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Advances of a Moving Four-Level Atom in a Nonlinear Medium

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Abstract: We study the interaction between a moving four-level atom and one-mode field in the presence of Kerr like medium and obtain the exact solution of the model. The atomic inversion, Field entropy, Purity and Fidelity have been examined under various values of detuning, the Kerr-like medium and the time-dependent coupling. We show that, the present system is a good candidate for building the quantum devices, where the mentioned parameters can be used controllers. Finally, playing with the initial states setting, one can obtain new features of the entanglement.

Keywords: Four-level atom, Kerr-like medium, entropy, purity, Fidelity.

1 Introduction

The interaction model between a four-level atom and a single quantized mode of a radiation field, when the rotating wave approximation (RWA) considered, is known as the Jaynes-Cummings model (JCM) [1]. This model was generalized to describe the multi-photon interaction and included arbitrary forms of both the field and the intensity-dependent atom-field coupling [2,3]. Also, the standard JCM has been extended in many directions, such as adding further levels [4] where the three-level atomic systems have been discussed [5,6,7,8]. The influence of the Kerr medium [9] and the intensity-dependent coupling [10] on the dynamics of a three-level atomic system have been investigated. the interaction between two three-level atoms and a single-mode field with multi-photon transition in the presence of Kerr medium and detuning parameter has been studied in [11]. Recently, there has been increasing interest in the study of the interaction between a four-level atom and cavity fields. Various types of a four-level atom [12,13,14,15,16,17,18,20,19] have been demonstrated.

Entanglement is a property found in the composite quantum systems, where the correlation between the subsystems cannot be discussed classically. Entanglement has been widely investigated in quantum information processing [22,23,24,25,26,27] and plays important roles in many potential applications, such as quantum communication, quantum teleportation, quantum cryptography, entanglement swapping, dense coding, and quantum computing, etc. [28,29,30,31,32,33]. The degree of entanglement (DEM) can be studied via different measures such as von Neumann entropy, linear entropy, purity and others measurements. Quantifying of the state requires knowledge whether the state is pure or mixed. The state of the system is a pure, then it is sufficient to use the von-Neumann entropy. The time evolution of atomic (field) entropy reflects the time evolution of the degree of entanglement thus when the entropy is high, the degree of entanglement is strong.

We aim at extending the previously cited treatments to study the problem of a four-level atom in the consider configuration interacting with a single-mode field to investigate the properties of the degree of entanglement of the above mentioned systems from the view point of the Phoenix-Knight [34,35,36]. The structure of the levels is given in figure 1. The assumed model contains, in fact, three three-level subsystems with a common fourth level ; one can distinguish here two Fig. 1. Energy-level scheme. subsystems in the ladder configuration (levels 1-4-3 and 2-4-3) and one subsystem in the lambda configuration (1-4-2). Here, we study a moving atomic system of a four-level atom coupled to one mode electromagnetic cavity field in the presence of both Kerr medium and the detuning parameters. We describe the Hamiltonian and derive the constants of motion. Also, this generalization

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takes into account the multi-photon processes. We derive the general form of the probability amplitudes for the considered system. Also, we calculate some statistical aspects of the model such as the atomic inversion, the field entropy, purity and fidelity, also, to examine the influences of the field nonlinearity (Kerr-like medium), detuning and the time dependent coupling on the degree of entanglement.

2 Description of the model

The model consists of a four-level system consists of a moving with states $|j\rangle$, $(j = 1,2,3,4)$, interacting with single quantized field modes with the frequency Ω and various coupling constants [37]. The atom has excited state $|3\rangle$, intermediate states $|4\rangle$ and $|2\rangle$ and ground state $|1\rangle$, with energies ω_3, ω_4, ω_2 and ω_1, respectively. We suppose that the allowed transitions $|3\rangle \longleftrightarrow |4\rangle$ and $|4\rangle \longleftrightarrow |1\rangle(|2\rangle)$ while the transition $|1\rangle \longleftrightarrow |2\rangle$ is forbidden as shown in Fig.(1). The Hamiltonian describing the non-resonant atom-field interaction including the center of mass of the atom beside the presence of the Kerr-like medium is given by,

$$\hat{H} = \hat{H}_K + \hat{H}_{A+F} + \hat{H}_I, \tag{1}$$

where $\hat{H}_K$ is the non-linearity Hamiltonian, $\hat{H}_{A+F}$ is the Hamiltonian of the atom (field) and $\hat{H}_I$ is the interaction Hamiltonian, for simplicity, we set $\hbar = 1$. In the RWA these terms are

$$\begin{aligned} \hat{H}_K &= \chi \hat{a}^{\dagger 2}\hat{a}^2, \\ \hat{H}_{A+F} &= \frac{\hat{p}^2}{2M} + \sum_j \omega_j \hat{\sigma}_{jj} + \Omega \hat{a}^{\dagger}\hat{a}, \end{aligned} \tag{2}$$

where p is the momentum operator, M is the mass of the atom, and $a(a^{\dagger})$ is the annihilation (creation) operator, respectively. Also, χ is the dispersive part of the fourth-order nonlinearity of the Kerr-like medium and $\widehat{\sigma}_{j\ell} = |j\rangle\langle\ell|$; ($\ell$=1,2,3,4) are the level occupation number when $j = \ell$ and otherwise are the transition operators from level j to ℓ. It is important to mention that the operators $\hat{\sigma}_{j\ell}$ are 4×4 matrices, the generators of the unitary group $U(4)$ [38,39]. On the other hand the interaction Hamiltonian $\widehat{H}_I$ for the considered system is given as

$$\begin{aligned} \hat{H}_I = {} & \lambda_1(t)(\hat{a}^m e^{im\vec{k}.\vec{r}}\hat{\sigma}_{41} + \hat{a}^{\dagger m} e^{-im\vec{k}.\vec{r}}\hat{\sigma}_{14}) + \lambda_2(t)(\hat{a}^m e^{im\vec{k}.\vec{r}}\hat{\sigma}_{42} + \hat{a}^{\dagger m} e^{-im\vec{k}.\vec{r}}\hat{\sigma}_{24}) \\ & + \lambda_3(t)(\hat{a}^m e^{im\vec{k}.\vec{r}}\hat{\sigma}_{34} + \hat{a}^{\dagger m} e^{-im\vec{k}.\vec{r}}\hat{\sigma}_{43}), \end{aligned} \tag{3}$$

where, $\vec{k}$ and $\vec{r}$ are the propagation vector and the position vector, respectively. $\lambda_s(t)$ is the effective coupling parameter and taking $\lambda_s(t) = \eta_s \sin(\delta_s t + \phi)$, where η_s(s=1,2,3) is an arbitrary constant (the constant coupling parameter), δ_s is the fluctuation frequency while ϕ is the relative phase and m is multiplicity of photons. In the following, we present some interesting properties of the operators of the considered model. Firstly, for the

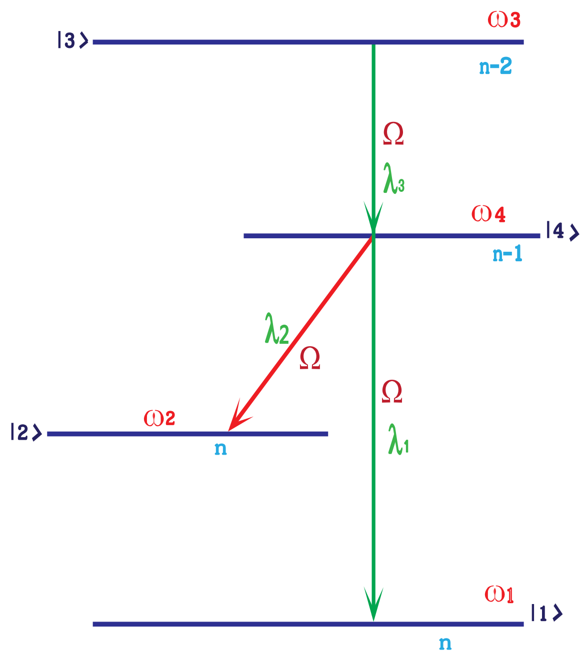

Fig. 1: The scheme for the considered atomic system

Fermion operators $\hat{\sigma}_{ab}$ and Boson operators ($\hat{a}$ and $\hat{a}^{\dagger}$), we can establish the following commutation relation in su(4), for the atomic operator [38,39]:

$$[\hat{\sigma}_{ab}, \hat{\sigma}_{cd}] = \hat{\sigma}_{ad}\delta_{bc} - \hat{\sigma}_{cb}\delta_{da}, \quad \hat{\sigma}_{ab}|b\rangle = |a\rangle, \quad [\hat{a}^m, \hat{\sigma}_{ab}] = [\hat{a}^{\dagger m}, \hat{\sigma}_{ab}] = 0. \tag{4}$$

where δ_{da} is the Kroneker delta. The operators $\hat{a}^{\dagger}$ and $\hat{a}$ are the Bose operators for the quantized field mode which obey

$$[\hat{a}, \hat{a}^{\dagger}] = 1, \qquad [\hat{a}, \hat{a}] = [\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0, \quad [\hat{a}, \hat{n}^{\dagger}] = \hat{a}, \quad [\hat{a}^{\dagger}, \hat{n}^{\dagger}] = -\hat{a}^{\dagger}. \tag{5}$$

