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Inverted Generalized Linear Exponential Distribution As A Lifetime Model

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Abstract: This paper concerns with a new lifetime model named the inverted generalized linear exponential distribution (IGLED). Statistical properties like moments, quantile and modes are introduced. The classification of the behavior of IGLED based on reliability analysis like mean residual life (MRL) time, the mean waiting time (MWT), the hazard rate (HR) function and the reversed hazard rate (RHR) function are discussed. Bonferroni curve (B_c), Lorentz curve, the scaled total time on test (TTT) transform curve and the measures of income inequality are also studied. The heavy-weight property is proved for IGLED under the shape parameter ξ. The explanation of the other two shape parameter in the sense of economic is shown. Furthermore, maximum likelihood estimation is used to estimate the parameters of the new model. Four applications are used to show whether the IGLED is better than other well-known distribution in modeling lifetime data.

Keywords: Reliability analysis, Unimodal hazard rate, Lorentz curve, Bonferroni curve, Inverted distribution

1 Introduction

In reliability theory, the HR and the RHR are important widely measures. Also, it is well-known that the residual life time, Ω_t and the reversed residual life time (time since failure) $\underline{\Omega}_t$ play an important role in reliability theory. The HR function and the RHR function are based on Ω_t and $\underline{\Omega}_t$ respectively, where for a system of age t, $\Omega_t = (T-t)|(T \geq t)$ is the remaining life time after t and $\underline{\Omega}_t = (t-T)|(T \leq t)$ is the time elapsed after failure till time t, given that the unit has already failed by time t. Another ageing measures widely used in reliability analysis are MRL time and MWT. Recently, the variance residual life (VRL) and the variance reversed residual life (VRRL) have an interest in reliability analysis [see [1], [2] and [3]]. The behavior of all these measures of the IGLED are discussed.

The generalized linear exponential distribution (GLED) is first proposed by [4]. [5] introduced a new generalization of GLED named exponentiated GLED. Recently, [6] provided some notes on GLED in [4]. In this article, we proposed a new inverted distribution named IGLED. IGLED is considered as a generalization of the inverted exponential distribution (IED), inverse Weibull distribution (IWD) and inverse Rayleigh distribution (IRD). There are many articles dealt with inverted distributions and its generalizations, see for example, [7], [8], [9] and [10]. The main theme of this paper is to obtain the IGLED and study its statistical properties and the properties in terms of reliability analysis and an income inequality.

The rest of this article is organized as follows. The probability density function(pdf), cumulative distribution function (cdf), hazard rate function, and survival function of IGLED are introduced in Section 2. In Section 3, some important statistical properties are proposed.Properties of the IGLED in terms of reliability analysis are given in Section 4. The behavior of the (B_c), the B, the Lorentz curve, the Gini coefficient and the scaled TTT transform curve are discussed in Section 5. In Section 6, the MLE and the ACIs are discussed. Analysis of four real data sets are presented in Section 7.

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2 Inverted Generalized Linear Exponential Distribution

For a random variable Y, the pdf of GLED is given by

$$f(y;c,b,\xi) = \xi\ e^{-(c\,y+\frac{b}{2}\,y^2)^{\xi}}\ (c\,y+\frac{b}{2}\,y^2)^{\xi-1}\ (c+b\,y),$$
$$c>0,\ b>0,\ \xi>0,\ y>0, \quad (1)$$

The pdf of IGLED with parameter vector $\Theta=(c,b,\xi)$ is given by setting $X=\frac{1}{Y}$ in (1) as

$$f(x;\Theta) = \xi\ e^{-(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi}}\ (\frac{c}{x}+\frac{b}{2\,x^2})^{\xi-1}(\frac{c}{x^2}+\frac{b}{x^3}),$$
$$c>0,\ b>0,\ \xi>0,\ x>0. \quad (2)$$

The cdf of the IGLED are given by;

$$F(x;\Theta) = e^{-(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi}} \quad , x>0. \quad (3)$$

The survival and hazard rate functions are given by:

$$S(t;\Theta) = 1\ -\ e^{-(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}, \quad (4)$$

and

$$h(t;\Theta) = \frac{\xi\ e^{-(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}\ (\frac{c}{t}+\frac{b}{2\,t^2})^{\xi-1}\ (\frac{c}{t^2}+\frac{b}{t^3})}{1\ -\ e^{-(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}} \quad , t>0, \quad (5)$$

respectively.

Remark 1.

From Equation (2), some special distributions can be obtained:

1. For $b=0$, and $\xi=1$, Equation (2) reduces to

$$f(x;c) = (\frac{c}{x^2})\ e^{-(\frac{c}{x})}, \quad x>0,\ c>0,$$

which is the pdf of the IED [8].

2. For $c=0$ and $\xi=1$, Equation (2) reduces to

$$f(x;b) = \frac{b}{x^3}\ e^{-(\frac{b}{2x^2})^2}, \quad x>0,\ b>0,$$

which is the pdf of the IRD [11].

3. For $b=0$, Equation (2) reduces to

$$f(x;c,\xi) = \xi\ e^{-(\frac{c}{x})^{\xi}}\ (\frac{c}{x})^{\xi-1}\ (\frac{c}{x^2}),\ \ x>0,\ c>0,\ \xi>0,$$

which is the pdf of the IWD.

Remark 2.

(1) Indeed, it is easy to show that the simulated data can be obtained from,

$$x = \frac{c+\sqrt{c^2+2\,b\,(-\ln u)^{\frac{1}{\xi}}}}{2\,(-\ln u)^{\frac{1}{\xi}}}, \quad (6)$$

where U follows a standard uniform distribution.
(2) From Equations (2) and (3), we get

$$x^3\,(\frac{c}{x}+\frac{b}{2x^2})\ f(x;\Theta) = \xi\, F(x;\Theta)\Big(-\ln F(x;\Theta)\Big)\ (c\,x+b). \quad (7)$$

3 Some Statistical Properties

In this section some statistical properties like, moment, quantiles and mode, are derived. In particular, the median is derived from the quantiles.

3.1 *Moments*

Moments play an important role in the applications of the statistical analysis. A probability distribution may be characterized by its moments. We now introduce an explicit form of the k-th moments of IGLED.

Theorem 3.1.

The k-th moments $\mu^{(k)}$ of IGLED; $k=1,2,3,\ldots$ is given by

$$\mu^{(k)} = \sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\binom{\frac{k-i}{2}}{j}\,(\frac{c}{2})^k\,(\frac{2\,b}{c^2})^j$$
$$\times\Big(\Gamma(\frac{j-k+\xi}{\xi})-\Gamma(\frac{j-k+\xi}{\xi},(\frac{c^2}{2\,b})^{\xi})\Big)$$
$$+\sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\binom{\frac{k-i}{2}}{j}\,(\frac{1}{2})^k\,(\frac{c^2}{2\,b})^j\,c^i\,(2\,b)^{\frac{k-i}{2}}$$
$$\times\Gamma\Big(\frac{2\,\xi-i-k-2\,j}{2\,\xi},(\frac{c^2}{2\,b})^{\xi}\Big),\ (j-k)>-\xi, \quad (8)$$

where $\Gamma(.)$ is gamma function and $\Gamma(.,.)$ is the upper incomplete gamma function.

