b,a)\eta(b,a))\}^2\right]\mathrm{d}t$$
$$\le \frac{1}{2}\int_0^1 \left[\{[f(a)]^{1-t}[f(b)]^t\}^2 + \{[w(a)]^{1-t}[w(b)]^t\}^2\right]\mathrm{d}t$$
$$= \frac{1}{2}\left[[f(a)]^2\int_0^1\left[\frac{f(b)}{f(a)}\right]^{2t}\mathrm{d}t + [w(a)]^2\int_0^1\left[\frac{w(b)}{w(a)}\right]^{2t}\mathrm{d}t\right]$$
$$= \frac{1}{4}\left[[f(a)]^2\int_0^2\left[\frac{f(b)}{f(a)}\right]^{u}\mathrm{d}u + [w(a)]^2\int_0^2\left[\frac{w(b)}{w(a)}\right]^{u}\mathrm{d}u\right]$$
$$= \frac{1}{4}\left[\frac{[f(a)+f(b)][f(b)-f(a)]}{\log f(b)-\log f(a)} + \frac{[w(a)+w(b)][w(b)-w(a)]}{\log w(b)-\log w(a)}\right]$$
$$= \frac{1}{4}[[f(a)+f(b)]L[f(b),f(a)] + [w(a)+w(b)]L[w(b),w(a)]]$$
$$\le \frac{1}{8}\left[[f(a)+f(b)]^2+[w(a)+w(b)]^2\right] \le \frac{1}{4}\Theta(a,b),$$

the required result. □

Theorem 12. *Let $f,w : I \subseteq \mathbb{R} \to \mathbb{R}$ be logarithmic α-preinvex functions, then we have*

$$\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x$$
$$\le \frac{[A(f(a),f(b))]^2+[A(w(a),w(b))]^2}{2}.$$

Proof. Since f,w be logarithmic α- preinvex functions, then we have

$$\frac{1}{\alpha(b,a)\eta(b,a)} \times \int_a^{a+\alpha(b,a)\eta(b,a)} f(x)w(2a+\alpha(b,a)\eta(b,a)-x)\mathrm{d}x$$
$$= \int_0^1 f(a+(1-t)\alpha(b,a)\eta(b,a))w(a+t\alpha(b,a)\eta(b,a))\mathrm{d}t$$
$$\le \int_0^1 [f(a)]^t[f(b)]^{1-t}[w(a)]^{1-t}[w(b)]^t\mathrm{d}t$$
$$= \int_0^1 [f(b)]\left[\frac{f(a)}{f(b)}\right]^t w(a)\left[\frac{w(b)}{w(a)}\right]^t\mathrm{d}t$$
$$= f(b)w(a)\int_0^1\left[\frac{f(a)w(b)}{f(b)w(a)}\right]^t\mathrm{d}t$$
$$= f(b)w(a)\frac{\frac{f(a)w(b)-f(b)w(a)}{f(b)w(a)}}{\log f(a)w(b)-\log f(b)w(a)}$$
$$= \frac{f(a)w(b)-f(b)w(a)}{\log f(a)w(b)-\log f(b)w(a)}$$
$$= L[f(a)w(b), f(b)w(a)]$$
$$\le \frac{f(a)w(b)+f(b)w(a)}{2} = A[f(a)w(b), f(b)w(a)]$$
$$\le \frac{1}{2}\int_0^1\left\{[f(a+(1-t)\alpha(b,a)\eta(b,a))]^2 + [w(a+t\alpha(b,a)\eta(b,a))]^2\right\}\mathrm{d}t$$
$$\le \frac{1}{2}\int_0^1\left\{[f(a)]^t[f(b)]^{1-t}\right\}^2\mathrm{d}t + \frac{1}{2}\int_0^1\left\{[w(a)]^{1-t}[w(b)]^t\right\}^2\mathrm{d}t$$
$$= \frac{[f(b)]^2}{2}\int_0^1\left[\frac{f(a)}{f(b)}\right]^{2t}\mathrm{d}t + \frac{[w(a)]^2}{2}\int_0^1\left[\frac{w(b)}{w(a)}\right]^{2t}\mathrm{d}t$$

$$=\frac{[f(b)]^2}{4}\int_0^2\left[\frac{f(a)}{f(b)}\right]^u\mathrm{d}u+\frac{[w(a)]^2}{4}\int_0^2\left[\frac{w(b)}{w(a)}\right]^u\mathrm{d}u$$