In the general form it is easy to show that:

$$[\hat{a}, \hat{a}^{\dagger m}] = m\hat{a}^{\dagger(m-1)}, \qquad [\hat{a}^{\dagger m}, \hat{a}] = -m\hat{a}^{\dagger(m-1)}, \tag{6}$$

Moreover, the field operators satisfy the following relations:

$$\begin{aligned} \hat{a}^m|n\rangle &= \sqrt{\frac{n!}{(n-m)!}}|n-m\rangle, \quad n > m, \\ \hat{a}^{\dagger m}|n\rangle &= \sqrt{\frac{(n+m)!}{(n)!}}|n+m\rangle, \end{aligned} \tag{7}$$

According to the previous relations, It is straightforward to show that the constant of motion in the atom-field system are given by

$$\begin{aligned}\hat{\sigma}_{11}+\hat{\sigma}_{22}+\hat{\sigma}_{33}+\hat{\sigma}_{44} &= \widehat{I},\\ \widehat{a}^{\dagger}\widehat{a}-m(\hat{\sigma}_{11}+\hat{\sigma}_{22}+\hat{\sigma}_{33}) &= \widehat{N}_1,\\ \overrightarrow{P}+\overrightarrow{k}\widehat{a}^{\dagger}\widehat{a} &= \widehat{N}_2.\end{aligned} \tag{8}$$

Now, we turn our attention to find the wave function of the system under consideration.

3 The wave function

To obtain the wave function $|\psi(t)\rangle$ at any time $t>0$, we write it as a linear combination of the states $|\overrightarrow{P}_0,1,n\rangle$, $|\overrightarrow{P}_0,2,n\rangle$, $|\overrightarrow{P}_0+2m\overrightarrow{k},3,n-2m\rangle$, and $|\overrightarrow{P}_0+m\overrightarrow{k},4,n-m\rangle$, where $|\overrightarrow{P}_0\rangle$ is the momentum eigen state, $|j\rangle$ denotes jth atom level and $|n\rangle$, $n=(0,1,2,...)$ are the eigen states of the number operator n. We consider the atom moving in a uniformly accelerated reference frame. Under these considerations, the momentum operator satisfies the following relations

$$\begin{aligned}[e^{\pm im\overrightarrow{k}.\overrightarrow{r}},\overrightarrow{P}] &= \pm m\overrightarrow{k}e^{\pm im\overrightarrow{k}.\overrightarrow{r}},\\ e^{\pm im\overrightarrow{k}.\overrightarrow{r}}|P_0\rangle &= |\overrightarrow{P_0}\pm m\overrightarrow{k}\rangle\\ \overrightarrow{P}|P_0\rangle &= \overrightarrow{P_0}|\overrightarrow{P_0}\rangle,\end{aligned} \tag{9}$$

Therefore, the state vector of the system takes the following form:

$$\begin{aligned}|\Psi(t)\rangle = \sum_n q_n\{&A(n,t)e^{-i\beta_1}|\overrightarrow{P}_0,1,n\rangle+B(n,t)e^{-i\beta_2}|\overrightarrow{P}_0,2,n\rangle\\ &+C(n-2m,t)e^{-i\beta_3}|\overrightarrow{P}_0+2m\overrightarrow{k},3,n-2m\rangle\\ &+D(n-m)e^{-i\beta_4}|\overrightarrow{P}_0+m\overrightarrow{k},4,n-m\rangle\},\end{aligned} \tag{10}$$

(11)

with

$$\begin{aligned}\beta_1 &= \frac{\overrightarrow{P}_0^2}{2M}+\omega_1+n\Omega,\\ \beta_2 &= \frac{\overrightarrow{P}_0^2}{2M}+\omega_2+n\Omega,\\ \beta_3 &= \frac{(\overrightarrow{P}_0+2m\overrightarrow{k})^2}{2M}+\omega_3+(n-2m)\Omega,\\ \beta_4 &= \frac{(\overrightarrow{P}_0+m\overrightarrow{k})^2}{2M}+\omega_4+(n-m)\Omega,\end{aligned} \tag{12}$$

where q_n describes the amplitude of state $|n\rangle$ which are the Fock states of the field modes and depend on the initial state of the atomic type and the quantities $A(n,t)$, $B(n,t)$, $C(n-2m,t)$ and $D(n-m,t)$ are the probability amplitudes which determined the initial state $|\Psi(0)\rangle$.

In our study, we consider the field to be initially in coherent state, thus the initial photon distribution q_n is given by

$$q_n = exp(-\overline{n}/2)\overline{n}^{n/2}/\sqrt{n!}. \tag{13}$$

According to the Schrödinger equation

$$i\frac{d}{dt}\Psi(t) = \hat{H}|\Psi(t)\rangle, \tag{14}$$

and field operators $\widehat{a}_m$ and $\widehat{a}_m^{\dagger}$ on the state vector (10), we obtain the following system of coupled ordinary differential equations:

$$\begin{aligned}i\frac{d}{dt}A(n,t) &= v_1A(n,t)+g_1\sin(\delta_1t+\phi)e^{i\Delta_1t}D(n-m,t),\\ i\frac{d}{dt}B(n,t) &= v_1B(n,t)+g_2\sin(\delta_2t+\phi)e^{i\Delta_2t}D(n-m,t),\\ i\frac{d}{dt}C(n-2m,t) &= v_2C(n-2m,t)+g_3\sin(\delta_3t+\phi)e^{-i\Delta_3t}D(n-m,t),\\ i\frac{d}{dt}D(n-m,t) &= g_1\sin(\delta_1t+\phi)e^{-i\Delta_1t}A(n,t)+g_2\sin(\delta_2t+\phi)e^{-i\Delta_2t}B(n,t)\\ &\quad +g_3\sin(\delta_3t+\phi)e^{i\Delta_3t}C(n-2m,t)+v_3D(n-m,t),\end{aligned} \tag{15}$$

with

$$\begin{aligned}&v_1=\chi n(n-1),\quad v_2=\chi(n-2m)(n-2m-1),\quad v3=\chi(n-m)(n-m-1),\\ &g_1=\eta_1\sqrt{\frac{n!}{(n-m)!}},\quad g_2=\eta_2\sqrt{\frac{n!}{(n-m)!}}\quad g_3=\eta_3\sqrt{\frac{(n-m)!}{(n-2m)!}}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\Delta_1 &= \omega_1-\omega_4+m\Omega-\frac{m^2k^2}{2M}-\frac{m\vec{P_0}.\overrightarrow{k}}{M},\\ \Delta_2 &= \omega_2-\omega_4+m\Omega-\frac{m^2k^2}{2M}-\frac{m\vec{P_0}.\overrightarrow{k}}{M},\\ \Delta_3 &= \omega_4-\omega_3+m\Omega-\frac{3m^2k^2}{2M}-\frac{m\vec{P_0}.\overrightarrow{k}}{M}.\end{aligned} \tag{17}$$