Proof. The k-th moments of IGLED can be written in the form

$$\mu^{(k)} = \xi\int_0^{\infty} x^k\ e^{-(\frac{c}{x}+\frac{b}{2x^2})^{\xi}}\ (\frac{c}{x}+\frac{b}{2x^2})^{\xi-1}\ (\frac{c}{x^2}+\frac{b}{x^3})\ dx$$

Upon using the substitution $v=(\frac{c}{x}+\frac{b}{2x^2})^{\xi}$, one can show that the k-th moments is given by

$$\mu^{(k)} = \int_0^{\infty}\Big(\frac{c+\sqrt{c^2+2\,b\,(v)^{\frac{1}{\xi}}}}{2\,(v)^{\frac{1}{\xi}}}\Big)^k\ e^{-v}\ dv.$$

Expanding $\Big(\frac{c+\sqrt{c^2+2\,b\,(v)^{\frac{1}{\xi}}}}{2\,(v)^{\frac{1}{\xi}}}\Big)^k$, yields

$$\mu^{(k)} = \int_0^{\infty}\sum_{i=0}^{k}\binom{k}{i}\,c^i\,(c^2+2\,b\,v^{\frac{1}{\xi}})^{\frac{k-i}{2}}\ e^{-v}\ dv. \quad (9)$$

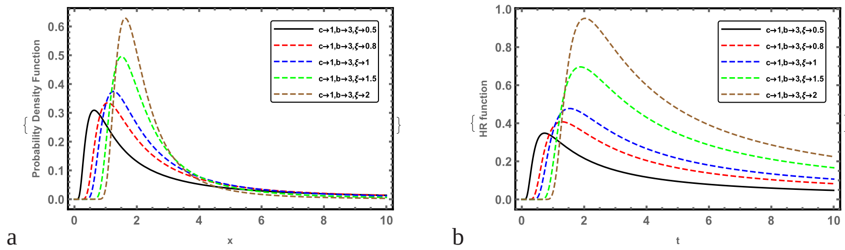

Fig. 1: a) The pdf of IGLED with different values of parameters b) The hazard rate function of IGLED with different values of parameters

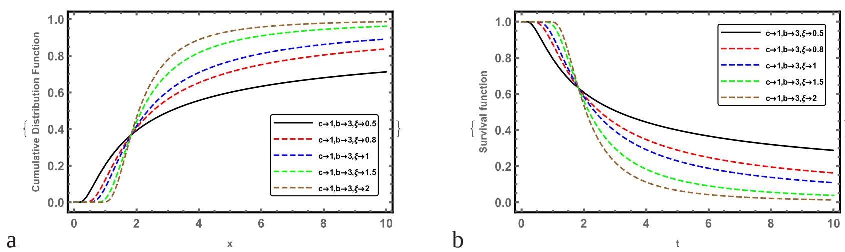

Fig. 2: a) The cdf function of IGLED with different values of parameters b) The survival function of IGLED with different values of parameters

One can show that, $\left|\frac{2\,b\,v^{\frac{1}{\xi}}}{c^2}\right| < 1$ when $v < (\frac{c^2}{2\,b})^{\xi}$ and $\left|\frac{c^2}{2\,b\,v^{\frac{1}{\xi}}}\right| < 1$ when $v > (\frac{c^2}{2\,b})^{\xi}$.
Hence, (9) should be written as

$$\mu^{(k)} = \sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\binom{\frac{k-i}{2}}{j}\,(\frac{c}{2})^k(\frac{2\,b}{c^2})^j\int_0^{(\frac{c^2}{2\,b})^{\xi}} v^{\frac{j-k}{\xi}}\,e^{-v}\,dv$$
$$+\sum_{i=0}^{k}\sum_{j=0}^{\infty}\binom{k}{i}\binom{\frac{k-i}{2}}{j}\,(\frac{1}{2})^k\,(\frac{c^2}{2\,b})^j\,c^i\,(2\,b)^{\frac{k-i}{2}}$$
$$\times\int_{(\frac{c^2}{2\,b})^{\xi}}^{\infty} v^{\frac{-2j-i-k}{2\,\xi}}\,e^{-v}\,dv. \qquad (10)$$

Then the proof is completed.

3.2 Mode and quantile

Theorem 3.2.

The pdf of IGLED has a unimodal shape in the interval $[x_1, x_2]$ where $x_1 = \frac{c\,+\,\sqrt{c^2\,+\,2\,b\,(1+\frac{1}{\xi})^{\frac{1}{\xi}}}}{2\,(1+\frac{1}{\xi})^{\frac{1}{\xi}}}$ and $x_2 = \frac{c\,+\,\sqrt{c^2\,+\,2\,b\,(1+\frac{1}{2\,\xi})^{\frac{1}{\xi}}}}{2\,(1+\frac{1}{2\,\xi})^{\frac{1}{\xi}}}$.

Proof.

The first derivative w.r.t. x of the pdf of the IGLED can be written as

$$\frac{d}{dx}f(x;\Theta) \;=\; g_1(x;\Theta)\,g(x;\Theta), \qquad (11)$$

where

$$g_1(x;\Theta) = \frac{\xi}{2\,(c\,x+b)^2}\,e^{-(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi}}\,(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi-2}\,(\frac{c}{x^2}+\frac{b}{x^3})^2,$$

and

$$g(x;\Theta) = (c\,x+b)^2\left(2\,\xi\,(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi}\,-2\,\xi-2\right)+b^2.$$

Equating (11) by zero, and it is clear that $g_1(x;\Theta) > 0$, then

$$(c\,x+b)^2\left(2\,\xi\,(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi}\,-2\,\xi-2\right)+b^2 = 0. \qquad (12)$$

It is clear that (12) can also be written as

$$(c\,x+b)^2\left(2\,\xi\,(\frac{c}{x}+\frac{b}{2x^2})^{\xi}\,-2\,\xi-1\right)-c^2\,x^2-2\,c\,b\,x = 0. \qquad (13)$$

One can show from (12) that $f(x;\Theta) > 0$ when $x \leq x_1$ and from (13) that $f(x;\Theta) < 0$ when $x \geq x_2$. Now, define $g(x;\Theta)$ on a closed interval $[x_1, x_2]$. Clearly, $g(x;\Theta)$ is a continuous function on a closed interval $[x_1, x_2]$. Furthermore, $g(x_1;\Theta) = b^2$ and $g(x_2;\Theta) = -c^2\,x_2^2 - 2\,c\,b\,x_2$. Then, there exists

$x_0 \in [x_1, x_2]$ such that $g(x_0;\Theta) = 0$. Since $g(x;\Theta)$ is a differentiable function on an open interval (x_1, x_2) and

$$\frac{d}{dx}g(x;\Theta) = 4\,c\,(c\,x+b)\left(\xi\,(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi} - \xi - 1\right) - 2\,(c\,x+b)^2\left(\xi^2\,(\frac{c}{x}+\frac{b}{2\,x^2})^{\xi-1}\,(\frac{c}{x^2}+\frac{b}{x^3})\right)$$

is always negative on (x_1, x_2), then it is clear that this root is unique.

Remark 3.

Some special cases can be obtained from Equation (11).

1. For $\xi = 1$ and $b = 0$, Equation (11) reduces to $2\,c\,x - c^2 = 0$ which leads to the mode $\widetilde{x} = \frac{c}{2}$ of IED.
2. For $\xi = 1$ and $c = 0$, Equation (11) reduces to $3\,b\,x^2 - b^2 = 0$ which leads to the mode $\widetilde{x} = \sqrt{\frac{b}{3}}$ of IRD.

