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Inference on the Parameters and Reliability Characteristics of three parameter Burr Distribution based on Records

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Abstract: A three parameter Burr distribution is considered. Two measures of reliability are discussed, namely $R(t) = P(X > t)$ and $P = P(X > Y)$. Point and interval estimation procedures are developed for the parameters, $R(t)$ and P based on records. Two types of point estimators are developed – uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES). A comparative study of different methods of estimation is done through simulation studies and asymptotic confidence intervals of the parameters based on MLE and log(MLE) are constructed. Confidence intervals for the MLE and UMVUE of the parametric functions are obtained. Testing procedures are also developed for various hypotheses. Real example is used to illustrate the results.

Keywords: Three parameter Burr distribution; point estimation; confidence intervals; records; simulation studies

1 Introduction

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item or a system, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength setup is the probability $P = P(X > Y)$, which represents the reliability of an item or a system of random strength X subject to random stress Y. A lot of work has been done in the literature for the point estimation and testing of $R(t)$ and P. For example, Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Kelley, Kelley and Schucany (1976), Sathe and Shah (1981), Chao (1982), Chaturvedi and Surinder (1999) developed inferential procedures for $R(t)$ and P for exponential distribution. Constantine, Karson and Tse (1986) derived UMVUE and MLE for P associated with gamma distribution. Awad and Gharraf (1986) estimated P for Burr distribution. For estimation of $R(t)$ corresponding to Maxwell and generalized Maxwell distributions, one may refer to Tyagi and Bhattacharya (1981) and Chaturvedi and Rani (1998), respectively. Inferences have been drawn for $R(t)$ and P for some families of lifetime distributions by Chaturvedi and Rani (1997), Chaturvedi and Tomer (2003), Chaturvedi and Singh (2006, 2008), Chaturvedi and Kumari (2015) and Chaturvedi and Malhotra (2016). Chaturvedi and Tomer (2002) derived UMVUE for $R(t)$ and P for negative binomial distribution. For exponentiated Weibull and Lomax distributions, the inferential procedures are available in Chaturvedi and Pathak (2012, 2013, 2014). Many authors have studied the estimation of Burr type XII parameters. Burr and Cislak (1968), Rodriguez (1977) and Tadikamalla (1980) summarized its properties and verified relations with some other distributions. Shao (2004) expanded an extended three parameter Burr type XII distribution and used it for flood frequency analysis. Based on type II censored, Wingo (1993) obtained the maximum likelihood estimators and discussed the necessary and sufficient condition to guarantee the existence, uniqueness and fitness. Wang et al. (1996) presented the methodology to obtain the maximum likelihood and interval estimation of Burr type XII distribution using censored and uncensored data. Soliman (2005) derived maximum likelihood, Bayes and empirical Bayes estimators (BEs) of Burr type XII distribution based on progressive censored samples using various loss functions and Wang and Shi (2010) considered empirical Bayes inference for the Burr model based on record values.

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Chandler (1952) introduced the concept of record values. Based on records, inferential procedures for the parameters of different distributions have been developed by Glick (1978), Nagaraja (1988a, 1988b), Balakrishan, Ahsanullah and Chan (1995), Arnold, Balakrishan and Nagaraja (1992), Habibi Rad, Arghami and Ahmadi (2006), Arashi and Emadi (2008), Razmkhah and Ahmadi (2011), Arabi Belaghi, Arashi and Tabatabaey (2015) and others.

In this article, Section 2 discusses a three parameter Burr distribution by introducing a scale parameter to the Burr XII model (1942). In Section 3, we develop point estimation procedures based on records when one parameter is unknown and the remaining are known and we also discuss the case when all the parameters are unknown. As far as point estimation is concerned, we derive UMVUES and MLES. A new technique of obtaining these estimators is developed, in which first of all the estimators of powers of parameter are obtained. These estimators are used to obtain estimators of $R(t)$. Using the derivatives of the estimators of $R(t)$, the estimators of sampled probability density function (pdf), at a specified point, are obtained which are subsequently used to obtain estimators of P. In Section 4, asymptotic confidence intervals (CIs) for scale and shape parameters and reliability function are constructed. Confidence intervals for the MLES and UMVUES of the parametric functions are also obtained. In Section 5, testing procedures are developed for the parameter of the distribution. In Section 6, we present numerical findings along with real data analysis and finally in Section 7 we discuss our results.

2 The Three Parameter Burr Distribution

The two parameter Burr type XII distribution was first introduced by Burr (1942). A random variable X is said to have a Burr type XII distribution with the shape parameters c and k, if its cumulative distribution function (cdf) and probability density function (pdf) have the forms, respectively, given by

$$F(x;k,c) = 1-(1+x^c)^{-k}; \quad x>0, c, k>0$$

and

$$f(x;k,c) = kcx^{c-1}(1+x^c)^{-(k+1)}; \quad x>0, c, k>0$$

Hogg and Klugman (1984) discussed a three parameter Burr distribution by introducing a scale parameter α to the Burr type XII distribution. This distribution has pdf and cdf of the following form respectively:

$$f(x;k,c,\alpha) = \frac{kcx^{c-1}}{\alpha}\left(1+\frac{x^c}{\alpha}\right)^{-(k+1)}; \; x>0, c, k, \alpha>0 \tag{2.1}$$

and

$$F(x;k,c,\alpha) = 1-\left(1+\frac{x^c}{\alpha}\right)^{-k}; \; x>0, c, k, \alpha, >0 \tag{2.2}$$

From (2.2), the reliability function at a specified time t (>0) is:

$$R(t) = P(X>t) = \left(1+\frac{t^c}{\alpha}\right)^{-k} \tag{2.3}$$

From (2.1)and (2.3), the hazard rate is given by:

$$h(t) = \frac{kct^{c-1}}{\alpha\left(1+\frac{t^c}{\alpha}\right)} \tag{2.4}$$

It follows from (2.4) and Figure 1 that the hazard rate is a decreasing function of t for every value of the parameters k, c and α except for when $c>1$. For $c>1$, the hazard rate increases upto time $t=2$ and then decreases.

3 Point estimation Procedures

Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed (iid) rvs from (2.1). An observation X_j will be called an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i$ for every $i<j$.

The record time sequence $\{T_n, n\geq 0\}$ is defined as:

$$\begin{cases} T_0 = 1; & \text{with probability 1} \\ T_n = \min\{j : X_j > X_{T_{n-1}}; & n\geq 1 \end{cases}$$

The record value sequence $\{R_n\}$ is then defined by:

$$R_n = X_{T_n}; \quad n = 0,1,2,\ldots$$

We can rewrite (2.1) as follows:

$$f(x;k,c,\alpha) = \frac{kcx^{c-1}}{\alpha(1+\frac{x^c}{\alpha})}\exp\left(-k\log\left(1+\frac{x^c}{\alpha}\right)\right); \quad x>0, k, c, \alpha>0$$

Assuming α and c to be known, the likelihood function of the first $n+1$ upper record values $R_0, R_1, R_2, \ldots, R_n$ is:

$$L(k|R_0,R_1,R_2,\ldots,R_n) = f(R_n;k,c,\alpha)\prod_{i=0}^{n-1}\frac{f(R_i;k,c,\alpha)}{1-F(R_i;k,c,\alpha)}$$

It is easy to see that

$$L(k|R_0,R_1,R_2,\ldots,R_n) = \left(\frac{kc}{\alpha}\right)^{n+1}\exp\left(-k\log\left(1+\frac{R_n^c}{\alpha}\right)\right)\prod_{i=0}^{n}\frac{R_i^{c-1}}{(1+\frac{R_i^c}{\alpha})} \tag{3.1}$$