As one can see there are two exponential terms in each equation: one is rapidly oscillating terms $\exp[i(\delta_1+\Delta_s)t+\phi]$ and the other is slowly varying terms $\exp[i(\delta_1-\Delta_s)t+\phi]$). In this case if we neglect the rapidly varying term compared with the slowly varying term, then Eq.15 reduces to:

$$\begin{aligned}i\frac{d}{dt}A(n,t) &= v_1A(n,t)-\tilde{g}_1\exp i[\tilde{\Delta}_1t-\phi]D(n-m,t),\\ i\frac{d}{dt}B(n,t) &= v_1B(n,t)-\tilde{g}_2\exp i[\tilde{\Delta}_2t-\phi]D(n-m,t),\\ i\frac{d}{dt}C(n-2m,t) &= v_2C(n-2m,t)+\tilde{g}_3\exp -i[\tilde{\Delta}_3t-\phi]D(n-m,t),\\ i\frac{d}{dt}D(n-m,t) &= \tilde{g}_1\exp -i[\tilde{\Delta}_1t-\phi]A(n,t)+\tilde{g}_2\exp\\ &\quad -i[\tilde{\Delta}_2t-\phi]B(n,t)\\ &\quad -\tilde{g}_3\exp i[\tilde{\Delta}_3t-\phi]C(n-2m,t)\\ &\quad +v_3D(n-m,t),\end{aligned} \tag{18}$$

where

$$\begin{aligned}\tilde{\Delta}_s &= \Delta_s-\delta_s,\quad s=1,2,3.\\ \tilde{g}_s &= (\frac{-i}{2})g_s,\end{aligned} \tag{19}$$

To solve the coupled system Eq.(18), we consider that the atom and the field are initially prepared in upper state and coherent state, respectively. In this case the initial wave function can be written as

$$|\Psi_{AF}(t=0)\rangle = |\Psi_A(t=0)\rangle \otimes |\Psi_F(t=0)\rangle. \tag{20}$$

where $|\Psi_A(t=0)\rangle$, is the initial state of the atom and $|\Psi_F(t=0)\rangle$, is the initial state of the field. Then the initial state is given by

$$|\Psi_{AF}(t=0)\rangle = \sum_{n=0} q_n |\overrightarrow{P}_0 + 2m\overrightarrow{k}, 3, n-2m\rangle, \tag{21}$$

Now, we will resolve the above system in the non-resonant case as follow.

4 The solution in the non-resonance case

In what follows, under the initial condition Eq.(21) at the non-resonance case. It is obvious that the coefficients of the coupled system differential equations (18) are time-dependent, we can avoid this problem through the following transformation:

$$\begin{aligned}
A(t,n) &= \bar{G}_1(t,n)e^{\frac{i\tilde{\Delta}_1}{2}t},\\
B(t,n) &= \bar{G}_2(t,n)e^{i(\tilde{\Delta}_2-\frac{\tilde{\Delta}_1}{2})t},\\
C(t,n-2m) &= \bar{G}_3(t,n-2m)e^{-i(\tilde{\Delta}_3+\frac{\tilde{\Delta}_1}{2})t},\\
D(t,n-m) &= \bar{G}_4(t,n-m)e^{\frac{-i\tilde{\Delta}_1}{2}t}.
\end{aligned} \tag{22}$$

According to this transformation, we have

$$i\frac{d}{dt}\begin{pmatrix}\bar{G}_1(t)\\ \bar{G}_2(t)\\ \bar{G}_3(t)\\ \bar{G}_4(t)\end{pmatrix} = \begin{pmatrix}\bar{\upsilon}_1 & 0 & 0 & -\tilde{g}_1e^{-i\phi}\\ 0 & \bar{\upsilon}_2 & 0 & -\tilde{g}_2e^{-i\phi}\\ 0 & 0 & \bar{\upsilon}_3 & -\tilde{g}_3e^{i\phi}\\ \tilde{g}_1e^{i\phi} & \tilde{g}_2e^{i\phi} & -\tilde{g}_3e^{-i\phi} & \bar{\upsilon}_4\end{pmatrix}\begin{pmatrix}\bar{G}_1(t)\\ \bar{G}_2(t)\\ \bar{G}_3(t)\\ \bar{G}_4(t)\end{pmatrix} \tag{23}$$

where

$$\begin{aligned}
&\bar{\upsilon}_1 = \upsilon_1 + \frac{\tilde{\Delta}_1}{2}, \quad \bar{\upsilon}_2 = \upsilon_1 + (\tilde{\Delta}_2 - \frac{\tilde{\Delta}_1}{2}),\\
&\bar{\upsilon}_3 = \upsilon_2 - (\tilde{\Delta}_3 + \frac{\tilde{\Delta}_1}{2}), \qquad \bar{\upsilon}_4 = \upsilon_3 - \frac{\tilde{\Delta}_1}{2},
\end{aligned} \tag{24}$$

Under this initial condition (21), the solution of (23) are in the form:

$$\bar{G}_j(t) = \sum_{x=1}^{4} \Im_{jx} e^{i\mu_x t}, \tag{25}$$

where

$$\begin{pmatrix}\Im_{1x}\\ \Im_{2x}\\ \Im_{3x}\\ \Im_{4x}\end{pmatrix} = \frac{1}{\mu_{xk}\mu_{kj}\mu_{jq}}\begin{pmatrix}\Re_1 & -\tilde{g}_1\tilde{g}_3e^{-2i\phi} & 0 & 0\\ \Re_2 & -\tilde{g}_2\tilde{g}_3e^{-2i\phi} & 0 & 0\\ \Re_3 & -(\bar{\upsilon}_3-\tilde{g}_3^2) & -\bar{\upsilon}_3 & -1\\ \Re_4 & \tilde{g}_3(\bar{\upsilon}_3+\bar{\upsilon}_4)e^{i\phi} & \tilde{g}_3e^{-i\phi} & 0\end{pmatrix}\begin{pmatrix}1\\ \mu_k+\mu_x+\mu_q\\ \mu_k\mu_x+\mu_x\mu_q+\mu_q\mu_k\\ \mu_k\mu_x\mu_q\end{pmatrix} \tag{26}$$

where $\mu_{xk} = \mu_x - \mu_k$, $x \neq k \neq j \neq q = 1,2,3,4$ and

$$\begin{aligned}
\Re_1 &= -(\bar{\upsilon}_1+\bar{\upsilon}_3+\bar{\upsilon}_4)\tilde{g}_1\tilde{g}_3e^{-2i\phi},\\
\Re_2 &= -(\bar{\upsilon}_2+\bar{\upsilon}_3+\bar{\upsilon}_4)\tilde{g}_2\tilde{g}_3e^{-2i\phi},\\
\Re_3 &= -\bar{\upsilon}_3^3+\tilde{g}_3^2(2\bar{\upsilon}_3+\bar{\upsilon}_4),\\
\Re_4 &= -(\tilde{g}_1^2+\tilde{g}_2^2+\tilde{g}_3^2-\bar{\upsilon}_3^2-\bar{\upsilon}_4^2-\bar{\upsilon}_3\bar{\upsilon}_4)\tilde{g}_3e^{-i\phi},
\end{aligned} \tag{27}$$

It is worth to mentioning that, μ_j satisfy the quadratic equation

$$\mu^4 + z_1\mu^3 + z_2\mu^2 + z_3\mu + z_4 = 0, \tag{28}$$

where

$$\begin{aligned}
z_1 &= \bar{\upsilon}_1+\bar{\upsilon}_2+\bar{\upsilon}_3+\bar{\upsilon}_4,\\
z_2 &= \bar{\upsilon}_3\bar{\upsilon}_4+\bar{\upsilon}_1(\bar{\upsilon}_3+\bar{\upsilon}_4)+\bar{\upsilon}_2(\bar{\upsilon}_3+\bar{\upsilon}_4)+\bar{\upsilon}_1\bar{\upsilon}_2+\tilde{g}_1^2+\tilde{g}_2^2+\tilde{g}_3^2,\\
z_3 &= \bar{\upsilon}_3\bar{\upsilon}_4(\bar{\upsilon}_1+\bar{\upsilon}_2)+\bar{\upsilon}_1\bar{\upsilon}_2(\bar{\upsilon}_3+\bar{\upsilon}_4)+\tilde{g}_3^2(\bar{\upsilon}_1+\bar{\upsilon}_2)+\tilde{g}_2^2(\bar{\upsilon}_1+\bar{\upsilon}_3)+\tilde{g}_1^2(\bar{\upsilon}_2+\bar{\upsilon}_3),\\
z_4 &= \bar{\upsilon}_1\bar{\upsilon}_2\bar{\upsilon}_3\bar{\upsilon}_4+\bar{\upsilon}_1\bar{\upsilon}_2\tilde{g}_3^2+\bar{\upsilon}_1\bar{\upsilon}_3\tilde{g}_2^2+\bar{\upsilon}_2\bar{\upsilon}_3\tilde{g}_1^2.
\end{aligned} \tag{29}$$

The general expressions for these roots are given by using MATHEMATICA. Having obtained the wave function $|\Psi(t)\rangle$. Now, we are able to study the quantum dynamical properties of the atom and field such as Atomic population inversion, field entropy, purity, fidelity. It is worth mentioning that choosing different values of the detuning, the Kerr-like medium and the time-dependent coupling leads to different physical results.