$$=\frac{[f(b)]^2}{4}\left[\frac{\left\{\frac{f(a)}{f(b)}\right\}^u}{\log\frac{f(a)}{f(b)}}\right]_0^2+\frac{[w(a)]^2}{4}\left[\frac{\left\{\frac{w(b)}{w(a)}\right\}^u}{\log\frac{w(b)}{w(a)}}\right]_0^2$$

$$=\frac{1}{4}\frac{[f(a)]^2-[f(b)]^2}{\log f(a)-\log f(b)}+\frac{1}{4}\frac{[w(a)]^2-[w(b)]^2}{\log w(a)-\log w(b)}$$

$$=\frac{1}{2}\left[\frac{f(a)+f(b)}{2}\frac{f(a)-f(b)}{\log f(a)-\log f(b)}\right]+\frac{1}{2}\left[\frac{w(a)+w(b)}{2}\frac{w(a)-w(b)}{\log w(a)-\log w(b)}\right]$$

$$=\frac{1}{2}[A[f(a),f(b)]L[f(a),f(b)]]+\frac{1}{2}[A[w(a),w(b)]L[w(a),w(b)]]$$

$$\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}\frac{f(a)+f(b)}{2}\right]+\frac{1}{2}\left[\frac{w(a)+w(b)}{2}\frac{w(a)+w(b)}{2}\right]$$

$$=\frac{[A(f(a),f(b))]^2+[A(w(a),w(b))]^2}{2},$$

which is the required result. □

Theorem 13. *Let $f_1,f_2,\dots,f_n$ be logarithmic α-preinvex functions. Then for $\mu_1,\mu_2,\dots,\mu_n>0$ and $\sum_{i=1}^n\mu_i=1$, we have*

$$\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)}\sum_{i=1}^n f_i(x)\mathrm{d}x\le\sum_{i=1}^n\left\{\mu_i\frac{f_i(a)+f_i(b)}{2}\left[L_{(\frac{1}{\mu_i}-1)}(f_i(b),f_i(a))\right]^{\frac{1-\mu_i}{\mu_i}}\right\}.$$

Proof. Since $f_1,f_2,\dots,f_n$ be logarithmic α-preinvex functions and using inequality

$$f_1.f_2\dots f_n\le\mu_1(f_1)^{\frac{1}{\mu_1}}+\mu_2(f_2)^{\frac{1}{\mu_2}}+\dots+\mu_n(f_n)^{\frac{1}{\mu_n}},\quad \mu_1,\mu_2,\dots,\mu_n>0,\sum_{i=1}^n\mu_i=1.$$

Then

$$\frac{1}{\alpha(b,a)\eta(b,a)}\int_a^{a+\alpha(b,a)\eta(b,a)}\sum_{i=1}^n f_i(x)\mathrm{d}x$$

$$\le\int_0^1\left\{\sum_{i=1}^n\mu_i(f_i(a+t\alpha(b,a)\eta(b,a)))^{\frac{1}{\mu_i}}\right\}\mathrm{d}t$$

$$\le\int_0^1\left\{\sum_{i=1}^n\mu_i[(f_i(a))^{1-t}(f_i(b))^t]^{\frac{1}{\mu_i}}\right\}\mathrm{d}t$$

$$=\sum_{i=1}^n\mu_i(f_i(a))^{\frac{1}{\mu_i}}\int_0^1\left(\frac{f_i(b)}{f_i(a)}\right)^{\frac{t}{\mu_i}}\mathrm{d}t$$

$$=\sum_{i=1}^n(\mu_i)^2(f_i(a))^{\frac{1}{\mu_i}}\int_0^{\frac{1}{\mu_i}}\left(\frac{f_i(b)}{f_i(a)}\right)^u\mathrm{d}u$$

$$=\sum_{i=1}^n(\mu_i)^2\frac{(f_i(b))^{\frac{1}{\mu_i}}-(f_i(a))^{\frac{1}{\mu_i}}}{\log f_i(b)-\log f_i(a)}$$

$$=\sum_{i=1}^n(\mu_i)^2\frac{(f_i(b))^{\frac{1}{\mu_i}}-(f_i(a))^{\frac{1}{\mu_i}}}{f_i(b)-f_i(a)}L(f_i(b),f_i(a))$$

$$=\sum_{i=1}^n\mu_i\left[L_{(\frac{1}{\mu_i}-1)}(f_i(b),f_i(a))\right]^{\frac{1-\mu_i}{\mu_i}}L(f_i(b),f_i(a))$$

$$\le\sum_{i=1}^n\mu_i\frac{f_i(a)+f_i(b)}{2}\left[L_{(\frac{1}{\mu_i}-1)}(f_i(b),f_i(a))\right]^{\frac{1-\mu_i}{\mu_i}}.$$