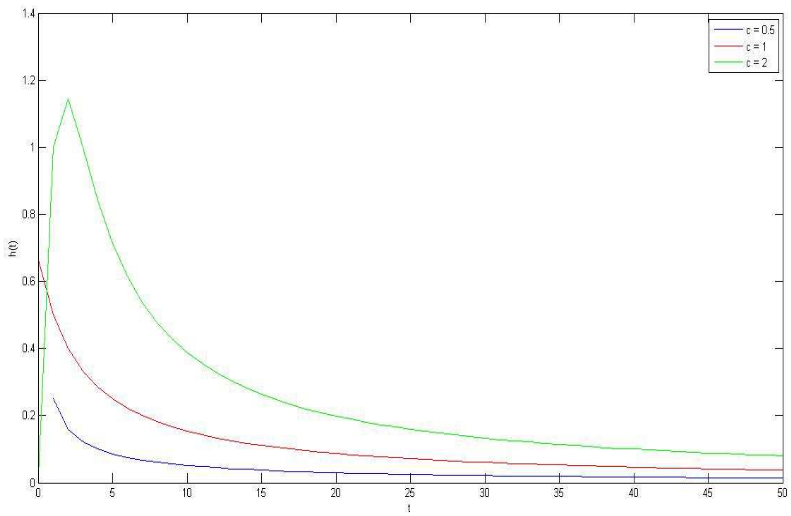

Fig. 1: Hazard Rate for $k = 2$ and $\alpha = 3$

The following theorem provides UMVUE of powers of k. This estimator will be utilized to obtain the UMVUE of reliability functions. For simplicity, we define:

$$U(x) = \log\left(1 + \frac{x^c}{\alpha}\right)$$

Theorem 1. For $q \in (-\infty, \infty)$, $q \neq 0$, the UMVUE of k^q is given by:

$$\tilde{k}^q = \begin{cases} \left\{\dfrac{\Gamma(n+1)}{\Gamma(n-q+1)}\right\}(U(R_n))^{-q}; & n > q-1 \\ 0; & \text{otherwise} \end{cases}$$

Proof. It follows from (3.1) and factorisation theorem [see Rohtagi and Saleh (2012, p. 361)] that $U(R_n)$ is a sufficient statistic for k and the *pdf* of $U(R_n)$ is:

$$h(U(R_n)|k) = \frac{k^{n+1}U(R_n)^n}{\Gamma(n+1)} \exp(-kU(R_n)) \tag{3.2}$$

From (3.2), since the distribution of $U(R_n)$ belongs to exponential family, it is also complete [see Rohtagi and Saleh (2012, p. 367)]. The result now follows from (3.2) that

$$E[U(R_n)^{-q}] = \left\{\frac{\Gamma(n-q+1)}{\Gamma(n+1)}\right\}k^q$$

In the following theorem, we obtain UMVUE of the reliability function.

Theorem 2. The UMVUE of the reliability function is

$$\tilde{R}(t) = \begin{cases} \left[1 - \dfrac{U(t)}{U(R_n)}\right]^n; & U(t) < U(R_n) \\ 0; & \text{otherwise} \end{cases}$$

Proof. It is easy to see that

$$\begin{aligned} R(t) &= \exp\{-kU(t)\} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\{kU(t)\}^i \end{aligned} \tag{3.3}$$

Applying Theorem 1, it follows from (3.3) that

$$\begin{aligned} \tilde{R}(t) &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} U(t)^i \tilde{k}^i \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left\{\frac{U(t)}{U(R_n)}\right\}^i \end{aligned}$$

and the theorem follows.

The following corollary provides UMVUE of the sampled *pdf*. This estimator is derived with the help of Theorem 2.

Corollary 1. The UMVUE of the sampled *pdf* (2.1) at a specified point x is

$$\tilde{f}(x;k,c,\alpha) = \begin{cases} \frac{ncx^{c-1}}{\alpha(1+\frac{x^c}{\alpha})U(R_n)}\left[1 - \dfrac{U(x)}{U(R_n)}\right]^{n-1}; \\ \qquad U(x) < U(R_n) \\ 0; \quad \text{otherwise} \end{cases}$$

Proof. We note that the expectation of $\int_t^{\infty} \tilde{f}(x;k,c,\alpha)dx$ with respect to R_n is $R(t)$. Hence,

$$\tilde{R}(t) = \int_t^{\infty} \tilde{f}(x;k,c,\alpha)dx$$

The result follows from Theorem 2.

In the following theorem, we obtain expression for the variance of $\tilde{R}(t)$, which will be needed to study its efficiency.

Theorem 3. The variance of $\tilde{R}(t)$ is given by:

$$\begin{aligned}
&Var\tilde{R}(t)\\
&=\frac{1}{n!}\{kU(t)\}^{(n+1)}\exp\{-kU(t)\}\\
&\quad\cdot\left[\frac{a_n}{kU(t)}-a_{n-1}\exp\{kU(t)\}E_i(-kU(t))\right.\\
&\quad+\sum_{i=0}^{n-2}a_i\left\{\sum_{m=1}^{n-i-1}\frac{(m-1)!}{(n-i-1)!}(-kU(t))^{n-i-m-1}\right.\\
&\quad\left.-\frac{1}{(n-i-1)!}(-kU(t))^{n-i-1}\exp(kU(t))E_i(-kU(t))\right\}\\
&\quad\left.+\sum_{i=n+1}^{2n}a_i(i-n)!\left(\frac{1}{kU(t)}\right)^{i-n+1}\sum_{r=0}^{i-n}\frac{1}{r!}(kU(t))^r\right]\\
&\quad-\exp\{-2kU(t)\}, \qquad (3.4)
\end{aligned}$$

where $a_i=(-1)^i\binom{2n}{i}$ and $-E_i(-x)=\int_x^\infty \frac{e^{-u}}{u}du$.

Proof. Using (3.2) and Theorem 2,

$$\begin{aligned}
&E\{\tilde{R}(t)^2\}\\
&=\frac{k^{n+1}}{\Gamma(n+1)}\int_{U(t)}^{\infty}\left[1-\frac{U(t)}{U(R_n)}\right]^{2n}\\
&\quad\{U(R_n)\}^n\exp\{-kU(R_n)\}dU(R_n)\\
&=\frac{1}{\Gamma(n+1)}(kU(t))^{n+1}\exp(-kU(t))\\
&\quad\cdot\int_0^\infty\frac{z^{2n}}{(1+z)^n}\exp(-zkU(t))dz\\
&=\frac{1}{\Gamma(n+1)}(kU(t))^{n+1}\exp(-kU(t))I,\ (\text{say}) \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned}
I&=\sum_{i=0}^{n}a_i\int_0^\infty\frac{1}{(z+1)^{n-i}}\exp(-zkU(t))dz\\
&\quad+\sum_{i=n+1}^{2n}a_i\int_0^\infty(z+1)^{i-n}\exp(-zkU(t))dz \qquad (3.6)
\end{aligned}$$