4.1 Atomic inversion

Here, the atomic inversion is defined in the consider system as the difference between the probabilities of finding the atom in the upper state $|3\rangle$ and in the lower state $|1\rangle$. The maximal state defined as the probabilities of fining the atom in excited state or ground state are equal. Using the wave function $|\Psi(t)\rangle$ or the matrix ,$\rho_{A(t)}$, and assuming that the atom starts from its excited state, the atomic inversion $W(t)$ is giving by [41,42]:

$$W(t) = \sum_{n=0}^{\inf} P_n\left\{|C(t)|^2 - |A(t)|^2\right\} = \langle C|C\rangle - \langle A|A\rangle, \tag{30}$$

where P_n is the distribution function for the field initially in the coherent state. In what follows, we shall investigate numerically the influence of the detuning parameters and non-linear Kerr-like medium on the dynamical behavior of the atomic inversion ,$W(t)$, when the atom is initially prepared in its exited state. In our computations, we choose the initial conditions of the system as follows: the mean photon number $\overline{n} = 40$, the relative phase $\phi = \frac{\Pi}{4}$, the coupling constants are equal $\lambda_1 = 0.7$, $\lambda_2 = 0.02$, $\lambda_3 = 0.7$ and the detuning parameters $\chi = 0$.

In Fig.(2) we have plot the atomic inversion against the scaled λt, where Figs.(2a-2d) show the effect of the detuning parameter. We show that in the absence of both the detuning parameter and the Kerr-like medium see Fig.(2a), the atomic inversion shows fluctuations between negative and positive values. The amplitude of oscillations of atomic inversion decreases when the detuning is increased, see Fig.(2b,2c). One observes that the negative fluctuation of the atomic inversion is revoked when the detuning parameters are increasing see Fig.(2d), the oscillations of $W(t)$ is shifted upward which means that more energy is stored in the atomic system.

Moreover, the amplitude of the atomic inversion oscillations increases when the Kerr-like medium is increased, see Figs.(3a,3b). It is shown that the period of revival increases as soon as the value of the fluctuation frequency ,δ, parameter increases, see Figs.(4a-4d). It is interesting to note that the disappearance of the negative values of the fluctuations means that the energy is stored mainly in the atomic system and very little energy is shared with the field (see from Fig.(4d)), the behaviour of $W(t)$ changed drastically, however it is shown an increase of the collapse time of the collapse which means that the atom reach's the maximal state.

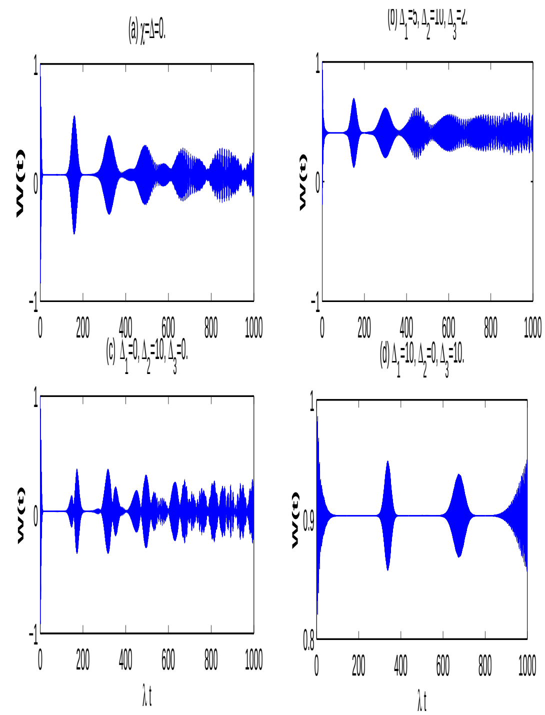

Fig. 2: The evolution of the Atomic inversion of the atomic system versus the scaled time λt for the coherent state with $m=1$, $\bar{n}=40$, $\phi=\pi/4$, $\chi=0, \eta_1=0.7, \eta_2=0.02, \eta_3=0.7$, $\delta=0.5$ and different values of Δ.

4.2 The field entropy

In this section, we use the field entropy as a measure the degree of entanglement between the field and the atom of the system under consideration. Quantum entanglement is one of the main parts for the execution of quantum information processing devices [34,43]. However, According to Araki and Leib theorem [44], for any two components of quantum systems (for instance, the one under consideration), the entropies are limited by the following triangle inequality:

$$|S_A - S_F| \leq S \leq |S_A + S_F| \tag{31}$$

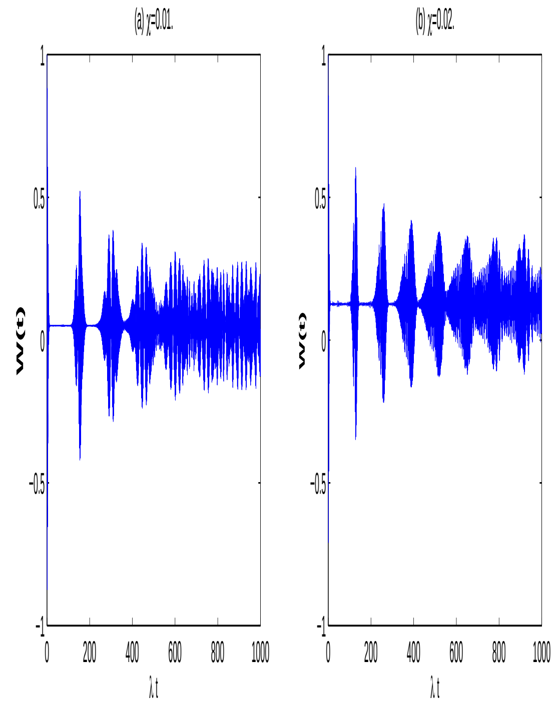

Fig. 3: The same as in (Fig 2) but for $\Delta=0$, and different values of χ for $W(t)$.

where the subscripts A and F refer to the atom and the field, respectively. The total entropy of the atom field system is denoted by S. If the atom is in a pure state, then in a suitable bases the density matrix is diagonal and has a single unit element. For this cases, $S=0$, while if the atom is in a mixed state, $S\neq 0$. Since the initial state is a pure state, then $S=0$, either S_f the field entropy or S_A the atomic entropy is used to measure the amount of entanglement between the two subsystems. When $S_f=S_A=0$, the system is disentangled or separable and both the field and atomic subsystems are in pure states. [35,36]. Therefore, instead of the evaluation of the field entropy, we can obtain the entropy of the atom. The entropies of the atom and the field, are defined through the corresponding reduced density operators by:

$$S_{A(F)} = -\mathbf{Tr}_{A(F)}(\rho_{A(F)} \ln \rho_{A(F)}). \tag{32}$$

The reduced density matrix of the atom required for evaluating Eq.(35) is given by

$$\rho_A = \mathbf{Tr}_F|\Psi(t)\rangle\langle\Psi(t)| = \begin{pmatrix} \rho_{33} & \rho_{34} & \rho_{32} & \rho_{31} \\ \rho_{43} & \rho_{44} & \rho_{42} & \rho_{41} \\ \rho_{23} & \rho_{24} & \rho_{22} & \rho_{21} \\ \rho_{13} & \rho_{14} & \rho_{12} & \rho_{11} \end{pmatrix} \tag{33}$$

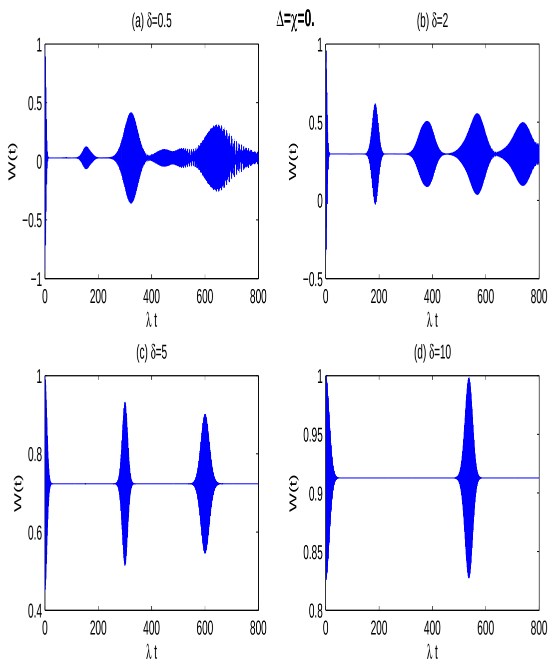

Fig. 4: The same as in (Fig 2) but for $\chi = \Delta_1 = \Delta_2 = \Delta_3 = 0$ for the $W(t)$.

