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# On General Complementarity Problems

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**Abstract:** Complementarity problems provide a unified and general framework for studying a wide class of unrelated problems. In this paper, we use the change of variables technique to establish the equivalence between the general complementarity problem and the fixed point problem. This equivalence is used to study the existence of a solution of general complementarity problems and suggest an iterative method. Our methods of analysis is very simple as compared with other techniques. These results can be viewed as significant refinements of the previously known results.

**Keywords:** Strongly mixed variational inequality, Iterative method, Convergence.

## 1 Introduction

Complementarity theory contains a wealth of new ideas and techniques, which was introduced and considered in early sixties by Lemke[6] and Cottle[1] independently. For the applications, motivation, numerical results and other aspects of complementarity problems, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein.
In recent years, complementarity problems have been generalized and extended in several directions using the novel and innovative ideas. Dolcetta [4] considered the implicit complementarity problems, which arise as the discretization of the variational inequalities. Noor [8] introduced and investigated a class of nonlinear complementarity problems, which is known as general complementarity problem. There are several techniques for solving complementarity problems including the change of variables method, the origin of which can be traced back to Van Bokhhoven [16]. Noor[7] and Noor et al.[10, 11, 12, 13] have used this technique to suggest some iterative for solving different classes of complementarity problems. We again us this technique to show that the general complementarity problems are equivalent to the fixed point problem. This equivalent formulation is used to consider the existence of the general complementarity problems. Our results can be viewed as significant refinement of the known results.

## 2 Formulations and basic facts

Let $R^n$ be the Euclidean space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively.
Let $T, g : R^n \to R$ be a continuous monotone nonlinear operators and let $K$ be a closed and convex set in $R^n$. We consider the problem of finding $u \in R^n$ such that

$$g(u) \geq 0, \quad Tu \geq 0, \quad \langle g(u), Tu\rangle = 0. \tag{1}$$

which is called the general complementarity problems, mainly introduced and studied by Noor [8]. A wide class of problems arising in pure and applied sciences can be studied via general complementarity problems (1).
If $g(u) = m - m(u)$, then problem (1) reduces to finding $u \in R^n$ such that

$$u \geq m(u), \quad Tu \geq 0, \quad \langle Tu, u - m(u)\rangle = 0, \tag{2}$$

which is known as implicit(quasi) complementarity problem, which were introduced by Dollcetta [4]. For he applications and other aspects of the complementarity problems in engineering and applied sciences, see [4, 10, 11, 12, 13, 14, 15] and the references therein.
If $g = I$, the identity operator, then problem (1) is called the nonlinear complementarity problem, introduced and investigated by Cottle [1], that is, find $u \in R^n$ such that

$$u \geq o, \quad Tu \geq 0, \quad \langle Tu, u\rangle = 0. \tag{3}$$

For the applications, formulation, numerical methods, other aspects and various generalizations of the complementarity problems, see, [1, 3, 5, 6, 7, 4, 8, 9, 10, 11, 12, 13, **?**, 14, 15, 16] and the references therein.

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To obtain the main results, we recall some well-known concepts and results.

**Definition 1.** *An operator $T : H \to H$ with respect to an arbitrary function g is said to be:*

*1. Strongly g-monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq \alpha \|g(u) - g(v)\|^2, \quad \forall u, v \in H.$$

*2. Lipschitz g-continuous, if there exist a constant $\beta > 0$, such that*

$$\|Tu - Tv\| \leq \beta \|g(u) - g(v)\|, \quad \forall u, v \in H.$$

*3. g-monotone, if*

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

If $g^{-1}$ exist, then Definition 2.1 reduces to:

**Definition 2.** *An operator $Tg^{-1} : H \to H$ said to be*

*1. Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle Tg^{-!}u - Tg^{-1}v, u - v)\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

*2. Lipschitz continuous, if*

$$\|Tg^{-1}u - Tg^{-1}v\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

*3. g-monotone, if*

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

*Remark.* If the operator $Tg^{-1}$ is both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, then $\alpha \leq \beta$.

## 3 Main results

In this section, we use the technique of change of variables, which is mainly due to Van Bokhoven [16] as developed by Noor et al.[10,11,12,13], to establish the equivalence between the general complementarity problem and the fixed point problem. We use this equivalent formulation to investigate the existence of a solution of the general complementarity problem.

Using the technique of Van Bokhoven [16] and Noor et al.[10,11,12,13], we rewrite the problem (1) as;

$$w = g(u) = \frac{|x| + x}{2}, \quad v = Tu = \frac{|x| - x}{2\rho}, \quad \langle w, v\rangle = 0. \quad (4)$$

This equivalent formulation plays a crucial role in the derivation of our main results.

If $g^{-1}$ exits, then (4), can be written in the following equivalent form

$$x = \frac{|x| + x}{2} - \rho Tg^{-1}(\frac{|x| + x}{2}). \quad (5)$$

In fact, we have the following result.

**Theorem 1.** *$u \in R^n$ satisfies 1, if and only if, $x \in R^n$ satisfies (5).*

Theorem 3.1 implies that the problem (1) is equivalent to the fixed point problem (5). In order to prove the existence of a solution of problem (1), it is enough to show that the mapping $x$ defined by

$$F(x) = \frac{|x| + x}{2} - \rho Tg^{-1}(\frac{|x| + x}{2}). \quad (6)$$

is a contraction mapping and hence it has a fixed point satisfying problem (1).