Using a result of Erdélyi (1954) that

$$\begin{aligned}
&\int_0^\infty\frac{\exp(-up)}{(u+a)^n}du\\
&=\sum_{m=1}^{n-1}\frac{(m-1)!(-p)^{n-m-1}}{(n-1)!a^m}\\
&\quad-\frac{(-p)^{n-1}}{(n-1)!}\exp(ap)E_i(-ap)
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^\infty\frac{1}{(z+1)^{n-i}}\exp(-zkU(t))dz\\
&=\sum_{m=1}^{n-i-1}\frac{(m-1)!}{(n-i-1)!}(-kU(t))^{n-i-m-1}\\
&\quad-\frac{1}{(n-i-1)!}(-kU(t))^{n-i-1}\\
&\quad\exp(kU(t))E_i(-kU(t)),\quad i=0,1,2,\ldots,n-2 \qquad (3.7)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_0^\infty\frac{1}{(1+z)}\exp(-zkU(t))dz\\
&=\exp(kU(t))\int_0^\infty\frac{1}{(z+1)}\exp(-kU(t)(z+1))dz\\
&=\exp(kU(t))\int_{(kU(t))}^\infty\frac{e^{-y}}{y}dy\\
&=-\exp(kU(t))E_i(-kU(t)). \qquad (3.8)
\end{aligned}$$

We have

$$\int_0^\infty\exp(-zkU(t))du=\left(\frac{1}{(kU(t)}\right) \qquad (3.9)$$

Finally,

$$\begin{aligned}
&\int_0^\infty(1+z)^{i-n}\exp(-zkU(t))dz\\
&=\sum_{r=0}^{i-n}\binom{i-n}{r}\int_0^\infty z^{i-n-r}\exp(-zkU(t))dz\\
&=\sum_{r=0}^{i-n}\binom{i-n}{r}\left\{\frac{1}{kU(t)}\right\}^{i-n-r+1}\Gamma(i-n-r+1) \qquad (3.10)
\end{aligned}$$

The theorem now follows on making substitutions from (3.7), (3.8), (3.9) and (3.10) in (3.6) and then using (3.5).

Theorem 4. The MLE of $R(t)$ is given by:

$$\hat{R}(t)=\exp\left\{\frac{-(n+1)U(t)}{U(R_n)}\right\}$$

Proof. It can be easily seen from (3.1) that the MLE of k is $\hat{k}=\frac{(n+1)}{U(R_n)}$. The theorem now follows from invariance property of MLE.

In the following corollary, we obtain the MLE of sampled *pdf* with the help of Theorem 4. This will be used to obtain MLE of P.

Corollary 2. The MLE of $f(x;k,c,\alpha)$ at a specified point x is

$$\hat{f}(x;k,c,\alpha)=\frac{(n+1)cx^{c-1}}{\alpha U(R_n)(1+\frac{x^c}{\alpha})}\exp\left\{\frac{-(n+1)U(x)}{U(R_n)}\right\}$$

Proof. The result follows from Theorem 4 on using the fact that

$$\hat{f}(x;k,c,\alpha) = -\frac{d}{dt}\hat{R}(t).$$

In the following theorem, we obtain the expression for variance of $\hat{R}(t)$.

Theorem 5. The variance of $\hat{R}(t)$ is given by:

$$Var\{\hat{R}(t)\}$$
$$= \frac{2}{n!}\{2(n+1)kU(t)\}^{\frac{n+1}{2}} K_{n+1}(2\sqrt{2(n+1)kU(t)}\,)$$
$$-\left[\frac{2}{n!}\{(n+1)kU(t)\}^{\frac{n+1}{2}} K_{n+1}(2\sqrt{(n+1)kU(t)}\,)\right]^2$$

where $K_r(\cdot)$ is modified Bessel function of second kind of order r.

Proof. Using (3.2) and Theorem 4, we have

$$E\{\hat{R}(t)\}$$
$$= \frac{k^{n+1}}{\Gamma(n+1)}\int_0^\infty \exp\left[-\left\{kU(R_n)+\frac{(n+1)U(t)}{U(R_n)}\right\}\right]$$
$$\{U(R_n)\}^n dU(R_n)$$
$$= \frac{1}{\Gamma(n+1)}\int_0^\infty \exp\left[-\left\{y+\frac{(n+1)kU(t)}{y}\right\}\right] y^n dy \tag{3.11}$$

Applying a result of Watson (1952) that

$$\int_0^\infty u^{-m}\exp\left\{-\left(au+\frac{b}{u}\right)\right\}du$$
$$= 2\left(\frac{a}{b}\right)^{(m-1)} 2K_{m-1}(2\sqrt{ab})$$

[it is to be noted that $K_{-m}(\cdot) = K_m(\cdot)$ for $m = 0,1,2,\ldots$], we obtain from (3.11) that

$$E\{\hat{R}(t)\} = \frac{2}{n!}\{(n+1)kU(t)\}^{\frac{(n+1)}{2}} K_{n+1}(2\sqrt{(n+1)kU(t)})$$

Similarly, we can obtain the expression for $E\{\hat{R}(t)^2\}$ and the result follows.

Let X and Y be two independent rvs from three parameter Burr distribution $f(x;k_1,c_1,\alpha_1)$ and $f(y;k_2,c_2,\alpha_2)$ respectively, i.e.

$$f(x;k_1,c_1,\alpha_1)$$
$$= \frac{k_1c_1x^{c_1-1}}{\alpha_1(1+\frac{x^{c_1}}{\alpha_1})}\exp\left\{-k_1\log\left(1+\frac{x^{c_1}}{\alpha_1}\right)\right\};$$
$$x>0, k_1, c_1, \alpha_1 > 0$$

and

$$f(y;k_2,c_2,\alpha_2)$$
$$= \frac{k_2c_2y^{c_2-1}}{\alpha_2(1+\frac{y^{c_2}}{\alpha_2})}\exp\left\{-k_2\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)\right\};$$
$$y>0, k_2, c_2, \alpha_2 > 0$$

Let $\{R_n\}$ and $\{R_m^*\}$ be the record value sequences for X's and Y's respectively. For simplicity, we define:

$$U(x) = \log\left(1+\frac{x^{c_1}}{\alpha_1}\right), \quad V(y) = \log\left(1+\frac{y^{c_2}}{\alpha_2}\right)$$

Theorem 6. The UMVUE of P is given by

$$\tilde{P} = \begin{cases} m\int_0^{\frac{V(R_n)}{V(R_m^*)}}(1-z)^{m-1}[1-U(R_n)^{-1} \\ \quad U\{(\alpha_2(e^{zV(R_m^*)}-1))^{\frac{1}{c_2}}\}]^n dz; \quad R_n < R_m^* \\ m\int_0^1(1-z)^{m-1}[1-U(R_n)^{-1} \\ \quad U\{(\alpha_2(e^{zV(R_m^*)}-1))^{\frac{1}{c_2}}\}]^n dz; \quad R_n \geq R_m^* \end{cases}$$

Proof. It follows from Corollary 1 that the UMVUES of $f(x;k_1,c_1,\alpha_1)$ and $f(y;k_2,c_2,\alpha_2)$ at specified points x and y are respectively:

$$\tilde{f}(x;k_1,c_1,\alpha_1) = \begin{cases} \frac{nc_1x^{c_1-1}}{\alpha_1(1+\frac{x^{c_1}}{\alpha_1})}U(R_n)\left[1-\frac{U(x)}{U(R_n)}\right]^{n-1}; \\ \qquad U(x) < U(R_n) \\ 0; \quad \text{otherwise} \end{cases}$$

and

$$\tilde{f}(y;k_2,c_2,\alpha_2) = \begin{cases} \frac{mc_1y^{c_2-1}}{\alpha_2(1+\frac{y^{c_2}}{\alpha_2})}U(R_n)\left[1-\frac{V(y)}{U(R_m^*)}\right]^{m-1}; \\ \qquad V(y) < V(R_m^*) \\ 0; \quad \text{otherwise} \end{cases}$$

From the arguments similar to those used in the proof of Corollary 1,

$$\tilde{P} = \int_{y=0}^\infty\int_{x=y}^\infty \tilde{f}(x;k_1,c_1,\alpha_1)\tilde{f}(y;k_2,c_2,\alpha_2)dx\,dy$$
$$= \int_{y=0}^\infty \tilde{R}(y;c_1,\alpha_1)\left\{-\frac{d}{dy}\tilde{R}(y;c_2,\alpha_2)\right\}dy$$
$$= m\int_0^{\min\{R_n,R_m^*\}}\left[1-\frac{U(y)}{U(R_n)}\right]^n\left\{\frac{mc_2y^{c_2-1}}{\alpha_2V(R_m^*)(1+\frac{y^{c_2}}{\alpha_2})}\right\}$$
$$\times\left[1-\frac{V(y)}{V(R_m^*)}\right]^{m-1} dy$$

The theorem now follows on considering the two cases and putting $\frac{V(y)}{V(R_m^*)} = z$.