Moreover, the Quantile of IGLED can be given by

$$x_q = \frac{c+\sqrt{c^2+2\,b\,(-\ln q)^{\frac{1}{\xi}}}}{2\,(-\ln q)^{\frac{1}{\xi}}}, \quad 0<q<1. \tag{14}$$

Then the median of the IGLED is obtained by setting $q = 0.5$ in Equation (14) as

$$Med = \frac{c+\sqrt{c^2+2\,b\,(\ln 2)^{\frac{1}{\xi}}}}{2\,(\ln 2)^{\frac{1}{\xi}}}. \tag{15}$$

Some special cases of quantile and median for IED, IRD and IWD can be obtained.

4 Properties of the IGLED in Terms of Reliability Analysis

In this section some properties of the IGLED, which is important in reliability analysis, are studied. In particular, the behavior of the HR, the RHR, the MRL time, the MWT, the variance of residual life (VRL) and the variance of reversed residual life (VRRL) are discussed.

4.1 Behavior of hazard rate function

From Equation (5), it is easy to prove that

$$\lim_{t\to 0^+} h(t;\Theta) = 0, \tag{16}$$

and

$$\lim_{t\to\infty} h(t;\Theta) = 0. \tag{17}$$

Since $h(t;\Theta) > 0$ and from Equations (16) and (17), one can see that $h(t;\Theta)$ is a non-monotonic function. This property makes the IGLED distribution widely applicable. Now, we want to show that the HR of IGLED is a unimodal.

Theorem 4.1.

The HR function of IGLED has a unimodal shape.
Proof.
Due to [12], $\eta(t)$ can be written as

$$\eta(t) = \frac{1}{t\,(2\,c\,t+b)\,(c\,t+b)}\Big(2\,(c\,t+b)^2 + 2\,\xi\,(c\,t+b)^2 - 2\,\xi\,(c\,t+b)^2\,(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi} - b^2\Big) \tag{18}$$

The first derivative of $\eta(t)$ can be obtained as

$\acute{\eta}(t) = p_1(t;\Theta)\;p(t;\Theta)$,

where

$$p_1(t;\Theta) = \frac{1}{t^2\,(2\,c\,t+b)^2\,(c\,t+b)^2},$$

and

$$p(t;\Theta) = -\Big(b^4\,(1+2\,\xi)+c\,b^3\,t\,(6+12\,\xi)+c^2\,b^2\,t^2\,(16+22\,\xi)+c^3\,b\,t^3\,(16+16\,\xi)+c^4\,t^4\,(4+4\,\xi) - \xi\,(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}\,\big(2\,b^4\,(1+2\,\xi)+c\,b^3\,t\,(12+16\,\xi)+c^2\,b^2\,t^2\,(22+24\,\xi)+c^3\,b\,t^3\,(16+16\,\xi)+c^4\,t^4\,(4+4\,\xi)\big)\Big).$$

Since $\xi > 0$, $c > 0$, $b > 0$, and $t > 0$, one can show that $\acute{\eta}(t) > 0$ whenever $t \le t_1$ and $\acute{\eta}(t) < 0$ whenever $t \ge t_2$ where $t_1 = \frac{c+2\,b\,\sqrt{c^2+2\,b\,(\frac{1}{\xi})^{\frac{1}{\xi}}}}{2\,(\frac{1}{\xi})^{\frac{1}{\xi}}}$ and $t_2 = \frac{c+2\,b\,\sqrt{c^2+2\,b\,(\frac{1}{2\,\xi})^{\frac{1}{\xi}}}}{2\,(\frac{1}{2\,\xi})^{\frac{1}{\xi}}}$. Define $p(t;\Theta)$ on a closed interval $[t_1, t_2]$. One can show that $p(t_1;\Theta) > 0$, $p(t_2;\Theta) < 0$ and $\acute{p}(t;\Theta) < 0$ on the interval (t_1, t_2). Then as in Theorem (3.2), there exist t_0 such that $p(t_0;\Theta) = 0$ and this root is a unique.

4.2 Behavior of reversed hazard rate function

The reversed hazard rate function of IGLED is given by

$$r(t;\Theta) = \frac{f(t;\Theta)}{F(t;\Theta)} = \xi\,(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi-1}\,(\frac{c}{t^2}+\frac{b}{t^3}) \quad ,t>0. \tag{19}$$

It is easy to prove that

$$\lim_{t\to 0^+} r(t;\Theta) = \infty, \tag{20}$$

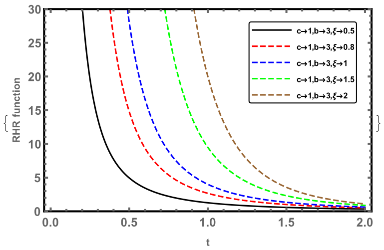

Fig. 3: Behavior of reversed hazard rate function with different values of parameters.

and

$$\lim_{t\to\infty} r(t;\Theta) = 0. \quad (21)$$

The first derivative of Equation (19) is given by

$$\frac{d}{dt} r(t;\Theta) = -\,\xi\, \frac{1}{2\,t^6}\,(\frac{c}{t} + \frac{b}{2t^2})^{\xi-2}\Big(2\,\xi(c\,t\,+\,b)^2 + c\,t\,(c\,t\,+b)^2 + b\,(c\,t\,+b) + c\,b\,t\Big) \quad (22)$$

which is always negative for $t > 0,\ c > 0,\ b > 0$ and $\xi > 0$. Then the reversed hazard rate function is decreasing.

Remark 5.

Since the reversed hazard rate function is decreasing, it is clearly that the distribution function of IGLED $F(t;\Theta)$ is log concave.

4.3 Behavior of mean residual life time

The MRL time of positive continuous random variable T is defined as

$$m(t;\Theta) = E[\Omega_t;\Theta] = \frac{1}{S(t;\Theta)} \int_t^\infty (x-t)\ f(x;\Theta)\,dx \quad (23)$$

Theorem 4.2.

Using the Equations (2), (4) and (23), the explicit forms for MRL time of IGLED are given by:

$$m(t;\Theta) = \begin{cases} \frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big[\frac{c}{2}\Big(\Gamma(\frac{\xi-1}{\xi}) - \Gamma\big((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\big)\Big) \\ +\frac{c}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2\,b}{c^2})^i\Big(\Gamma(\frac{\xi+i-1}{\xi}) - \Gamma\big((\frac{\xi+i-1}{\xi}),(\frac{c^2}{2\,b})^{\xi}\big)\Big) \\ +\frac{1}{2}\,(2\,b)^{\frac{1}{2}}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2\,b})^i\Big(\Gamma\big((\frac{2\,\xi-2\,i-1}{2\,\xi}),(\frac{c^2}{2\,b})^{\xi}\big) \\ -\Gamma\big((\frac{2\,\xi-2\,i-1}{2\,\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\big)\Big) - t\Big(1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}\Big)\Big], \\ \qquad (\frac{c^2}{2\,b})^{\xi} < (\frac{c}{t}+\frac{b}{2t^2})^{\xi}; \\ \frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big[\frac{c}{2}\Big(\Gamma(\frac{\xi-1}{\xi}) - \Gamma\big((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\big)\Big) \\ +\frac{c}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2\,b}{c^2})^i\Big(\Gamma(\frac{\xi+i-1}{\xi}) - \Gamma\big((\frac{\xi+i-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\big)\Big) \\ -t\Big(1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}\Big)\Big], \qquad (\frac{c^2}{2\,b})^{\xi} > (\frac{c}{t}+\frac{b}{2t^2})^{\xi}. \end{cases} \quad (24)$$