This completes the proof. □

Theorem 14. *Let $f_1,f_2,\dots,f_n$ be differentiable logarithmic α-preinvex functions on I^0(interior of I). Then, we have*

$$\left.\begin{array}{l}\int_a^{a+\alpha(b,a)\eta(b,a)}\sum_{i=1}^n f_i(v)dv\\ \ge\mu_1f_1\left(\frac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\\ \quad\times\int_a^{a+\alpha(b,a)\eta(b,a)}f_2(v)f_3(v)\dots f_n(v)\exp(\Psi_1)dv\\ \quad+\mu_2f_2\left(\frac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\\ \quad\times\int_a^{a+\alpha(b,a)\eta(b,a)}f_1(v)f_3(v)\dots f_n(v)\exp(\Psi_2)dv\\ \quad\vdots\\ \quad+\mu_nf_n\left(\frac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\\ \quad\times\int_a^{a+\alpha(b,a)\eta(b,a)}f_1(v)f_2(v)\dots f_{n-1}(v)\exp(\Psi_n)dv\end{array}\right\},$$

where

$$\Psi_i=\left\langle\alpha\left(v,\frac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\frac{f_i'(\frac{2a+\alpha(v,u)\eta(v,u)}{2})}{f_i(\frac{2a+\alpha(v,u)\eta(v,u)}{2})},\ \eta\left(v,\frac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\right\rangle.$$

Proof. Since $f_1,f_2\dots,f_n$ be differentiable logarithmic α-convex functions, so we have

$$f_1(v)\ge f_1(u)\exp\left[\left\langle\alpha(v,u)\frac{f_1'(u)}{f_1(u)},\eta(v,u)\right\rangle\right],\tag{7}$$

$$f_2(v)\ge f_2(u)\exp\left[\left\langle\alpha(v,u)\frac{f_2'(u)}{f_2(u)},\eta(v,u)\right\rangle\right],\tag{8}$$

$\vdots$

$$f_n(v) \geq f_n(u)\exp\left[\left\langle \alpha(v,u)\frac{f_n'(u)}{f_n(u)}, \eta(v,u)\right\rangle\right], \quad (9)$$

Multiplying (7) by $\mu_1 f_2(v)f_3(v)\dots f_n(v)$, (8) by $\mu_2 f_1(v)f_3(v)\dots f_n(v)$ and (9) by $\mu_n f_1(v)f_2(v)\dots f_{n-1}(v)$ respectively and then adding the resultant, we have

$$\left.\begin{aligned} &\sum_{i=1}^{n} f_i(v) \\ &\geq \mu_1 f_1(u)f_2(v)f_3(v)\dots f_n(v) \\ &\quad\times\exp\left[\left\langle \alpha(v,u)\frac{f_1'(u)}{f_1(u)}, \eta(v,u)\right\rangle\right] \\ &\quad+\mu_2 f_2(u)f_1(v)f_3(v)\dots f_n(v) \\ &\quad\times\exp\left[\left\langle \alpha(v,u)\frac{f_2'(u)}{f_2(u)}, v-u\right\rangle\right] \\ &\quad\vdots \\ &\quad+\mu_n f_n(u)f_1(v)f_2(v)\dots f_{n-1}(v) \\ &\quad\times\exp\left[\left\langle \alpha(v,u)\frac{f_n'(u)}{f_n(u)}, v-u\right\rangle\right] \end{aligned}\right\}. \quad (10)$$

Putting $u = \frac{2a+\alpha(v,u)\eta(v,u)}{2}$ in (10), we have

$$\left.\begin{aligned} &\sum_{i=1}^{n} f_i(v) \\ &\geq \mu_1 f_1\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)f_2(v)f_3(v)\dots f_n(v) \\ &\quad\times\exp\left[\left\langle \alpha\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\frac{f_1'(\frac{2a+\alpha(v,u)\eta(v,u)}{2})}{f_1(\frac{2a+\alpha(v,u)\eta(v,u)}{2})},\right.\right. \\ &\qquad\left.\left.\eta\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\right\rangle\right] \\ &\quad+\mu_2 f_2\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)f_1(v)f_3(v)\dots f_n(v) \\ &\quad\times\exp\left[\left\langle \alpha\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\frac{f_2'(\frac{2a+\alpha(v,u)\eta(v,u)}{2})}{f_2(\frac{2a+\alpha(v,u)\eta(v,u)}{2})},\right.\right. \\ &\qquad\left.\left.\eta\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\right\rangle\right] \\ &\quad\vdots \\ &\quad+\mu_n f_n\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)f_1(v)f_2(v)\dots f_{n-1}(v) \\ &\quad\times\exp\left[\left\langle \alpha\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\frac{f_n'(\frac{2a+\alpha(v,u)\eta(v,u)}{2})}{f_n(\frac{2a+\alpha(v,u)\eta(v,u)}{2})},\right.\right. \\ &\qquad\left.\left.\eta\left(v,\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right)\right\rangle\right] \end{aligned}\right\}.$$