The matrix in Eq.(33) are given, for instance, by

$$\begin{aligned}
\rho_{11} &= \textstyle\sum_{n=0}^{\infty} p_n |A(n,t)|^2, \\
\rho_{22} &= \textstyle\sum_{n=0}^{\infty} p_n |B(n,t)|^2, \\
\rho_{33} &= \textstyle\sum_{n=0}^{\infty} p_n |C(n-2,t)|^2, \\
\rho_{44} &= \textstyle\sum_{n=0}^{\infty} p_n |D(n-1,t)|^2, \\
\rho_{12} &= \textstyle\sum_{n=0}^{\infty} p_n A(n,t) B^*(n,t), \\
\rho_{13} &= \textstyle\sum_{n=0}^{\infty} q_{n-2} q_n^* A(n-2,t) C^*(n-2,t), \\
\rho_{14} &= \textstyle\sum_{n=0}^{\infty} q_{n-1} q_n^* A(n-1,t) D^*(n-1,t), \\
\rho_{23} &= \textstyle\sum_{n=0}^{\infty} q_{n-2} q_n^* B(n-2,t) C^*(n-2,t), \\
\rho_{24} &= \textstyle\sum_{n=0}^{\infty} q_{n-1} q_n^* B(n-1,t) D^*(n-1,t), \\
\rho_{34} &= \textstyle\sum_{n=0}^{\infty} q_n q_{n-1}^* C(n-2,t) D^*(n-2,t),
\end{aligned} \tag{34}$$

where in all of the above relations, $P_n = |q_n|^2$ is the distribution of the initial radiation field, and A, B, C and D are the atomic probability amplitudes derived in Eq.(22). Hence, the entropy of the field or atom can be obtained by the following relation:

$$S_F = S_A = -\sum_{j=1}^{4} \Gamma_j \ln \Gamma_j. \tag{35}$$

where Γ_j, the eigenvalues of the reduced atomic density matrix in Eq.(33), which given by following equation:

$$\Gamma^4 - \Gamma^3 + \Re_1 \Gamma^2 + \Re_2 \Gamma + \Re_3 = 0, \tag{36}$$

where

$$\begin{aligned}
\Re_1 &= \sum_j [\rho_{jj}\rho_{kk} - |\rho_{jk}|^2], \qquad j < k, \\
\Re_2 &= \sum_{j \neq k \neq \ell} \rho_{jj} |\rho_{k\ell}|^2 - \sum_{j<k<\ell} \rho_{jj}\rho_{kk}\rho_{\ell\ell} \\
&\quad - [\rho_{12}\rho_{23}\rho_{31} + \rho_{12}\rho_{24}\rho_{41} + \rho_{13}\rho_{34}\rho_{41} + \rho_{23}\rho_{34}\rho_{42} + h.c.], \\
\Re_3 &= \rho_{11}\rho_{22}\rho_{33}\rho_{44} + \sum_{j \neq k<\ell<m} \rho_{jj} [\rho_{k\ell}\rho_{\ell m}\rho_{mk} + h.c.] + \sum_{j<k<\ell<m} |\rho_{jk}|^2 |\rho_{\ell m}|^2 \\
&\quad - \sum_{j<k \neq \ell \neq m} \rho_{jj}\rho_{kk} |\rho_{\ell m}|^2 - [\rho_{12}\rho_{23}\rho_{34}\rho_{41} + \rho_{12}\rho_{24}\rho_{43}\rho_{31} + \rho_{13}\rho_{34}\rho_{42}\rho_{21} + h.c.],
\end{aligned} \tag{37}$$

It is worth to mention that the four roots of Eq.(36) are given as shown previously by using MATHEMATICA program. Now, we turn our attention to examine numerically the effect of the detuning operator and the non-linear Kerr-like medium on the dynamical behavior of the field entropy, In Fig.(5) the maximal states are occurring at the collapses and the maximum degree of entanglement occurring for different values of detuning parameter. We see that the entropy is stable for a long time. This mean that the time of maximal state increases as the detuning parameter is increased (see Fig.(5b-5d)). In Fig.(6) it is shown that the entanglement increased when the atom is papered in maximal state with a non-zero the Kerr like medium. The results in Fig.(7) indicate that the evolution of the field entropy against the scald time (λt) from which the DEM is studied for different values of δ, we have considered the absence of the detuning parameters and Kerr like medium , we observe that when the entropy is dynamically reduced to the minimum values for absence the detuning and Kerr like medium and the field can not reach to the pure state, we see that for $\delta = 0$ there is increase of the number of oscillation, we have maximal state, so we have high entanglement see Fig.(7a), but for increasing δ we arrive to pure state and the entropy field reich to zero value. see Fig.(7b-7d).

5 The Purity

The purity defines a measure on quantum states, giving information on how much a state is mixed. It is well known that state entanglement and mixture state are properties central to quantum information theory. The relation between entanglement and mixture state have attracted much attention. The purity $P(t)$ of the system may be used as a good tool to give information about the entanglement of the components of the system. For this reason we devote the present section to discuss the purity of the system under consideration. The purity of the field state can be determined from the quantity [45,46]

$$P(t) = Tr(\rho^2(t)), \tag{38}$$

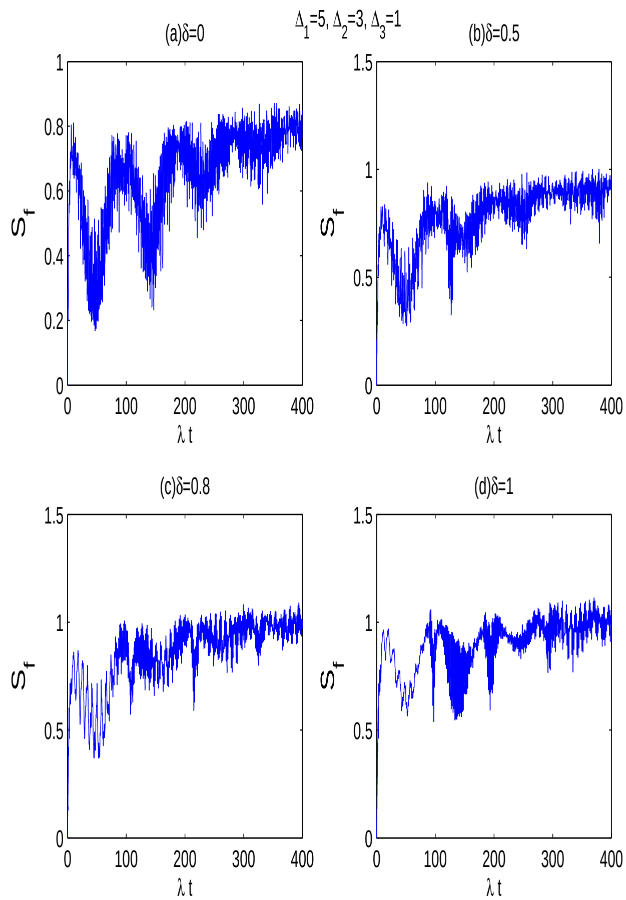

Fig. 5: The evolution of the Field entropy of the atomic system versus the scaled time λt for the coherent state with $m=1$, $\overline{n}=40$, $\phi=\pi/4$, $\chi=0$, $\eta_1=0.7, \eta_2=0.02, \eta_3=0.7$, $\delta=0.5$ and different values of Δ.