Theorem 7. When $c_1 = c_2 = 1$ and $\alpha_1 = \alpha_2 = 1$, the UMVUE of P is:

$$\tilde{P} = \begin{cases} m\sum_{i=0}^{m-1}(-1)^i\frac{m!n!}{(m-1-i)!(n+1+i)!}\left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{i+1}; \\ \qquad R_n < R_m^* \\ m\sum_{i=0}^{n}(-1)^i\frac{m!n!}{(m+i)!(n-i)!}\left\{\frac{U(R_m^*)}{U(R_n)}\right\}^{i}; \\ \qquad R_n \geq R_m^* \end{cases}$$

Proof. Taking $U(\cdot) = V(\cdot)$ in Theorem 6, then for $R_n < R_m^*$

$$\tilde{P} = m\int_0^{\frac{U(R_n)}{U(R_m^*)}} (1-z)^{m-1}\left\{1-\frac{zU(R_m^*)}{U(R_n)}\right\}^n dz$$
$$= m\left\{\frac{U(R_n)}{U(R_m^*)}\right\}\int_0^1 \left\{1-\frac{wU(R_n)}{U(R_m^*)}\right\}^{m-1}(1-w)^n dw$$
$$= m\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{i+1}\int_0^1 w^i(1-w)^n dw$$

and the first assertion follows. Similarly, we can prove the second assertion.

Theorem 8. The MLE of P is

$$\hat{P} = \int_0^\infty e^{-z}\exp\left\{\frac{-(n+1)}{U(R_n)}U((\alpha_2(e^{\frac{zV(R_m^*)}{m+1}}-1))^{\frac{1}{c_2}})\right\}dz$$

Proof. We have,

$$\hat{P} = \int_{y=0}^\infty\int_{x=y}^\infty \hat{f}(x;k_1,c_1,\alpha_1)\hat{f}(y;k_2,c_2,\alpha_2)dx\,dy$$
$$= \int_{y=0}^\infty \hat{R}(y;c_1,\alpha_1)\hat{f}(y;k_2,c_2,\alpha_2)dy$$
$$= \int_{y=0}^\infty \exp\left\{\frac{-(n+1)U(y)}{U(R_n)}\right\}\left\{\frac{(m+1)c_2y^{c_2-1}}{\alpha_2V(R_m^*)(1+\frac{y^{c_2}}{\alpha_2})}\right.$$
$$\left.\cdot\exp\left\{\frac{-(m+1)V(y)}{V(R_m^*)}\right\}dy\right.$$

The result now follows on putting $\left\{\frac{(m+1)V(y)}{V(R_m^*)}\right\} = z$.

Theorem 9. When $c_1 = c_2 = 1$ and $\alpha_1 = \alpha_2 = 1$, the MLE of P is given by

$$\hat{P} = \frac{(m+1)U(R_n)}{(m+1)U(R_n)+(n+1)U(R_m^*)}$$

Now we consider the case when all the parameters k, c and α are unknown. From (2.1), the log-likelihood function is given as:

$$\begin{aligned} &l(k,c,\alpha)\\ &= L(k,c,\alpha|R_0,R_1,R_2,\ldots,R_n)\\ &= (n+1)\log(k)+(n+1)\log(c)-(n+1)\log(\alpha)\\ &\quad -k\log\left(1+\frac{R_n^c}{\alpha}\right)+(c-1)\sum_{i=0}^n\log(R_i)\\ &\quad -\sum_{i=0}^n\log\left(1+\frac{R_i^c}{\alpha}\right) \end{aligned} \tag{3.12}$$

The MLEs of k, c and α are the solutions of the three simultaneous equations given below:

$$\frac{n+1}{k} - \log\left(1+\frac{R_n^c}{\alpha}\right) = 0 \tag{3.13}$$

$$\frac{-(n+1)}{\alpha} + k\frac{R_n^c}{\alpha^2(1+\frac{R_n^c}{\alpha})} + \frac{1}{\alpha^2}\sum_{i=0}^n\frac{R_i^c}{(1+\frac{R_i^c}{\alpha})} = 0 \tag{3.14}$$

and

$$\frac{n+1}{\alpha} - k\frac{R_n^c\log(R_n)}{\alpha(1+\frac{R_n^c}{\alpha})} - \sum_{i=0}^n\frac{1}{(1+\frac{R_i+\sum_{i=0}^n\log(R_i)^c}{\alpha})}\frac{R_i^c}{\alpha}\log(R_i) = 0 \tag{3.15}$$

From (3.13), we get

$$\hat{k} = \frac{n+1}{\log(1+\frac{R_n^{\hat{c}}}{\hat{\alpha}})} \tag{3.16}$$

where $\hat{k}$, $\hat{c}$ and $\hat{\alpha}$ are the MLEs of k, c and α respectively. Since these non-linear equations don't have a closed form solution, therefore we apply Newton Raphson algorithm to compute MLEs of c and α. These values of MLEs of c and α so obtained can be substituted in (3.16) to obtain MLE of k.

It is to be noted that from Theorem 4, Theorem 8 and invariance property of MLE, the MLE of $R(t)$ is given as:

$$\hat{R}(t) = \exp\left\{\frac{-(n+1)U(t)}{U(R_n)}\right\}$$

where $U(x) = \log(1+\frac{x^{\hat{c}}}{\hat{\alpha}})$. Whereas the MLE of P is given by:

$$\hat{P} = \int_0^\infty e^{-z}\exp\left\{\frac{-(n+1)}{U(R_n)}U((\hat{\alpha}_2(e^{\frac{zV(R_m^*)}{m+1}}-1))^{\frac{1}{\hat{c}_2}})\right\}dz$$

where $U(x) = \log(1+\frac{x^{\hat{c}_1}}{\hat{\alpha}_1})$, $V(x) = \log\left(1+\frac{x^{\hat{c}_2}}{\hat{\alpha}_2}\right)$.