**Theorem 2.** *Let the operator $Tg^{-1}$ be the strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If there exists a constant $\rho >$ such that*

$$0 < \rho < \frac{2\alpha}{\beta^2}, \quad (7)$$

*then there exists a solution of problem (1).*

*Proof.* Let $u \in R^n$ be a solution of problem (1). Then, from Theorem 3.1, we can define the mapping $F(x)$ by (6). We rewrite (1) as:

$$w = \frac{|x| + x}{2} \quad (8)$$

$$F(x) = w - \rho Tg^{-1}(w). \quad (9)$$

Consider

$$\begin{aligned}
&\|F(x_1) - F(x_2)\|^2 \\
&= \|w_1 - w_2 - \rho(Tg^{-1}(w_1) - Tg^{-1}w_2\|^2 \\
&= \langle w_1 - w_2, w_1 - w_2\rangle - 2\rho\langle(Tg^{-1}(w_1) - Tg^{-1}(w_2), w - z\rangle \\
&\quad + \langle Tg^{-1}(w_1) - Tg^{-1}(w_2), Tg^{-1}(w_1) - Tg^{-1}(w_2)\rangle \\
&\leq \|w - z\|^2 - 2\rho\alpha\|w - z\|^2 + \rho^2\beta^2\|w_1 - w_2\|^2 \\
&= (1 - 2\rho\alpha + \rho^2\beta^2)\|w_1 - w_2\|^2,
\end{aligned}$$

which implies that

$$\|F(x_1) - F(x_2)\| \leq \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\|w_1 - w_2\|. \quad (10)$$

Also, from (8), we have

$$\begin{aligned}
\|w_1 - w_2\| &= \|\frac{|x_1| + x_1}{2} - \frac{|x_2| + x_2}{2}\| \\
&\leq \frac{1}{2}\|(|x_1| - |x_2|) + (x_1 - x_2)\| \\
&\leq \{\|x_1 - x_2\| + \|x_1 - x_2\||\} = \|x_1 - x_2\|. \quad (11)
\end{aligned}$$

From (10) and (11), we have

$$\begin{aligned}\|F(x_1)-F(x_2)\| &\leq \sqrt{1-2\rho\alpha+\beta^2\rho^2}\|x_1-x_2\| \\ &= \theta\|x_1-x_2\|,\end{aligned}$$

where

$$\theta = \sqrt{1-2\rho\alpha+\beta^2\rho^2}.$$

From (7), it follows that $\theta < 1$, so the mapping $F(x)$ is a contraction mapping and consequently, it has a fixed point satisfying the problem (1).

We would like to mention that this alternative fixed point formulation enables us to suggest the following iterative methods for solving the general complementarity problem (1). Algorithm 3.1. For a given initial value $x_0$, compute the approximate solution $x_{n+1}$ by the iterative scheme

$$x_{n+1} = \frac{|x_n|+x_n}{2} - \rho T g^{-1}(\frac{|x_n|+x_n}{2}), \quad n = 0,1,2,\ldots$$

Using the technique of Noor et al.[7,11,12,13], one can study the convergence criteria of Algorithm 3.1. The comparison of this method with other numerical methods is an interesting problem for future research.

## Conclusion

In this paper we have used the change of variables technique to establish the equivalence between the general complementarity problem and the fixed point problem. This alternative formulation is used to study the existence of a solution of the general complementarity problem under some suitable conditions. It is expected that this technique may be used to suggest some iterative methods for solving the general complementarity and related problems.

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## References

[1] R. W. Cottle, Nonlinear programs with positively bounded Jacobians, J, SIAM Appl. Math. 14(1966), 147-158.
[2] R. W. Cottle, Complementarity and variational problems, Symposia Math. 19(1976), 177-208.
[3] R.W. Cottle, J.S. Pang and R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, (1992).
[4] I. Dolcetta, Sistemi di complementarita a diseqagliance variationali, PhD Thesis, University of Rome, 1972.
[5] S. Karamardian, Generalized complementarity problem, J. Optim. Theory Appl. 8(1971). 161-168.
[6] C. E. Lemke Bimatrix equilibrium points, and mathematical programming, Munugemenr Sci. 11 (1965), 681489.
[7] M.A. Noor, Fixed point approach for complementarity problems, J. Math. Anal. AppL 133, 437-448, (1988).
[8] M. A. Noor, General variational inequalities, Appl. Math. letts. 1(1988), 119-121.
[9] Noor, M. A. (2004). Some developments in general variational inequalities, Appl. Math. Comput. 251, 199-277.
[10] M. A. Noor and S. Zarae, Linear quasi complementarity problems, Utilifus Math. 27 (1985), 249-260.
[11] M. A. Noorand S. Zarae, S., An Iterative Scheme for Complementarity Problems, Engineering Analysis, 3(1986), 240243.
[12] M. A. Noor and E. A. Al-Said, An Iterative Technique for Generalized Strongly Nonlinear Complementarity Problems, Applied Mathematics Letters, Vol. 11, 1998
[13] M. A. Noor and E. A. Al-Said, Change of variable method for generalized complementarity problems, J. Opt. Theory Appl. 100(2)(1999), 389-395.
[14] M. A. Noor, K. I. Noor, and T. M.Rassias, Some Aspects of Variational Inequalities, Journal of Computational and Applied Mathematics, 47(1993),285312.
[15] J. S. Pang, On the convergence of a basic iterative method for the implicit complementarity problem, J. Optim. Theory Appl. 37 (1982), 149-162.
[16] W. M. Van Bokhoven, A class of linear complementarity problems solvable in polynomial time, Technical Report, Department of Electrical Engineering, Technical University, Eindhoven, Holland, 1980.

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