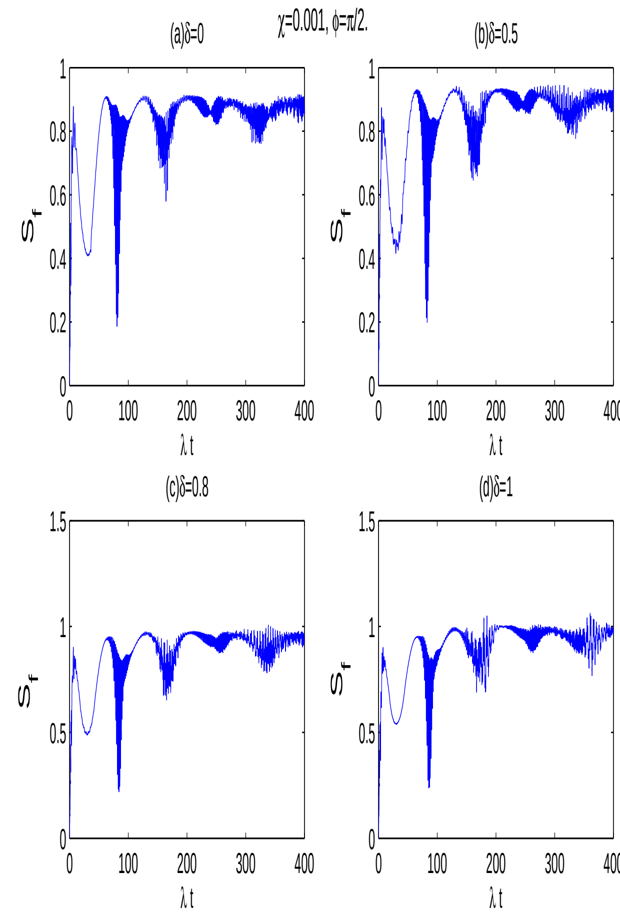

Fig. 6: The same as in (Fig. 5) but for $\Delta=0$, and different values of χ for S_f.

where ρ is the field-reduced density matrix. For a pure state, we have $P(t)=1$ i.e. the purity takes its maximum value of one if the state is a one-dimensional projector, while for a maximally mixed state, i.e., for a total mixture $\rho=\frac{1}{d}$, the purity reaches its minimal value $P=\frac{1}{d}$ where d is the dimension of ρ, i.e, the minimum value of this quantity is bounded by the inverse of the dimension of the system Hilbert space. A necessary and sufficient condition for the ensemble to be described in terms of a pure state is that $Tr(\rho^2(t))=1$, in this case clearly a state-vector description of each individual system of the ensemble is possible. For the case $Tr(\rho^2(t))<1$, the field will be in a statistical mixture state. However, for a maximally mixed state ensemble corresponds to $Tr(\rho^2(t))=\frac{1}{2}$. From Eq.(33), it is easy to show that

$$P(t)=\rho_{11}^2+\rho_{22}^2+\rho_{33}^2+\rho_{44}^2+2|\rho_{12}|^2+2|\rho_{13}|^2+2|\rho_{14}|^2+2|\rho_{23}|^2+2|\rho_{24}|^2+2|\rho_{34}|^2.$$

To do the analysis and discuss the purity, we have plotted the purity in Figs.(8,9,10) against the scaled λt for both the atomic and field subsystems for some chosen parameters, in Fig.(8a) Corresponds to the absence of both Kerr effect and detuning, shown that irregular oscillatory behavior for the time evolution of the purity, it is obvious that the purity of the system takes longer time to be pulled down compared with the effect of detuning parameter only. To visualize the influence of the detuning on the purity, we set three different values of detuning ($\chi=0$, $\Delta_1=\Delta_2=\Delta_3=0$), with all other parameters, we notice that the purity becomes stable and less than 0.5, and the field is in statistically mixed state. Generally, we note that $P(t)\simeq 0.5$ for the specifically values of the detuning parameters seen in Figs.(8b-8d). This means that the purity occurs in both subsystems at the same time and precisely at the same rate.

To visualize the influence of Kerr-like medium on the purity, see Figs.(9a,9b) respectively; Corresponds to the

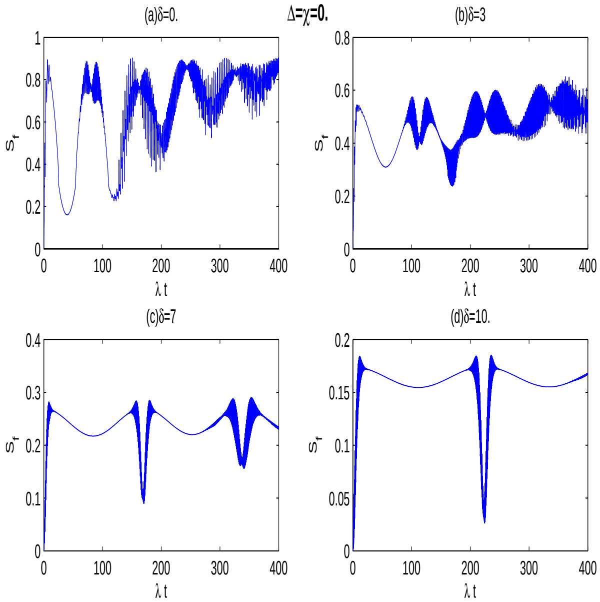

Fig. 7: The same as in (Fig. 5) but for $\chi = \Delta = 0$ for the Field entropy.

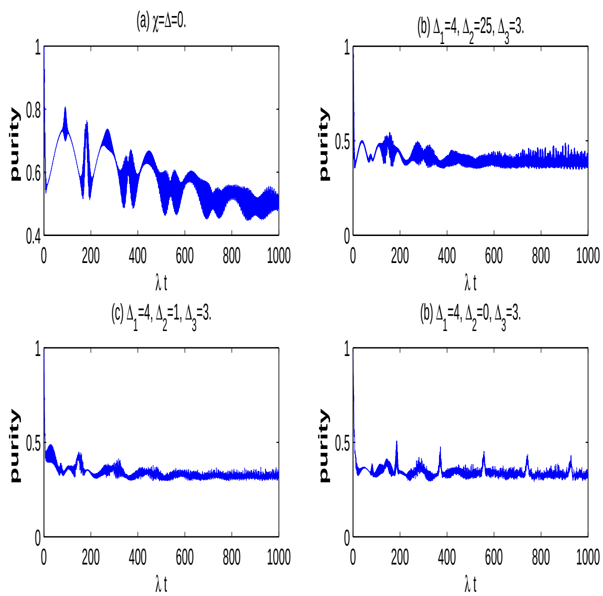

Fig. 8: The time evolution of Purity of the atomic system versus the scaled time λt for the coherent state with $m = 1$, $\overline{n} = 40$, $\phi = \pi/3$, $\chi = 0, \eta_1 = 0.7, \eta_2 = 0.4, \eta_3 = 0.8$, $\delta = 3$ and different values of Δ.

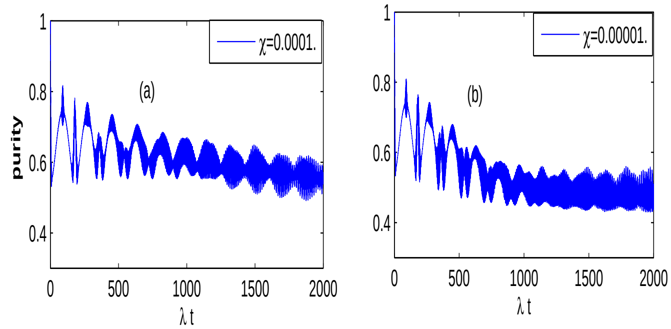

Fig. 9: The same as in (Fig 8) but for $\Delta = 0$, and different values of χ .

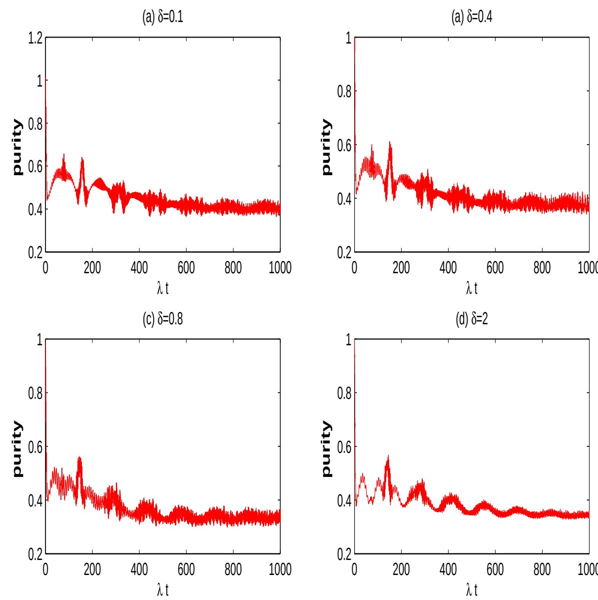

Fig. 10: The same as in Fig.(8) but for $\Delta_1 = 2, \Delta_2 = 1, \Delta_3 = 2$, and for different values of δ.

presence of Kerr medium and exact resonant condition ($\chi \neq 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$). We plotted purity in Fig.(9a) By taking $\chi = 0.001$. we see that purity has increased and many sharp peaks of high values appear with some kind of periodicity. Also, we note that purity has more revivals, and there is not entanglement between subsystem. But for $\chi = 0.00001$, it is noticed that in Fig.(9b) after a sufficient time, the purity becomes stable and less than 0.5, which leads to decrease in the value of purity and the field is in statistically mixed state, So the amplitude of the purity decreases as the non-linear Kerr-like medium decreases.