4 Confidence Intervals

The Fisher information matrix of $\theta = (k,c,\alpha)^T$ is:

$$I(\theta) = -E\begin{bmatrix} \frac{\partial^2 l}{\partial k^2} & \frac{\partial^2 l}{\partial k\partial c} & \frac{\partial^2 l}{\partial k\partial\alpha}\\ \frac{\partial^2 l}{\partial c\partial k} & \frac{\partial^2 l}{\partial c^2} & \frac{\partial^2 l}{\partial c\partial\alpha}\\ \frac{\partial^2 l}{\partial\alpha\partial k} & \frac{\partial^2 l}{\partial\alpha\partial c} & \frac{\partial^2 l}{\partial\alpha^2} \end{bmatrix}$$

where $\frac{\partial^2 l}{\partial k^2} = \frac{-(n+1)}{k^2}$, $\frac{\partial^2 l}{\partial c\partial k} = \frac{\partial^2 l}{\partial k\partial c} = \frac{-R_n^c\log(R_n)}{\alpha(1+\frac{R_n^c}{\alpha})}$, $\frac{\partial^2 l}{\partial\alpha\partial k} = \frac{\partial^2 l}{\partial k\partial\alpha} = \frac{R_n^c}{\alpha^2(1+\frac{R_n^c}{\alpha})}$,

$$\frac{\partial^2 l}{\partial\alpha\partial c} = \frac{\partial^2 l}{\partial c\partial\alpha} = \frac{kR_n^c\log(R_n)}{\alpha^2(1+\frac{R_n^c}{\alpha})^2} + \sum_{i=0}^n\frac{R_i^c\log(R_i)}{\alpha^2(1+\frac{R_i^c}{\alpha})^2},$$

$$\frac{\partial^2 l}{\partial\alpha^2} = \frac{n+1}{\alpha^2} - \frac{kR_n^c}{\alpha^4}\left[\frac{2\alpha(1+\frac{R_n^c}{\alpha})-R_n^c}{(1+\frac{R_n^c}{\alpha})^2}\right] - \sum_{i=0}^n\frac{R_i^c}{\alpha^4}\left[\frac{2\alpha(1+\frac{R_i^c}{\alpha})-R_i^c}{(1+\frac{R_n^c}{\alpha})^2}\right],$$

$$\frac{\partial^2 l}{\partial c^2} = \frac{-(n+1)}{c^2} - \frac{kR_n^c(\log(R_n))^2}{\alpha(1+\frac{R_n^c}{\alpha})^2} - \sum_{i=0}^n\frac{R_i^c(\log(R_i))^2}{\alpha(1+\frac{R_i^c}{\alpha})^2}$$

Since it is a complicated task to obtain the expectation of the above expressions, therefore we use observed Fisher information matrix which is obtained by dropping the expectation sign. The asymptotic variance-covariance matrix of the MLEs is the inverse of $I\ (\hat{\theta})$. After obtaining the inverse matrix, we get variance of $\hat{k}$, $\hat{c}$ and $\hat{\alpha}$. We use these values to construct confidence intervals of k, c and α respectively.

Assuming asymptotic normality of the MLEs, CIs for k, c and α are constructed. Let $\hat{\sigma}^2(\hat{k})$, $\hat{\sigma}^2(\hat{c})$ and $\hat{\sigma}^2(\hat{\alpha})$ be the estimated variances of $\hat{k}$, $\hat{c}$ and $\hat{\alpha}$ respectively. Then $100(1-\varepsilon)\%$ asymptotic CIs for k, c and α are respectively given by:

$$(\hat{k}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{k}),\hat{k}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{k})),\ (\hat{c}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{c}),\hat{c}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{c}))$$
$$\text{and}\ \ (\hat{\alpha}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha}),\hat{\alpha}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha}))$$

where $Z_{\frac{\varepsilon}{2}}$ is the upper $100(1-\varepsilon)$ percentile point of standard normal distribution. Using these CIs, one can easily obtain the $100(1-\varepsilon)\%$ aymptotic *CI* for $R(t)$ as follows:

$$\Bigg(\exp\Bigg(-(\hat{k}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{k}))\log\Bigg(1+\frac{t^{\hat{c}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{c})}}{\hat{\alpha}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha})},$$
$$\exp\Bigg(-(\hat{k}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{k}))\log\Bigg(1+\frac{t^{\hat{c}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{c})}}{\hat{\alpha}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha})}\Bigg)\Bigg)\Bigg)$$

Meeker and Escober (1998) reported that the asymptotic CI based on log(MLE) has better coverage probability. An approximate $100(1-\varepsilon)\%$ CI for $\log(k)$, $\log(c)$ and $\log(\alpha)$ are:

$$(\log(\hat{k})-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{k})),\log(\hat{k})+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{k}))),$$
$$(\log(\hat{c})-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{c})),\log(\hat{c})+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{c})))$$
$$\text{and}\ (\log(\hat{\alpha})-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\alpha})),\log(\hat{\alpha})+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\alpha})))$$

where $\hat{\sigma}^2(\log(\hat{\alpha}))$ is the estimated variance of $\log(\alpha)$ and is approximated by $\hat{\sigma}^2(\log(\hat{\alpha}))=\frac{\hat{\sigma}^2(\hat{\alpha})}{\hat{\alpha}^2}$. Similarly, $\hat{\sigma}^2(\log(\hat{k}))$ and $\hat{\sigma}^2(\log(\hat{c}))$ are the estimated variance of $\log(k)$ and $\log(c)$ and are approximated by $\hat{\sigma}^2(\log(\hat{k}))=\frac{\hat{\sigma}^2(\hat{k})}{\hat{k}^2}$ and $\hat{\sigma}^2(\log(\hat{c}))=\frac{\hat{\sigma}^2(\hat{c})}{\hat{c}^2}$ respectively. Hence, approximate $100(1-\varepsilon)\%$ CI for k, c and α are:

$$(\hat{k}e^{-Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{k})}{\hat{k}}},\hat{k}e^{Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{k})}{\hat{k}}}),\ (\hat{c}e^{-Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{c})}{\hat{c}}},\ \hat{c}e^{Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{c})}{\hat{c}}})\ \text{and}$$
$$(\hat{\alpha}e^{-Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\alpha})}{\hat{\alpha}}},\hat{\alpha}e^{Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\alpha})}{\hat{\alpha}}})$$

Now, we construct interval estimates of UMVUE and MLE of k. From (3.2) it follows that $2k\log\left(1+\frac{R_n^c}{\alpha}\right)\sim\chi^2_{2(n+1)}$. Thus $100(1-\varepsilon)\%$ CI for $\tilde{k}$ and $\hat{k}$ is respectively obtained as: $(\frac{2kn}{\chi^2_{2(n+1)}(1-\frac{\varepsilon}{2})},\frac{2kn}{\chi^2_{2(n+1)}(\frac{\varepsilon}{2})})$ and $(\frac{2k(n+1)}{\chi^2_{2(n+1)}(1-\frac{\varepsilon}{2})},\frac{2k(n+1)}{\chi^2_{2(n+1)}(\frac{\varepsilon}{2})})$.

Similarly, we construct interval estimates of UMVUE and MLE of $R(t)$. Thus $100(1-\varepsilon)\%$ CI for $\tilde{R}(t)$ and $\hat{R}(t)$ is respectively obtained as:

$$\left(\left[1-\frac{2kU(t)}{\chi^2_{2(n+1)}(\frac{\varepsilon}{2})}\right]^n,\left[1-\frac{2kU(t)}{\chi^2_{2(n+1)}(1-\frac{\varepsilon}{2})}\right]^n\right)$$

and

$$\left(\exp\left(-\frac{2k(n+1)U(t)}{\chi^2_{2(n+1)}(\frac{\varepsilon}{2})}\right),\exp\left(-\frac{2k(n+1)U(t)}{\chi^2_{2(n+1)}(1-\frac{\varepsilon}{2})}\right)\right)$$