Proof.
To derive the explicit forms of the MRL time of IGLED, the integral $\int_t^\infty x\ f(x;\Theta)\ dx$ must be calculated (see Appendix). The MRL time satisfies the following:

$$\lim_{t\to 0} m(t;\Theta) = \frac{c}{2}\,\Gamma(\frac{\xi-1}{\xi}) + \frac{c}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\,(\frac{2\,b}{c^2})^i \times \Bigg(\gamma\Big((\frac{\xi+i-1}{\xi}),(\frac{c^2}{2\,b})^{\xi}\Big)\Bigg) + \frac{1}{2}\,(2\,b)^{\frac{1}{2}} \times \sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\,(\frac{c^2}{2\,b})^i\,\Gamma(\frac{2\,\xi-2\,i-1}{2\,\xi}), \quad (25)$$

,where $\gamma(.,.)$ is the lower incomplete gamma, which agrees with the first moment, and

$$\lim_{t\to\infty} m(t;\Theta) = \infty. \quad (26)$$

On the other hand, as in [13], Equation (23) can be rewritten as

$$m(t;\Theta) = \int_t^\infty e^{-\int_t^{t+x} h(t;\Theta)\,dt}\,dx \quad (27)$$

where $h(t;\Theta)$ is given by Equation (5). From Equation (27), it is clear that $m(t;\Theta)$ is decreasing at first and then starts increasing.

4.4 Behavior of mean waiting time

The MRL time has a mirror image, called MWT. The MWT of a positive continuous random variable T is defined as

$$\bar{m}(t;\Theta) = E[\bar{\Omega}_t;\Theta] = t - \frac{1}{F(t;\Theta)}\int_0^t x\ f(x;\Theta)\,dx. \quad (28)$$

Theorem 4.3.

Using the Equations (2), (3) and (28), the explicit forms for MWT of IGLED are given by:

$$\bar{m}(t;\Theta)=\begin{cases} t-\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big[\Gamma\Big((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)+\frac{1}{2}\sqrt{2b} \\ \Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^i\Big(\Gamma\Big((\frac{2\xi-2i-1}{2\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)\Big], \\ \qquad (\frac{c^2}{2b})^{\xi}<(\frac{c}{t}+\frac{b}{2t^2})^{\xi}; \\ \\ t-\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big[\Gamma\Big((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)+\frac{c}{2}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^i \\ \Big(\Gamma\Big((\frac{\xi+i-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)-\Gamma\Big((\frac{\xi+i-1}{\xi}),(\frac{c^2}{2b})^{\xi}\Big)\Big) \\ +\frac{1}{2}(2b)^{\frac{1}{2}}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^i\Gamma\Big((\frac{2\xi-2i-1}{2\xi}),(\frac{c^2}{2b})^{\xi}\Big)\Big], \\ \qquad (\frac{c^2}{2b})^{\xi}>(\frac{c}{t}+\frac{b}{2t^2})^{\xi}. \end{cases} \tag{29}$$

Proof.
To derive the explicit forms of the MWT of IGLED, the integral $\int_0^t x\, f(x;\Theta)\, dx$ must be calculated (see Appendix).

By Theorem (5) of [14], we can say that $\bar{m}(t;\Theta)$ is monotone increasing because $F(t;\Theta)$ is log-concave.

4.5 Behavior of the variance of residual life

In this subsection, the variance of r.v. Ω_t and its monotonic properties are studied.

Theorem 4.4.

Let T be a positive continuous r.v., then the explicit forms for VRL of IGLED are given by:

$$Var(\Omega_t;\Theta)=\begin{cases} \Big(\frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big)\Big[\frac{c^2}{2}\Big(\Gamma(\frac{\xi-2}{\xi})-\Gamma\Big(\frac{\xi-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big) \\ +\frac{b}{2}\Big(\Gamma(\frac{\xi-1}{\xi})-\Gamma\Big(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)+\frac{c^2}{2}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^i \\ \Big(\Gamma(\frac{\xi+i-2}{\xi})-\Gamma\Big(\frac{\xi+i-2}{\xi},(\frac{c^2}{2b})^{\xi}\Big)\Big)+\frac{c}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^i \\ \Big(\Gamma\Big(\frac{2\xi-2i-3}{2\xi},(\frac{c^2}{2b})^{\xi}\Big)-\Gamma\Big(\frac{2\xi-2i-3}{2\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)\Big] \\ -\Big[\Big(\frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big)\Big(\frac{c}{2}\Big(\Gamma(\frac{\xi-1}{\xi})-\Gamma(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi})\Big) \\ +\frac{c}{2}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^i\Big(\Gamma(\frac{\xi+i-1}{\xi})-\Gamma(\frac{\xi+i-1}{\xi},(\frac{c^2}{2b})^{\xi})\Big)\Big) \\ +\frac{1}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^i\Big(\Gamma\Big(\frac{2\xi-2i-1}{2\xi},(\frac{c^2}{2b})^{\xi}\Big)- \\ \Gamma\Big(\frac{2\xi-2i-1}{2\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)\Big]^2, \qquad (\frac{c^2}{2b})^{\xi}<(\frac{c}{t}+\frac{b}{2t^2})^{\xi}; \\ \\ \Big(\frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big)\Big[\frac{c^2}{2}\Big(\Gamma(\frac{\xi-2}{\xi})-\Gamma\Big(\frac{\xi-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)+ \\ \frac{b}{2}\Big(\Gamma(\frac{\xi-1}{\xi})-\Gamma\Big(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)+ \\ \frac{c^2}{2}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^i\Big(\Gamma(\frac{\xi+i-2}{\xi})-\Gamma\Big(\frac{\xi+i-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\Big)\Big)\Big] \\ -\Big[\Big(\frac{1}{1-e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\Big)\Big(\frac{c}{2}\Big(\Gamma(\frac{\xi-1}{\xi})-\Gamma(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi})\Big)+ \\ \frac{c}{2}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^i\Big(\Gamma(\frac{\xi+i-1}{\xi})-\Gamma(\frac{\xi+i-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi})\Big)\Big)\Big]^2, \\ \qquad (\frac{c^2}{2b})^{\xi}>(\frac{c}{t}+\frac{b}{2t^2})^{\xi}. \end{cases} \tag{30}$$

Proof.
The VRL can be defined as

$$Var(\Omega_t;\Theta)=E(T^2|T\geq t)-[E(T|T\geq t)]^2=$$
$$\int_t^{\infty}x^2\,\frac{f(x;\Theta)}{S(t;\Theta)}\,dx-\left(\int_t^{\infty}x\,\frac{f(x;\Theta)}{S(t;\Theta)}\,dx\right)^2. \tag{31}$$

To derive the explicit forms for the VRL of IGLED, the following integrals $\int_t^{\infty}x\, f(x;\Theta)\, dx$ and $\int_t^{\infty}x^2\, f(x;\Theta)\, dx$ must be calculated (see Appendix).

To study the behavior of VRL for IGLED, it is important to study the following relations:

$$Var(\Omega_t;\Theta)-m(t;\Theta)^2=\frac{2}{S(t;\Theta)}$$
$$\int_t^{\infty}S(x;\Theta)\,[m(x;\Theta)-m(t;\Theta)]\,dx \tag{32}$$

[see [1]], and

$$\frac{\partial}{\partial t}Var(\Omega_t;\Theta)=h(t;\Theta)\,m(t;\Theta)^2\,[\frac{Var(\Omega_t;\Theta)}{m(t;\Theta)^2}-1] \tag{33}$$

[see [2]]. It is clear from Equation (33) that $Var(\Omega_t;\Theta)$ is increasing if $Var(\Omega_t;\Theta)>m(t;\Theta)^2$; moreover, from Equation (32) $Var(\Omega_t;\Theta)>m(t;\Theta)^2$ if and only if $m(t,\Theta)$ is increasing $(x>t)$. On the other hand, it is clear from Equation (33) that $Var(\Omega_t;\Theta)$ is decreasing if $Var(\Omega_t;\Theta)<m(t;\Theta)^2$; moreover, from Equation (32)

$Var(\Omega_t;\Theta) < m(t;\Theta)^2$ if and only if $m(t;\Theta)$ is decreasing $(x < t)$. Then, it easy to show that the VRL is a bathtub for IGLED given that the MRL for IGLED is bathtub.