In Fig.(10), the presence of time-dependent coupling leads to decrease in the value of purity. With the increase in the values of δ the value of purity also begins to increase seen in Fig.(10a), as well as Figs.(10b-10d) for $\chi = 0$ and $\Delta = 0$), after a short time, With the increase of the values of δ, the purity occurs in both subsystems, and it becomes stable and less than 0.5. So that the field is in statistically mixed state and there is an enhancement of exchanging energy between the atom and the field.

6 The Fidelity

The fidelity is an important concept in quantum optics and it is the good measure of distance between quantum states. It has been adopted broadly as an important physical parameter in quantum communication and quantum calculations. We calculate the fidelity of the transition between a pure state $|\Psi(0)\rangle$ and the final state described by $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$. This is equal to the square root of the overlap between the state $|\Psi(0)\rangle$ and the state defined by $\rho(t)$. The fidelity is given by the form [47,**?**],

$$F(t) = \sqrt{\langle\Psi(0)|\rho(t)|\Psi(0)\rangle} = |\langle\Psi(0)|\Psi(t)\rangle|, \tag{39}$$

Otherwise, the fidelity of quantum state is the degree to keep the information of the initial state in a final state and is given by the relation [50,49,51]

$$F(\rho_1,\rho_2) = [Tr(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{\frac{1}{2}}]^2 = \sum_i \sqrt{\lambda_i}, \tag{40}$$

where ρ_1 and ρ_2 are the density operators corresponding to initial and final states, λ_i is the eigen values of $\rho_1^{\frac{1}{2}}\rho_2\rho_2^{\frac{1}{2}}$. F varies between 0 and 1. When ,$F = 1$, then the initial state and the final state are coincided, indicating that two density matrices equal to each other. In this case, an ideal transmitting process takes place see Figs.(11a,12a). for $F = 0$, it means that corresponding to initial and final states are orthogonal each other, indicating that the quantum information (i.e. quantum state) is totally distorted in the transmission. If $0 < F(\rho_1,\rho_2) < 1$, indicating that certain distortion exists in the transmission process of information. if at $t = 0$ the atom is in the excited state and the cavity field is prepared in the coherent state, then the time evolution of the system is described by the state vector Eq.(10) the $F_s(t)$, $F_f(t)$ and $F_a(t)$, (s, f and a) denote the system, field and atom, respectively, as well as the initial condition of system, the $F_a(t)$, $F_f(t)$ and $F_s(t)$ can be respectively expressed as

$$\begin{aligned}
F_s(t) &= \left|\sum_{n=0}^{\infty} P_n C_{n-2}(t)\right|^2, \\
F_a(t) &= \sum_{n=0}^{\infty} P_n \left|C_{n-2}(t)\right|^2, \\
F_f(t) &= \left|\sum_{n=0}^{\infty} q_n q^\star_{n+2} A_n(t)\right|^2 + \left|\sum_{n=0}^{\infty} q_n q^\star_{n+2} B_n(t)\right|^2 \\
&\quad + \left|\sum_{n=0}^{\infty} p_n C_{n-2}(t)\right|^2 + \left|\sum_{n=0}^{\infty} q_n q^\star_{n+1} D_{n-1}(t)\right|^2,
\end{aligned} \tag{41}$$

It is obvious that the fidelity of quantum state for the atom, the field, and the system is related to the coupling constant λ of atom field, Kerr-like medium of the initial field. Since it is a kinetic system, the fidelity also depends on the time t. Substituting Eq.(10) into Eq.(41), one can get the resulting expressions of fidelity for the atoms, the field, and the system. To visualize the influence of the fluctuation frequency δ on the Fidelity as a function of scale time λt The numerical results are shown in Fig.(l1) for different values of δ, for the special case $\chi = 0, \Delta_1 = 5, \Delta_2 = 10, \Delta_3 = 1$, see Fig.(11a) for $\delta = 0$ it is observed that $F_s \approx 1$ (solid line)(Fig.11a) and $F_a \approx 1$ (solid line)(Fig.11c) there are high fidelity, almost there is no interaction between field and atom, namely the interaction of the field with atom is shielded by the detuning.

We find that the $F_s(t), F_f(t)$ dependence on time, the fidelity begin the value 1 but it sharply drops after short time $t = 3$ to almost a constant value (collapse period) (sudden death). seen in Figs.(11b,11e) but at $t = 10$ the fidelities go to along the collapse period, however, it builds up towards a peak during the revival period and drops again to the constant value until it reaches a second lower peak at the second main revival. But we find the effect of kerr and detuning on $F_a(t)$ are very important as we see in Figs.(11c,11f) the behavior of function completely changed by effect Kerr-like medium and detuning. In Fig.(11a,11c)for $\delta = 1$ $(dotline)$, we find a big deep gabs values as strong effect of fluctuation frequency δ on $F_s(t)$ and $F_a(t)$, fidelities.

To visualize the influence of the detuning parameter on the Fidelity, the evolutions of the fidelity for (the atom, the field and the system) with the coherent state. Fig.(12). When $\chi = 0, \Delta_1 = 4, \Delta_2 = 5, \Delta_3 = 0$ (dashed line), $\Delta_1 = 10, \Delta_2 = 5, \Delta_3 = 2$ (dot line)and $\Delta_1 = 0, \Delta_2 = 2, \Delta_3 = 0$(solid line). shown there is dramatic behavior of the three Fidelity as we see in fig.(12a-12c), it is observed that $F_s(t) \approx 1$ (solid line) there are high fidelity, almost there is no interaction between field and atom, namely the interaction of the field with the atom shielded by the effect of detuning. But $F_s(t)$ and $F_a(t)$ has strong interaction, and decrease fidelity. as in Fig.(12b) there is delay of $F_f(t)$ on the fidelity of quantum information for the field. The time evolutions of fidelity of quantum information obviously show the fidelity decreases gradually. It means that the distortion increases gradually, the interaction between the field and the detuning increases, and the fidelity of the system decreases.

To visualize the influence of the Kerr Medium on the Fidelity of Quantum State. The F_a, F_f and F_s with the different values of Kerr coefficient and are plotted in Fig.(13). It is obviously shown from Fig.(13) that the periodicity of the evolutions of fidelities of quantum information in the weak initial field is introduced by Kerr medium. With increasing the Kerr coefficient, the oscillation frequencies of the fidelities F_f and F_s increase, while that of F_a decreases obviously, indicating Kerr medium relatively weakens the relationship between the field and the atom. If $\chi = 0.03, 0.02, 0.01$, the evolutions of the fidelities of quantum information exhibit the oscillations and perfect in Figs.(13a-13c) it is observed that $F_s \approx 0$ see Figs.(13a,13b) (solid line) there are low fidelity, with little oscillations, implying that the strong initial coherent field greatly weakens the fidelities of quantum information. almost there is interaction of the field with the atom is shielded by effect of Kerr-like medium see Figs.(13a,13b). Our results show that the fidelity can provide more detailed information about the system behavior than some global quantities such as the field entropy.

7 Conclusion

We have examined the properties of the entanglement between a four-level atom and a single-mode cavity field. The model is generalized by assuming the existence of the detuning parameters, the multi-photon process, the Kerr-like medium and the intensity dependent coupling. The general expressions of the conservation quantities of motion are given. The exact solution for this model is given when the atom-field is initially prepared in a coherent superposition state. We have explored the temporal evolution of the atomic inversion, the field entropy, purity and fidelity are calculated. For the atomic inversion, we have shown that the atom stay in maximal entangled state when specially values of detuning and the Ker like medium. The degree of this entanglement has been measured by the field entropy. The purity occurs in both subsystems at the same time for a maximally mixed state and we found it in statistically mixed state. Our results show that the fidelity can provide more detailed information about the system behaviour than some global quantities such as the field entropy.