5 Testing of Hypotheses

Suppose for known values of c and α, we have to test the hypothesis $H_0: k=k_0$ against $H_1: k\neq k_0$. It follows from (3.1) that, under H_0,

$$\sup_{\theta_0}L(k|R_0,R_1,\ldots,R_n)$$
$$=\left(\frac{k_0c}{\alpha}\right)^{n+1}\exp\left\{-k_0\log\left(1+\frac{R_n^c}{\alpha}\right)\right\}$$
$$\prod_{i=0}^{n}\frac{R_i^c}{(1+\frac{R_i^c}{\alpha})};\quad \theta_0=\{k:k=k_0\}$$

and

$$\sup_{\theta}L(k|R_0,R_1,\ldots,R_n)$$
$$=\left(\frac{c}{\alpha}\right)^{n+1}\left\{\frac{n+1}{\log\left(1+\frac{R_n^c}{\alpha}\right)}\right\}^{n+1}\exp(-(n+1))$$
$$\prod_{i=0}^{n}\frac{R_i^c}{(1+\frac{R_i^c}{\alpha})};\ \theta=\{k:k>0\}$$

Therefore, the likelihood ratio (LR) is given by:

$$\theta(R_0,R_1,\ldots,R_n)$$
$$=\frac{\sup_{\theta_0}L(k|R_0,R_1,\ldots,R_n)}{\sup_{\theta}L(k|R_0,R_1,\ldots,R_n)}$$
$$=\left\{\frac{k_0\log(1+\frac{R_n^c}{\alpha})}{n+1}\right\}^{n+1}$$
$$\exp\left\{-k_0\log\left(1+\frac{R_n^c}{\alpha}\right)+(n+1)\right\}\qquad(5.1)$$

We note that the first term on the right hand side of (5.1) is monotonically increasing and the second term is monotonically decreasing in $\log(1+\frac{R_n^c}{\alpha})$. It follows from (3.2) that $2k_0\log(1+\frac{R_n^c}{\alpha})\sim\chi^2_{2(n+1)}$. Thus, the critical region is given by $\{0<\log(1+\frac{R_n^c}{\alpha})<l_0\}\cup\{l_0'<\log(1+\frac{R_n^c}{\alpha})<\infty\}$, where

l_0 and l_0' are obtained such that $l_0 = \frac{\chi^2_{2(n+1)}\left(\frac{\varepsilon}{2}\right)}{2k_0}$ and $l_0' = \frac{\chi^2_{2(n+1)}\left(1-\frac{\varepsilon}{2}\right)}{2k_0}$.

An important hypothesis in life-testing experiments is $H_0 : k \leq k_0$ against $H_1 : k > k_0$. It follows from (3.1) that for $k_1 > k_2$,

$$\frac{L(k_1|R_0,R_1,\ldots,R_n)}{L(k_2|R_0,R_1,\ldots,R_n)} = \left(\frac{k_1}{k_2}\right)^{n+1} \exp\left\{(k_2-k_1)\log\left(1+\frac{R_n^c}{\alpha}\right)\right\} \quad (5.2)$$

It follows from (5.2) that $f(x;k,c,\alpha)$ has monotone likelihood ratio in $\log(1+\frac{R_n^c}{\alpha})$. Thus, the uniformly most powerful critical region for testing H_0 against H_1 is given by [see Lehmann (1959, p.88)]

$$\theta(R_0,R_1,\ldots,R_n) = \begin{cases} 1; & \log\left(1+\frac{R_n^c}{\alpha}\right) \leq l_0'' \\ 0; & \text{otherwise} \end{cases}$$

where $l_0'' = \frac{\chi^2_{2(n+1)}(\varepsilon)}{2k_0}$.

6 Numerical Findings

In this section we use Monte Carlo simulation technique to obtain estimates under this scheme. It involves the following steps:

I. For known values of k_1, c_1, α_1, k_2, c_2 and α_2, we generate 1000 samples each from distribution of $X \sim Gamma(n+1,k_1)$ and $Y \sim Gamma(m+1,k_2)$ for specified values of n and m to obtain X_j and Y_j, $j = 1,2,\ldots,1000$, respectively.

II. Compute $U(R_n) = \frac{1}{1000}\sum_{j=1}^{1000} X_j$ and $V(R_m^*) = \frac{1}{1000}\sum_{j=1}^{1000} Y_j$.

For specified value of t, compute $R(t)$ for the three parameter Burr distribution with (k_1,c_1,α_1) and hence compute MLE and UMVUE of $R(t)$. It can be easily shown that

$$P = \frac{k_2c_2}{\alpha_2}\int_{y=0}^{\infty} \frac{y^{c_2-1}}{1+\frac{y^{c_2}}{\alpha_2}} \exp\left(-k_2\log\left(1+\frac{y^{c_2}}{\alpha_2}\right) - k_1\log\left(1+\frac{y^{c_1}}{\alpha_1}\right)\right)dy$$

In Table 1, for $\alpha_1 = 2$, $c_1 = 3$, $k_1 = 0.5$, $\alpha_2 = 5$, $c_2 = 2$ and $k_2 = 4$, we have shown MLE and UMVUE of k_1 for several values of n. For $t = 5$, $R(t) = 0.1254$ and the MLE and UMVUE of $R(t)$ are shown in Table 1 for several values of n. $P = 0.7701$ by the above expression and the MLE and UMVUE of P are shown in Table 1 for several values of n and m.

Figure 2 shows *pdf* plot of three parameter Burr distribution and also displays the MLE and UMVUE of sampled *pdf*.

In order to investigate the performance of the estimators obtained under this scheme, we have evaluated $Var(\tilde{R}(t))$ and $MSE(\hat{R}(t))$ for $c = 2$, $k = 3$, $\alpha = 6$. Table 2 gives $Var(\tilde{R}(t))$ and $MSE(\hat{R}(t))$ for $t = 1(1)20$ and $n = 6,12,18$ and 21. Figure 3 compares the variance UMVUE of reliability function with the mean square error of MLE of reliability function calculated in Table 2 as time t increases for $n = 17$.

For computations shown in Table 3 and Table 4, we have considered $k = 19$, $c = 25$ and $\alpha = 17$. For $t = 1$, $R(t) = 0.3375$. From Table 3 and Table 4 we observe that as sample size increases, the length of CIs based on MLE and log-transformed MLE initially decrease and then start increasing. As reported by Meeker and Escober we too observe that asymptotic CIs based on log-transformed MLE have better coverage probability.

For computations shown in Table 5 and Table 6, we have considered $\alpha = 2$, $c = 6$ and $k = 7$. For $t = 1$, $R(t) = 0.0585$ and we compute point estimate and interval estimate (CI) of UMVUE and MLE of k and $R(t)$. From Table 3 and Table 4 we observe that as sample size increases, the length of CI of UMVUE and MLE of k and $R(t)$ decreases.

In the theory developed in Section 5, we have considered record values from three parameter Burr distribution with $k = 3$, $c = 5$ and $\alpha = 6$.

1.438675 1.701992 1.710421 1.847597 1.923981 2.183058 2.240566 3.043976

The MLE and UMVUE of k are obtained as $\hat{k} = 3.8879$ and $\tilde{k} = 3.4019$. For testing the hypothesis $H_0 : k = k_0 = 3.8$ against $H_1 : k \neq k_0 = 3.8$ under this scheme, with the help of Chi-Square tables at 5% level of significance, we obtained $l_0 = 0.8883$ and $l_0' = 3.7095$. Hence, in this case we may accept H_0 at 5% level of significance since $U(R_n) = 2.0576$. Again, for testing $H_0 : k \leq k_0 = 3.8$ against $H_1 : k > k_0 = 3.8$, we obtained $l_0'' = 1.0238$ and hence, in this case we may accept H_0 at 5% level of significance. Now, for testing the hypothesis $H_0 : k = k_0 = 3.4$ against $H_1 : k \neq k_0 = 3.4$ under this scheme, with the help of Chi-Square tables at 5% level of significance, we obtained $l_0 = 1.0152$ and $l_0' = 4.2395$. Hence, in this case we may accept H_0 at 5% level of significance since $U(R_n) = 2.0576$. Again, for testing $H_0 : k \leq k_0 = 3.4$ against $H_1 : k > k_0 = 3.4$, we obtained $l_0'' = 1.1701$ and hence, in this case we may accept H_0 at 5% level of significance.