4.6 Behavior of the variance of reversed residual life

In this subsection, the variance of r.v. $\bar{\Omega}_t$ and its monotonic properties are studied.

Theorem 4.5.

Let T be a positive continuous r.v., then the explicit forms for VRRL of IGLED are given by:

$$Var(\bar{\Omega}_t;\Theta) = \begin{cases} \left(\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\right)\Big[\frac{c^2}{2}\left(\Gamma\left(\frac{\xi-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)+ \\ \frac{b}{2}\left(\Gamma\left(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)+\frac{c}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i} \\ \left(\Gamma\left(\frac{2\xi-2i-3}{2\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)\Big]- \\ \Big[\left(\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\right)(\frac{c}{2})\Gamma\left((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)+ \\ \frac{1}{2}\sqrt{2b}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i}\left(\Gamma\left(\frac{2\xi-2i-1}{2\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)\Big]^2, \\ \qquad (\frac{c^2}{2b})^{\xi} < (\frac{c}{t}+\frac{b}{2t^2})^{\xi}; \\ \\ \left(\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\right)\Big[\frac{c^2}{2}\left(\Gamma\left(\frac{\xi-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)+ \\ \frac{b}{2}\left(\Gamma\left(\frac{\xi-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)\right)+\frac{c^2}{2}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^{i} \\ \left(\Gamma\left(\frac{\xi+i-2}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)-\Gamma\left(\frac{\xi+i-2}{\xi},(\frac{c^2}{2b})^{\xi}\right)\right)+ \\ \frac{c}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i}\left(\Gamma\left(\frac{2\xi-2i-3}{2\xi},(\frac{c^2}{2b})^{\xi}\right)\right)\Big]- \\ \Big[\left(\frac{1}{e^{-(\frac{c}{t}+\frac{b}{2t^2})^{\xi}}}\right)(\frac{c}{2})\Gamma\left((\frac{\xi-1}{\xi}),(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)+\frac{c}{2}\Sigma_{i=0}^{\infty} \\ \binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^{i}\left(\Gamma\left(\frac{\xi+i-1}{\xi},(\frac{c}{t}+\frac{b}{2t^2})^{\xi}\right)-\Gamma\left(\frac{\xi+i-1}{\xi},(\frac{c^2}{2b})^{\xi}\right)\right) \\ +\frac{1}{2}\sqrt{2b}\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i}\left(\Gamma\left(\frac{2\xi-2i-1}{2\xi},(\frac{c^2}{2b})^{\xi}\right)\right)\Big]^2, \\ \qquad (\frac{c^2}{2b})^{\xi} > (\frac{c}{t}+\frac{b}{2t^2})^{\xi}. \end{cases} \quad (34)$$

Proof.
The VRRL can be defined as

$$Var(\bar{\Omega}_t;\Theta) = E(T^2|T<t) - [E(T|T<t)]^2 = \int_0^t x^2 \frac{f(x;\Theta)}{F(t;\Theta)}\,dx - \left(\int_0^t x\,\frac{f(x;\Theta)}{F(t;\Theta)}\,dx\right)^2. \quad (35)$$

To derive the explicit forms for the VRRL of IGLED, the following integrals $\int_0^t xf(x;\Theta)\,dx$ and $\int_0^t x^2 f(x;\Theta)\,dx$, must be calculated (see Appendix).

In order to show the behavior of VRRL, one can study the following relations:

$$Var(\bar{\Omega}_t;\Theta) - \bar{m}(t;\Theta)^2 = \frac{2}{F(t;\Theta)} \int_0^t F(x;\Theta)\,[\bar{m}(x;\Theta) - \bar{m}(t;\Theta)]dx \quad (36)$$

and

$$\frac{\partial}{\partial t}Var(\bar{\Omega}_t;\Theta) = r(t;\Theta)\,\bar{m}(t;\Theta)^2\,[1 - \frac{Var(\bar{\Omega}_t;\Theta)}{(\bar{m}(t;\Theta))^2}] \quad (37)$$

[see [3]]. From Equation (37), it is clear that the $Var(\bar{\Omega}_t;\Theta)$ is increasing if $Var(\bar{\Omega}_t;\Theta) < \bar{m}(t;\Theta)^2$; moreover, from Equation (36) $Var(\bar{\Omega}_t;\Theta) < \bar{m}(t;\Theta)^2$ if and only if $\bar{m}(t,\Theta)$ is increasing $(t > x)$. Then, it easy to show that the VRRL is increasing for IGLED given that the MWT for IGLED is increasing.

5 Measures of Income Inequality using IGLED

Bonferroni, Lorentz and the scaled TTT plot curves are widely used tools for analyzing and visualizing income inequality. The B and (B_c) have assumed relief not only in economics to study income and poverty, but also in other fields like reliability and medicine. Besides, the (B_c) uses to derive the Lorentz curve. The measures of income inequality like the (B_c), the B, the Lorentz curve and the Gini coefficient are studied using IGLED. Also, the scaled TTT transform curve is introduced to show the behavior of failure rate function of IGLED.

5.1 Lorentz curve

The Lorentz curve was presented first by [15] as a graphical representation of income distribution (for more details, see [16]). The Lorentz curve can be written as

$$L(q) = \frac{1}{\mu^{(1)}} \int_0^q x_q\,dq, \quad (38)$$

where x_q is the quantile of IGLED given by (14), $0 < q < 1$ and $\mu^{(1)}$ is the first moment given by (25).

Then, the Lorentz curve of IGLED can be presented in explicit form as,

$$L(q) = \begin{cases} \frac{1}{\mu^{(1)}}\Big[\frac{c}{2}\Gamma\left(\frac{\xi-1}{\xi},-\log q\right)+\frac{c}{2}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{2b}{c^2})^{i}\Big(\Gamma\left(\frac{i+\xi-1}{\xi},-\log q\right)- \\ \Gamma\left(\frac{i+\xi-1}{\xi},(\frac{c^2}{2b})^{\xi}\right)\Big)+\frac{1}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i}\,\Gamma\left(\frac{2\xi-2i-1}{2\xi},(\frac{c^2}{2b})^{\xi}\right)\Big], \\ \qquad e^{-(\frac{c^2}{2b})^{\xi}} < e^{-(\frac{c}{x_q}+\frac{b}{2x_q^2})^{\xi}}; \\ \\ \frac{1}{\mu^{(1)}}\Big[\frac{c}{2}\Gamma\left(\frac{\xi-1}{\xi},-\log q\right)+\frac{1}{2}\sqrt{2b}\,\Sigma_{i=0}^{\infty}\binom{\frac{1}{2}}{i}(\frac{c^2}{2b})^{i}\,\Gamma\left(\frac{2\xi-2i-1}{2\xi},-\log q\right)\Big], \\ \qquad e^{-(\frac{c^2}{2b})^{\xi}} > e^{-(\frac{c}{x_q}+\frac{b}{2x_q^2})^{\xi}}. \end{cases} \quad (39)$$

where $\xi > 1$.