Integrating both sides of above inequality with respect to v on $[a, a+\alpha(b,a)\eta(b,a)]$, we have

$$\left.\begin{aligned} &\int_a^{a+\alpha(b,a)\eta(b,a)}\sum_{i=1}^{n} f_i(v)dv \\ &\geq \mu_1 f_1\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right) \\ &\quad\times\int_a^{a+\alpha(b,a)\eta(b,a)} f_2(v)f_3(v)\dots f_n(v)\exp(\Psi_1)\,dv \\ &\quad+\mu_2 f_2\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right) \\ &\quad\times\int_a^{a+\alpha(b,a)\eta(b,a)} f_1(v)f_3(v)\dots f_n(v)\exp(\Psi_2)\,dv \\ &\quad\vdots \\ &\quad+\mu_n f_n\left(\tfrac{2a+\alpha(v,u)\eta(v,u)}{2}\right) \\ &\quad\times\int_a^{a+\alpha(b,a)\eta(b,a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp(\Psi_n)\,dv \end{aligned}\right\}.$$

This completes the proof. □

References

[1] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, A. G. Khan, Some new bounds for Simpsons rule involving special functions via harmonic *h*-convexity, J. Nonlinear Sci. Appl., 10, (2017), 1755-1766.

[2] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, Two point trapezoidal like inequalities involving hypergeometric functions, Filomat, 31 (2017), 2281-2292.

[3] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, On bounds involving *k*-Appells hypergeometric functions, J. Inequal. Appl., (2017), 2017:118.

[4] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, Some fractional extensions of trapezium inequalities via coordinated harmonic convex functions, J. Nonlinear Sci. Appl., 10, (2017), 1714-1730.

[5] G. Cristescu, L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[6] S. S. Dragomir, B. Mond, Integral inequalities of Hadamard's type for log-convex functions, Demonstration Math. (2), 354-364 (1998).

[7] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, (2000).

[8] C. Fulga C, V. Preda, Nonlinear programming with *E*-preinvex and local *E*-preinvex functions. European J. Operational Research. 192(2009), 737-743.

[9] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[10] A. Ben-Israel, B. Mond, What is invexity? J. Aust. Math. Soc. Ser. B 28 (1986) 19.

[11] V. Jeyakumar, B. Mond, On generalized convex mathematical programming, J. Aust. Math. Soc. Ser. B 34(1992), 43-53.

[12] K. S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, (1993).

[13] S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189(1995), 901-908.

[14] M. A. Noor, Variational like inequalities, Optimization, 30(1994), 323-330.

[15] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(2000), 217-229.

[16] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2(2007), 126-131.

[17] M. A. Noor, On Hermite-Hadamard integral inequalities for involving two log-preinvex functions, J. Inequal. Pure and Appl. Math., 3(2007), 75-81.

[18] M. A. Noor, Differentiable nonconvex functions and general variational inequalities, Appl. Math. Comput. 199(2008) 623-630.

[19] M. A. Noor, M. U. Awan, K. I. Noor, On some inequalities for relative semi-convex functions, J. Inequal. Appl. 2013, 2013:332.

[20] M. A. Noor, M. U. Awan, K. I. Noor, On weighted trapezoidal like inequalities having preinvexity property, J. Adv. Math. Stud., 9(3), (2016), 381-389.

[21] M. A. Noor, K. I. Noor, Some characterizations of strongly preinvex functions, J. Math. Anal. Appl. 316(2006), 697-706.
[22] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, On Hermite-Hadamard inequalities for *h*-preinvex functions, Filomat, 28(7), (2014), 1463-1474.
[23] M. Z. Sarikaya, E. Set E, M. E. Özdemir, On some new inequalities of Hadamard type involving *h*-convex functions. Acta Math. Univ. Comenianae. 2, 265-272 (2010).
[24] T. Weir T, B. Mond, Preinvex functions in multiobjective optimization, J. Math. Anal. Appl. 136(1988), 29-38.
[25] G. S. Yang, K. L. Tseng, H. T. Wang, A note on integral inequalities of hadamard type for log-convex and log-concave functions, Taiwanese Journal of Mathematics, 16(2), 479-496, (2012).

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