An Example on Real Data

To illustrate the estimation methods proposed in the preceding sections, we consider a data analysis of the maximum flood level (in millions of cubic feet per

Table 1: UMVUE of $k, R(t)$ and P

n, m	$\hat{k}$	$\tilde{k}$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{P}$	$\tilde{P}$
10,10	0.50417	0.458338	0.123337	0.121190	0.768618	0.776029
10,20	0.49715	0.451953	0.126986	0.125216	0.772746	0.782217
10,50	0.50641	0.460375	0.122195	0.119931	0.767525	0.778363
10,100	0.50093	0.455395	0.125005	0.123031	0.770633	0.781903
20,10	0.50077	0.476919	0.125093	0.124347	0.770937	0.772458
20,20	0.49633	0.472699	0.127416	0.126787	0.771640	0.775317
20,50	0.50456	0.480538	0.123136	0.122290	0.769361	0.774436
20,100	0.49889	0.475132	0.126072	0.125375	0.770301	0.775870
50,10	0.50038	0.490572	0.125291	0.125053	0.768763	0.766477
50,20	0.49984	0.490035	0.125577	0.125344	0.770284	0.770277
50,50	0.50163	0.491791	0.124647	0.124396	0.767785	0.769229
50,100	0.50155	0.491716	0.124686	0.124436	0.770136	0.772094
100,10	0.49969	0.494745	0.125652	0.125546	0.770622	0.767112
100,20	0.50136	0.496395	0.124785	0.124671	0.769943	0.768675
100,50	0.50036	0.495406	0.125304	0.125194	0.770147	0.770357
100,100	0.49892	0.493979	0.126056	0.125954	0.770663	0.771387

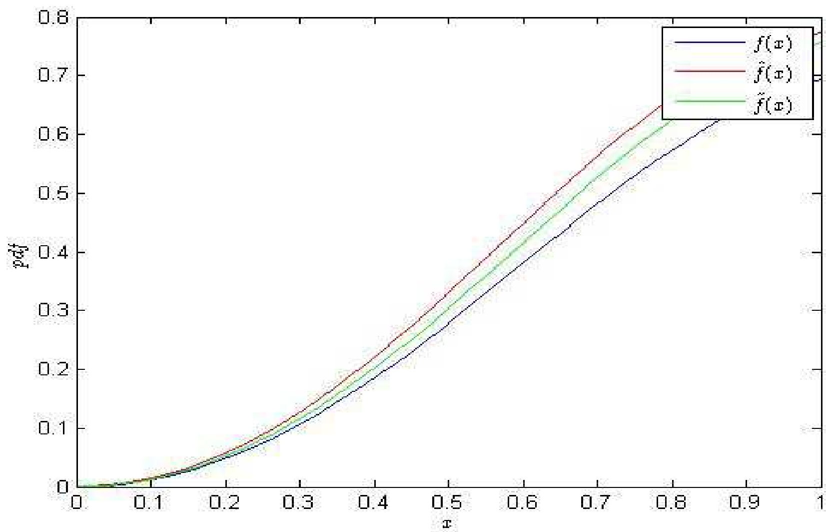

Fig. 2: MLE and UMVUE of sampled *pdf*

second) for the Susquehanna River of Harrisburg over 20 four-year periods (Dumonceaux and Antle, 1973) and is as follows:

0.6540.613 0.315 0.449 0.297 0.402 0.379

0.423 0.379 0.3235 0.269 0.740 0.418 0.412

0.494 0.416 0.338 0.392 0.484 0.265

The following are the upper record values obtained from it.

0.65400.7400

Shao (2004) showed that the MLE of k, c and α using the New-Raphson method are obtained as $\hat{k} = 0.142$, $\hat{c} = 6.434$ and $\hat{\alpha} = 1350.844$. Based on the estimates from this sample, we compute interval estimates of UMVUE and MLE of k and $R(t)$ and the results are shown in Table 7. We also compute point estimate, interval estimate and MSE of UMVUE and MLE of $R(t)$ for time $t = 0.3$ and the results are shown in Table 8.

7 Discussion

A lot of work has been done in the literature to estimate and test the hypotheses for the reliability functions. In the present paper, we have discussed a three parameter Burr distribution. Based on record values, estimation and testing procedures are developed for this distribution.

In Table 2, a comparative study of efficiencies of UMVUE and MLE of reliability function based on record

Table 2: Mean Square Error of MLE and UMVUE of Reliability function

n	6		12		18		21	
t	$Var(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$Var(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$Var(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$Var(\tilde{R}(t))$	$MSE(\hat{R}(t))$
1	0.01432	0.14522	0.00714	0.11452	0.00475	0.09094	0.00407	0.10254
2	0.01596	0.45428	0.00851	0.37184	0.00580	0.27874	0.00501	0.34401
3	0.00468	0.50920	0.00245	0.39112	0.00166	0.26042	0.00143	0.35702
4	0.00111	0.46048	0.00055	0.32348	0.00036	0.18835	0.00031	0.28675
5	0.00028	0.39893	0.00013	0.25663	0.00008	0.13138	0.00007	0.22058
6	0.00008	0.34526	0.00003	0.20496	0.00002	0.09330	0.00002	0.17122
7	0.00003	0.30163	0.00001	0.16660	0.00001	0.06822	0.00001	0.13566
8	0.00001	0.26649	0.00000	0.13796	0.00000	0.05136	0.00000	0.10979
9	0.00000	0.23798	0.00000	0.11620	0.00000	0.03968	0.00000	0.09058
10	0.00000	0.21455	0.00000	0.09934	0.00000	0.03135	0.00000	0.07600
11	0.00000	0.19505	0.00000	0.08602	0.00000	0.02526	0.00000	0.06470
12	0.00000	0.17861	0.00000	0.07532	0.00000	0.02069	0.00000	0.05577
13	0.00000	0.16459	0.00000	0.06659	0.00000	0.01720	0.00000	0.04859
14	0.00000	0.15252	0.00000	0.05937	0.00000	0.01448	0.00000	0.04274
15	0.00000	0.14202	0.00000	0.05332	0.00000	0.01232	0.00000	0.03791
16	0.00000	0.13281	0.00000	0.04821	0.00000	0.01059	0.00000	0.03387
17	0.00000	0.12468	0.00000	0.04384	0.00000	0.00919	0.00000	0.03045
18	0.00000	0.11746	0.00000	0.04007	0.00000	0.00803	0.00000	0.02754
19	0.00000	0.11099	0.00000	0.03680	0.00000	0.00706	0.00000	0.02504
20	0.00000	0.10517	0.00000	0.03393	0.00000	0.00626	0.00000	0.02288