Then the Gini coefficient for IGLED can be given as

$$G(L(q)) = 1 - \frac{2}{\mu^{(1)}} \left[\frac{-b}{c} \, (\frac{c^2}{2\,b})^{\xi} \left(E_{\frac{1}{\xi}} \left(2\,(\frac{c^2}{2\,b})^{\xi} \right) - e^{-(\frac{c^2}{2\,b})^{\xi}} E_{\frac{1}{\xi}} \left((\frac{c^2}{2\,b})^{\xi} \right) \right) - \frac{1}{2} \sqrt{2\,b} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\frac{c^2}{2\,b})^{\frac{2\,\xi-1}{2}} \left(E_{\frac{1+2\,i}{\xi}} \left(2\,(\frac{c^2}{2\,b})^{\xi} \right) - e^{-(\frac{c^2}{2\,b})^{\xi}} E_{\frac{1+2\,i}{\xi}} \left((\frac{c^2}{2\,b})^{\xi} \right) \right) + \frac{c}{4} \left((2-2^{\frac{1}{\xi}})\,\Gamma(\frac{\xi-1}{\xi}) + 2^{\frac{1}{\xi}}\,\Gamma\Big(\frac{\xi-1}{\xi}, 2\,(\frac{c^2}{2\,b})^{\xi}\Big) - 2\,e^{-(\frac{c^2}{2\,b})^{\xi}}\,\Gamma\Big(\frac{\xi-1}{\xi}, (\frac{c^2}{2\,b})^{\xi}\Big) \right) + \frac{c}{2} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\frac{2\,b}{c^2})^{i} \left(\Gamma(\frac{i+\xi-1}{\xi}) - \Gamma\Big((\frac{i+\xi-1}{\xi}), (\frac{c^2}{2\,b})^{\xi} \Big) + 2^{\frac{1-\xi-i}{\xi}} \left(\Gamma\Big((\frac{i+\xi-1}{\xi}), 2\,(\frac{c^2}{2\,b})^{\xi} \Big) - \Gamma(\frac{i+\xi-1}{\xi}) \right) \right) + \frac{1}{2} \sqrt{2\,b} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\frac{c^2}{2\,b})^{i} \, (1-e^{-(\frac{c^2}{2\,b})^{\xi}})\,\Gamma\Big(\frac{2\,\xi-2\,i-1}{2\,\xi}, (\frac{c^2}{2\,b})^{\xi} \Big) \right],$$

where $E_{.}(.)$ is an exponential integral function.

Remark 6.

1. One can show that

$$\lim_{\xi\to\infty} G(L(q)) = 0,$$

and

$$\lim_{\xi\to 1} G(L(q)) = 1.$$

5.2 Bonferroni curve

The B has appropriate properties, see [17]. Now, the (B_c) and B are used to analyze IGLED.

For the IGLED, the B_c can be presented as

$$B_c(q) = \frac{L(q)}{q} \tag{40}$$

Then the B can be written as

$$B = 1 - \int_0^1 B_c(q)\,dq = 1 - \frac{1}{\mu^{(1)}} \left[\frac{c}{2\,(\xi-1)} \, (\frac{c^2}{2\,b})^{\xi} \left(\xi\, e^{-(\frac{c^2}{2\,b})^{\xi}} + (\xi\,(\frac{c^2}{2\,b})^{\xi} + \xi - 1)\, E_{\frac{1-\xi}{\xi}} \left((\frac{c^2}{2\,b})^{\xi} \right) \right) + \frac{1}{2} \sqrt{2\,b} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \frac{(\frac{c^2}{2\,b})^{\frac{4\,\xi-1}{2}}}{1+2\,i-2\,\xi} \left((2\,\xi\,(\frac{c^2}{2\,b})^{\xi} + 2\,i+1-2\,\xi)\, E_{\frac{2\,i+1-2\,\xi}{2\,\xi}} \left((\frac{c^2}{2\,b})^{\xi} \right) - 2\,\xi\, e^{-(\frac{c^2}{2\,b})^{\xi}} \right) + \frac{c}{2} \left(\gamma\Big(\frac{2\,\xi-1}{\xi}, (\frac{c^2}{2\,b})^{\xi} \Big) + (\frac{c^2}{2\,b})^{\xi}\, \Gamma\Big(\frac{\xi-1}{\xi}, (\frac{c^2}{2\,b})^{\xi} \Big) \right) + \frac{c}{2} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\frac{2\,b}{c^2})^{i} \left(\gamma\Big((\frac{i+\xi-1}{\xi}), (\frac{c^2}{2\,b})^{\xi} \Big) \right) + \frac{1}{2} \sqrt{2\,b} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\frac{c^2}{2\,b})^{i+\xi}\, \Gamma\Big((\frac{2\,\xi-2\,i-1}{2\,\xi}), (\frac{c^2}{2\,b})^{\xi} \Big) \right].$$

Remark 7.

1. It is easy to show that

$$\lim_{\xi\to\infty} B = 0,$$

and

$$\lim_{\xi\to 1} B = 1.$$

5.3 Scaled total time on test transform curve

A graphical method using the scaled TTT transform curve was first proposed by [18]. The scaled TTT transform curve of IGLED can be written as

$$\phi_q = L(q) + \frac{(1-q)\,x_q}{\mu^{(1)}} \tag{41}$$

Remark 8.

It is clear from Equations (39), (40) and (41) that $\phi_q > B_c > L(q)$. This result agrees with the result in [19].

5.4 Interpretation of IGLED parameters

The shape parameter ξ is said to be index tail since it satisfies the heavy-tailed property for the IGLED. For $x > 0$, one can show that

$$\lim_{t\to\infty} \frac{1-F(tx;\Theta)}{1-F(t;\Theta)} = x^{-\xi}.$$

Furthermore, it is ease to show that the index tail satisfies

$$\lim_{t\to\infty}\frac{d\log S(t;\Theta)}{d\log x}=-\xi$$

The specification of the other two parameters c and b can be studied using the Gini coefficient and the elasticity function. For more details, see [20]. The elasticity function of the quantile function x_q with respect to the shape parameter c is given by

$$\varepsilon_c(x_q)=\frac{c}{x_q}\,\frac{\partial x_q}{\partial c}=\frac{c}{\sqrt{c^2+2\,b\,(-\log q)^{\frac{1}{\xi}}}}.$$

It is noticed that the elasticity function is an increasing function in q, since

$$\frac{\partial\varepsilon_c(x_q)}{\partial q}=\frac{c\,b\,(-\log q)^{\frac{1}{\xi}-1}}{\xi\,q\left(c^2+2\,b\,(-\log q)^{\frac{1}{\xi}}\right)^{\frac{3}{2}}},$$

is always positive.

The elasticity function of the quantile function x_q with respect to the shape parameter b is given by

$$\varepsilon_b(x_q)=\frac{b}{x_q}\,\frac{\partial x_q}{\partial b}=\frac{1}{2}-\frac{c}{2\sqrt{c^2+2\,b\,(-\log q)^{\frac{1}{\xi}}}}.$$

It is noticed that the elasticity function is an decreasing function in q, since

$$\frac{\partial\varepsilon_b(x_q)}{\partial q}=-\frac{c\,b\,(-\log q)^{\frac{1}{\xi}-1}}{2\,\xi\,q\left(c^2+2\,b\,(-\log q)^{\frac{1}{\xi}}\right)^{\frac{3}{2}}},$$

is always negative.