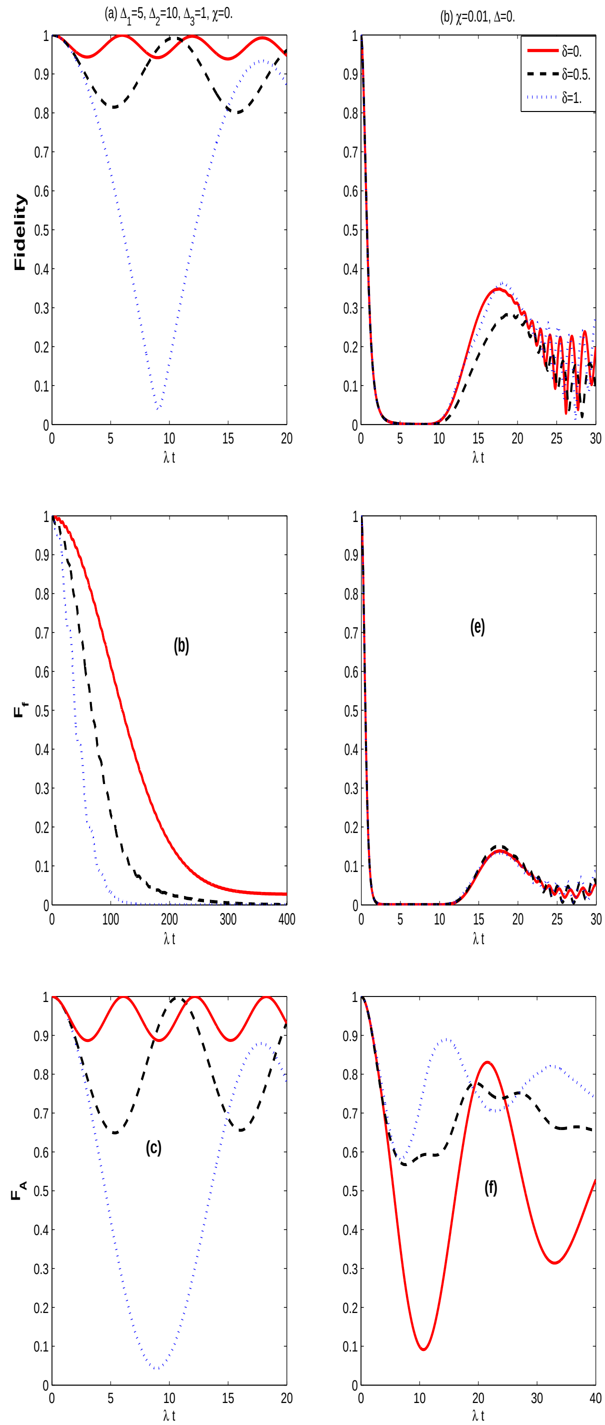

Fig. 11: The time evolution of the Fidelity for versus the scaled time λt with $m = 1$, $\bar{n} = 20$, $\eta_1 = 0.1, \eta_2 = 0.02, \eta_3 = .08$, $\phi = pi/4$ and for different values of δ.

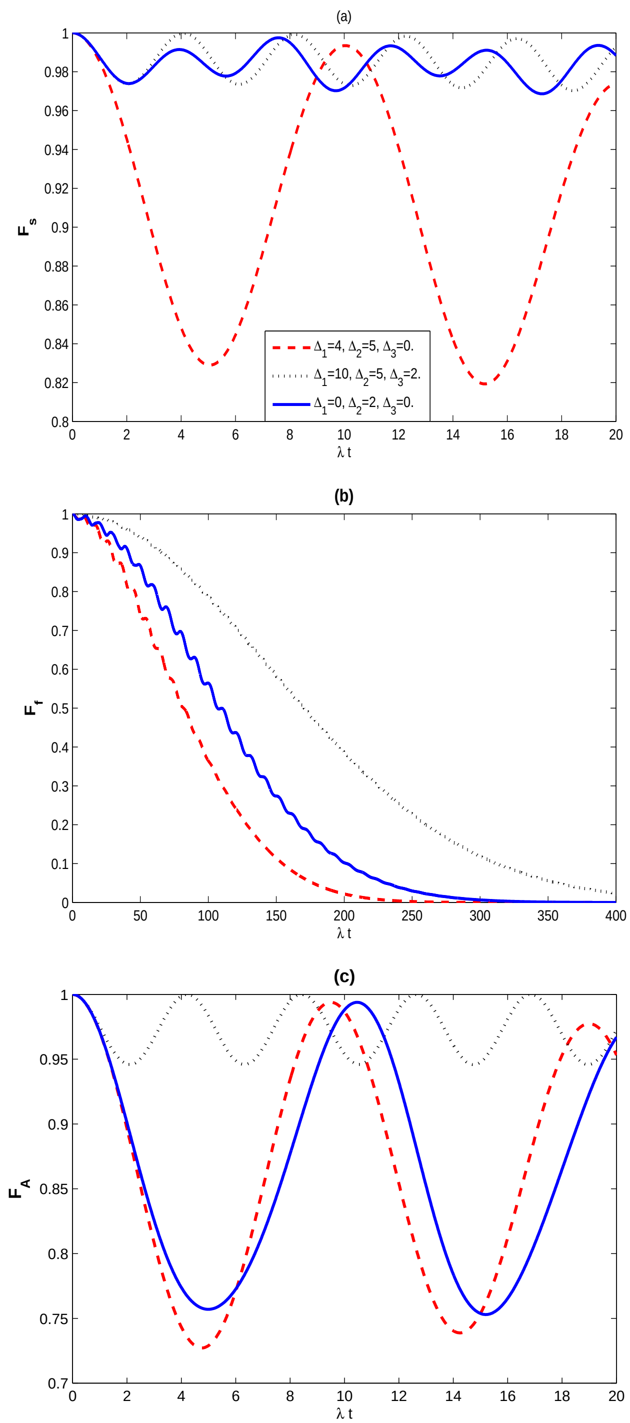

Fig. 12: The same as in (Fig 11) but for $\delta = 0.5$, $\chi = 0$ and different values of Δ for Fidelity different values of Δ.

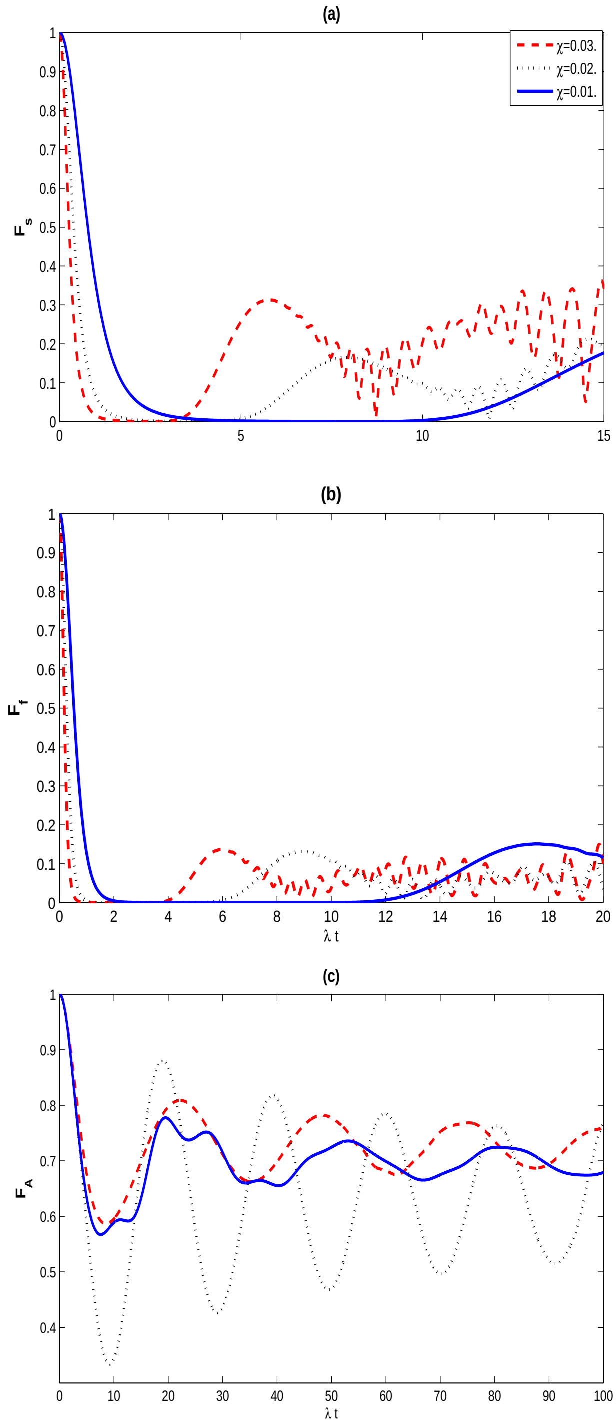

Fig. 13: The same as in (Fig 11) but for $\Delta = 0$, and different values of χ for Fidelity.

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