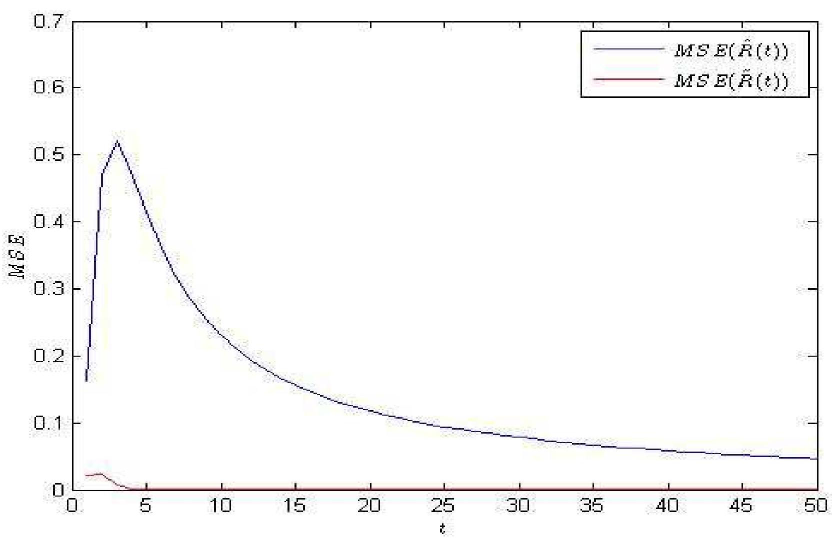

Fig. 3: Mean Square Error of MLE and UMVUE of Reliability function for sample size $n = 3$

values has been performed. It is clear from simulation results that UMVUES of the reliability function are more efficient than MLE of reliability function. We also observe that as sample size increases, the UMVUE of the reliability function based on records become more efficient, but such is not a case in MLE of the reliability function based on records. However, with the passage of time, the efficiency of these estimators initially decreases and then starts increasing. In Table 3 and Table 4, we established that the confidence interval based on log-transformed MLE have a better coverage probability. Table 5 and Table 6 shows the interval estimates based on UMVUE and MLE of the parameter and reliability function. We observe that as sample sizes increase, these interval estimates become more accurate. An analysis on real data is in conformity with the results produced by simulation. It hereby shows the relevance of the study.

Table 3: CI and length of CI based on MLE and log(MLE) of k and c at 95% and 90% level of significance

	k		$\log(k)$		c		$\log(c)$	
n	95%	90%	95%	90%	95%	90%	95%	90%
5	[8.8059 29.1940]	[10.4448 27.5551]	[11.1107 32.4911]	[12.1116 29.8059]	[21.6088 28.3911]	[22.1540 27.8459]	[21.8287 28.6319]	[22.3100 28.0143]
	20.3881	17.1102	21.3804	17.6942	6.7823	5.6919	6.8032	5.7042
7	[9.5595 28.4404]	[11.0773 26.9226]	[11.5602 31.2276]	[12.5216 28.8301]	[22.4869 27.5131]	[22.8909 27.1090]	[22.6090 27.6437]	[22.9774 27.2005]
	18.8809	15.8453	19.6674	16.3085	5.0262	4.2181	5.0346	4.2231
12	[5.0546 32.9453]	[7.2966 30.7033]	[9.1200 39.5831]	[10.2622 35.1773]	[21.3655 28.6344]	[21.9498 28.0501]	[21.6173 28.9119]	[22.1285 28.2440]
	27.8906	23.4066	30.4631	24.9150	7.2689	6.1002	7.2945	6.1154
20	[4.1254 33.8745]	[6.5169 31.4830]	[8.6847 41.5670]	[9.8496 36.6510]	[16.4456 33.5543]	[17.8209 32.1790]	[17.7556 35.2001]	[18.7597 33.3159]
	29.7490	24.9661	32.8822	26.8014	17.1086	14.3580	17.4444	14.5561

Table 4: CI and length of CI based on MLE and log(MLE) of α and CI and length of CI of $R(t)$ at 95% and 90% level of significance

	α		$\log(k)$		$R(t)$	
n	95%	90%	95%	90%	95%	90%
5	[5.9231 28.0768]	[7.7039 26.2960]	[8.8607 32.6157]	[9.8393 29.3718]	[0.0105 0.7347]	[0.0346 0.6771]
	22.1537	18.5920	23.7549	19.5325	0.7242	0.6425
7	[6.3719 27.6280]	[8.0806 25.9193]	[9.0978 31.7658]	[10.0597 28.7282]	[0.0158 0.7118]	[0.0432 0.6574]
	21.2561	17.8387	22.6680	18.6684	0.6960	0.6142
12	[5.3537 28.6462]	[7.2261 26.7738]	[8.5688 33.7266]	[9.5666 30.2092]	[0.0035 0.8407]	[0.0186 0.7652]
	23.2925	19.5477	25.1577	20.6462	0.8372	0.7465
20	[1.8186 32.1813]	[4.2594 29.7405]	[6.9600 41.5223]	[8.0346 35.9690]	[0.0005 1]	[0.0032 1]
	30.3626	25.4811	34.5623	27.9343	0.9995	0.9968

Table 5: Point estimate, Interval estimate and length of CI of UMVUE and MLE of k at 95% and 90% level of significance

n	$\tilde{k}$	$\tilde{k}$		$\hat{k}$	$\hat{k}$	
		95%	90%		95%	90%
5	6.0733	[2.9995 15.8954]	[3.3292 13.3944]	7.2879	[3.5994 19.0744]	[3.9950 16.0733]
		12.8958	10.0652		15.4750	12.0783
10	6.7769	[3.8063 12.7477]	[4.1268 11.3470]	7.4546	[4.1869 14.0225]	[4.5395 12.4817]
		8.9414	7.2202		9.8355	7.9422
15	7.5237	[4.2441 11.4812]	[4.5460 10.4623]	8.0253	[4.5270 12.2466]	[4.8490 11.1598]
		7.2371	5.9163		7.7195	6.3107
20	6.4420	[4.5324 10.7697]	[4.8172 9.9488]	6.7641	[4.7590 11.3082]	[5.0581 10.4462]
		6.2373	5.1315		6.5492	5.3881

Table 6: Point estimate, Interval estimate and length of CI of UMVUE and MLE of $R(t)$ at 95% and 90% level of significance

n	$\tilde{R}(t)$	$\tilde{R}(t)$		$\hat{R}(t)$	$\hat{R}(t)$	
		95%	90%		95%	90%
5	0.0336	[0 0.2481]	[0 0.2073]	0.0520	[0.0004 0.2323]	[0.0014 0.1979]
		0.2481	0.2073		0.2319	0.1964
10	0.0402	[0.0006 0.1870]	[0.0021 0.1602]	0.0486	[0.0033 0.1831]	[0.0063 0.1587]
		0.1863	0.1581		0.1797	0.1523
15	0.0330	[0.0037 0.1607]	[0.0068 0.1399]	0.0386	[0.0069 0.1595]	[0.0108 0.1399]
		0.1569	0.1330		0.1525	0.1291
20	0.0608	[0.0072 0.1454]	[0.0110 0.1280]	0.0644	[0.0102 0.1452]	[0.0144 0.1286]
		0.1382	0.1170		0.1349	0.1141

Table 7: CI of UMVUE and MLE of k and $R(t)$ at 95% level of significance

$\tilde{k}$	$\hat{k}$	$\tilde{R}(t)$	$\hat{R}(t)$
95%	95%	95%	95%
[0.0254 0.5862]	[0.0509 1.1725]	[0.9769 1]	[0.9126 1]

Table 8: Point estimate and MSE of UMVUE and MLE of $R(t)$

$\tilde{R}(t)$	$Var(\tilde{R}(t))$	$\hat{R}(t)$	$MSE(\hat{R}(t))$
0.9940	3.2751E-14	0.9969	3.5794E-05

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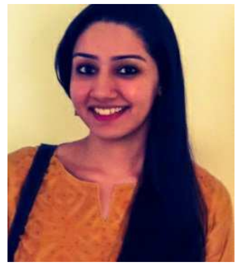

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