6 Maximum Likelihood Estimation

MLE is probably the most widely used method of estimation in statistics. Suppose that $X_1,\ldots,X_r$ be independent random sample of size r from IGLED. From 2, the log-likelihood function can be obtained as

$$\ell(\Theta)=r\log\,\xi-\sum_{i=1}^{r}(\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi}+\sum_{i=1}^{r}\log\left((\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi-1}\right)+\sum_{i=1}^{r}\log\left((\frac{c}{x_i^2}+\frac{b}{x_i^3})\right). \tag{42}$$

By taking the first derivative ($\ell_\Theta(\Theta)=\frac{\partial\ell}{\partial\Theta}$) of (42) with respect to c, b and ξ, we get

$$\ell_c(\Theta)=\sum_{i=1}^{r}\frac{1}{x_i^2(\frac{c}{x_i^2}+\frac{b}{x_i^3})}+(\xi-1)\sum_{i=1}^{r}\frac{1}{x_i(\frac{c}{x_i}+\frac{b}{2x_i^2})}-\sum_{i=1}^{r}\frac{\xi(\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi-1}}{x_i}, \tag{43}$$

$$\ell_b(\Theta)=\sum_{i=1}^{r}\frac{1}{x_i^3(\frac{c}{x_i^2}+\frac{b}{x_i^3})}+(\xi-1)\sum_{i=1}^{r}\frac{1}{2x_i^2(\frac{c}{x_i}+\frac{b}{2x_i^2})}-\sum_{i=1}^{r}\frac{\xi(\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi-1}}{2x_i^2}, \tag{44}$$

and

$$\ell_\xi(\Theta)=\frac{r}{\xi}+\sum_{i=1}^{r}\log\left(\frac{c}{x_i}+\frac{b}{2x_i^2}\right)-\sum_{i=1}^{r}\log\left(\frac{c}{x_i}+\frac{b}{2x_i^2}\right)(\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi}. \tag{45}$$

6.1 The parameters c and b are known

The normal equation $\ell_\xi(\Theta)=0$ can be written as

$$\frac{1}{\xi}=\frac{1}{r}\left(\sum_{i=1}^{r}\log\left(\frac{c}{x_i}+\frac{b}{2x_i^2}\right)\left((\frac{c}{x_i}+\frac{b}{2x_i^2})^{\xi}-1\right)\right) \tag{46}$$

It is clear that the first derivative of the right-side hand ($\Psi(\xi;x)$) of (46) with respect to ξ is always positive. This mean that the $\Psi(\xi;x)$ is increasing function. Then by graphical method [21] the MLE of ξ exists and unique see Figures (5a, 8a, 11a and 14a).

6.2 The parameters c, b and ξ are unknown

The MLE $\hat{\Theta}$ of Θ is given by solving the three normal equations $\ell_c(\Theta)=0$, $\ell_b(\Theta)=0$ and $\ell_\xi(\Theta)=0$. These nonlinear equations can not be solved analytically and a numerical method (Newton-Raphson method) can be used.

6.3 Fisher information matrix

Since the computation of Fisher information matrix (given by taking the expectation of the second derivative of (42)) is very difficult, so, it seems appropriate to approximate these expected values by their MLEs. Then, the asymptotic variance-covariance matrix is given as [see, [22]];

$$\begin{pmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{b}) & Cov(\hat{c},\hat{\xi})\\ Cov(\hat{b},\hat{c}) & Var(\hat{b}) & Cov(\hat{b},\hat{\xi})\\ Cov(\hat{\xi},\hat{c}) & Cov(\hat{\xi},\hat{b}) & Var(\hat{\xi})\end{pmatrix}=\begin{pmatrix}-\ell_{cc}(\Theta) & -\ell_{cb}(\Theta) & -\ell_{c\xi}(\Theta)\\ -\ell_{bc}(\Theta) & -\ell_{bb}(\Theta) & -\ell_{b\xi}(\Theta)\\ -\ell_{\xi c}(\Theta) & -\ell_{\xi b}(\Theta) & -\ell_{\xi\xi}(\Theta)\end{pmatrix}^{-1}_{(\hat{c},\hat{b},\hat{\xi})}, \tag{47}$$

where $\ell_{\Theta_i\Theta_j}(\Theta)=\frac{\partial^2\ell}{\partial\Theta_i\Theta_j}$, $i,j=1,2,3$. Accordingly, the ACIs based on the asymptotic variance-covariance matrix for the parameters c, b and ξ are, respectively given as:

$\hat{c} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{c})}$, $\hat{b} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{b})}$ and $\hat{\xi} \pm z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\xi})}$, where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

7 Real Data Analysis

In this section, four real data sets are presented for interpretative study. For every data set, we compare IGLED with its sub-models (IWD, IRD and IED) and with the generalized inverse Weibull (GIW) distribution given in [9], Log-normal distribution (Log-N) and inverse Gaussian distribution (IGD). For identifying the shapes of hazard rate for given data sets, the scaled TTT transform plot is given as

$$\phi_n(\frac{r}{n}) = \frac{\sum_{i=1}^{n} X_{i:n} + (n-r)\, X_{r:n}}{\sum_{i=1}^{n} X_i},$$

where r=1,...,n and $X_{i:n}$ is the order statistics of the data. Kolmogorov-Smirnov (K-S) distance test, Anderson Darling (A^*) test and *Cramér* Von-Mises (W^*) test are used for non-parametric test statistic. All computations are introduced by *Mathematica*11.
The pdf of the Log-N distribution is

$$f(x;c,b) = \frac{1}{\sqrt{2\,\pi}\,x\,b}\, e^{-\frac{(-c+\log x)^2}{2\,b^2}}, \quad x>0,$$

and the pdf of the IGD is

$$f(x;c,b) = \frac{\sqrt{b}}{\sqrt{2\,\pi\,x^3}}\, e^{-\frac{b\,(x-c)^2}{2\,c^2\,x}}, \quad x>0.$$

7.1 The intervals between successive failures data

Consider the following data set from [23] consisting of 15 observations of records kept for the time of successive failures of the air conditioning system of Boeing 720 airplane number 7910. The data are 502, 386, 326, 153, 74, 70, 59, 57, 48, 29, 29, 27, 26, 21, 12. The mean, the variance, standard deviation, the skewness and the kurtosis are 121.267, 23798.8, 154.269, 1.52307 and 3.82465 respectively. The measure of skewness indicated that the data are positively skewed. Furthermore, the TTT plot of the observed data show that the hazard rate of the intervals between successive failures data is unimodal which is first concave and then convex as shown in Fig. (4b).

From Table 1, based on the p-value associated with the k-s distance value, one can show that

1. The IRD must be rejected at $\alpha \geq 0.18$.
2. The IGLED, IWD, GIW, IED, IGD and log-normal distribution must not be reject at any considerable α.
3. The IGLED fits data better than another distributions because it has the highest p-value.

Furthermore, the IGLED is the best distribution fits the data based on (W^*) and (A^*).

7.2 Burning velocity data

In this subsection, the burning velocity of different chemical materials which used in [24] is analyzed. The burning velocity is the velocity of a laminar flame under stated conditions of composition, temperature, and pressure. A reference value of 46 cm/sec for the fundamental burning velocity of propane has been used. The data set are 68, 61, 64, 55, 51, 68, 44, 82, 60, 89, 61, 54, 166, 66, 50, 87, 48, 42, 58, 46, 67, 46, 46, 44, 48, 56, 47, 54, 47, 80, 38, 108, 46, 40, 44, 312, 41, 31, 40, 41, 40, 56, 45, 43, 46, 46, 46, 46, 52, 58, 82, 71, 48, 39, 41. The mean, the variance, standard deviation, the skewness and the kurtosis are 0.61, 0.614174, 0.405184, 4.76642 and 28.6962 respectively. The measure of skewness indicated that the data are positively skewed. Furthermore, the TTT plot of the observed data show that the hazard rate of the burning velocity data is unimodal which is first concave and then convex as shown in Fig. (7b).

From Table 2, based on the p-value associated with the k-s distance value, one can show that

1. The IRD and IED must be rejected at $\alpha \geq 0.001$.
2. The IGD must be rejected at $\alpha \geq 0.09$.
3. The log-normal distribution must be rejected at $\alpha \geq 0.13$.
4. The IGLED, IWD and GIW must not be reject at any considerable α.
5. The IGLED fits data better than another distributions because it has the highest p-value.

Also, the IGLED is the best distribution fits the data based on (W^*) and (A^*).

7.3 Fatigue lives data

[25] gave the data below which gives the fatigue lives in (hours) for 10 bearings tested in each of two testers. Here, the failures time for tester II is presented 152.7 , 172.0 , 172.5 , 173.3 , 193.0 , 204.7 , 216.5 , 234.9 , 262.6 , 422.6. The mean, the variance, standard deviation, the skewness and the kurtosis are 220.48, 6147.44, 78.4056, 1.86358 and 5.58507 respectively. The measure of skewness indicated that the data are positively skewed. Furthermore, the TTT plot of the observed data is presented in Fig. (10b). For computational ease, we consider the failure times in (days).

From Table 3, based on the p-value associated with the k-s distance value, one can show that

1. The IED must be rejected at $\alpha \geq 0.04$.
2. The IGLED, IWD, GIW, and IRD must not be reject at any considerable α.
3. The IGLED fits data better than another distributions because it has the highest p-value.

Clearly, the IGLED is the best distribution fits the data based on (W^*) and (A^*).

Table 1: The MLEs of unknown parameters, the K-S test with the corresponding P-value, the W^* test with the corresponding P-value and A^* test with the corresponding P-value for different models using the intervals between successive failures data

Model	MLEs	K-S test	p-value	(W^*)	p-value	(A^*)	p-value
IGLED	c=34.009, b=253.128, ξ=1.05339	0.139	0.934	0.0403	0.9314	0.286	0.948
IWD	c=38.357, ξ=1.146	0.1479	0.898	0.044	0.913	0.307	0.933
GIW(c,a,ξ)	c=3.765, a=14.296, ξ=1.1459	0.1479	0.898	0.044	0.913	0.307	0.933
IRD	b=1884.45	0.282	0.185	0.3001	0.135	2.925	0.0299
IED	c=40.2438	0.153	0.875	0.0537	0.8535	0.322	0.921
Log-Normal(c,b)	c=4.156, b=1.09139	0.1797	0.7179	0.0893	0.64	0.5582	0.6884
IGD(c,b)	c=121.267, b=60.233	0.1733	0.7585	0.076	0.716	0.4424	0.806

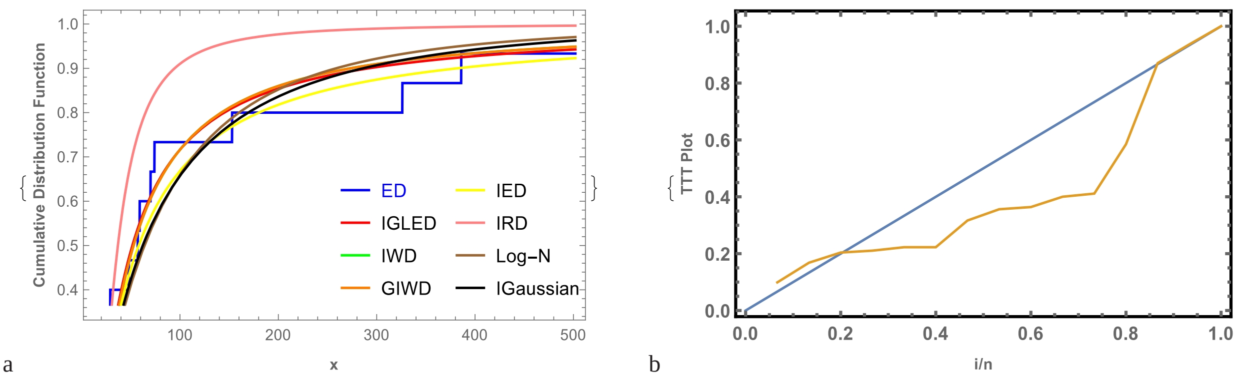

Fig. 4: a) Empirical distribution functions versus distribution functions of modeling distributions based on the intervals between successive failures data b) Scaled TTT transform of the intervals between successive failures data.

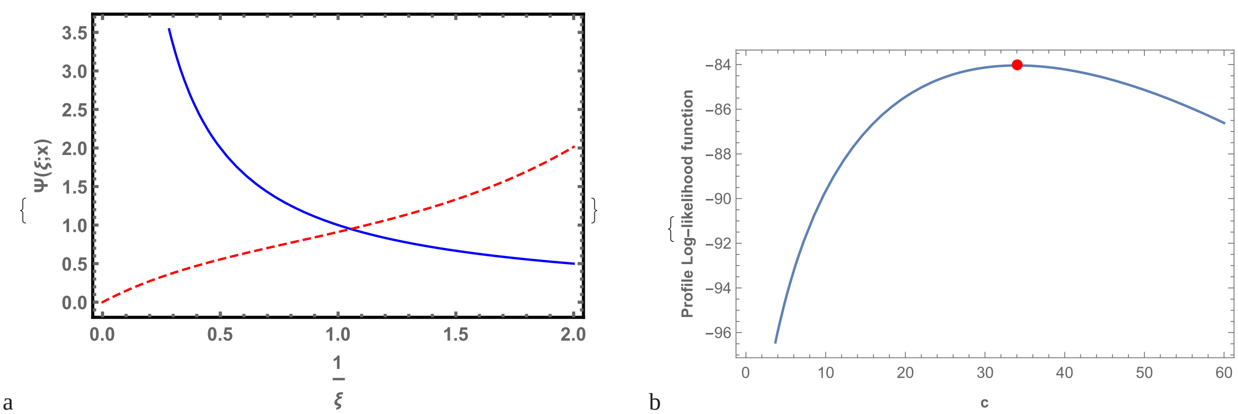

Fig. 5: (a) Plot of the $\frac{1}{\xi}$ and $\Psi(\xi;x)$ functions for the intervals between successive failures data. (b) The profile log-likelihood of the parameter c for the intervals between successive failures data

7.4 Annual wage data

The annual wage data (in multiple of 100 US dollars) from [26] which gave a random sample of 30 production-line workers under age 40 in a States industrial firm. [27] used this data for computing the Bayesian estimation of the survival function of Pareto distribution of the second kind. The data set are 101, 103, 103, 104, 104, 105, 106, 107, 108, 111, 112, 112, 112, 115, 115, 116, 119, 119, 119, 123, 125, 128, 132, 140, 151, 154, 156, 157, 158, 198. The mean, the variance, standard deviation, the skewness and the kurtosis are 123.767, 519.082, 22.7834, 1.47393 and 4.8866 respectively. The measure of skewness indicated that the data are positively skewed. Furthermore, the TTT plot of the observed data is given in Fig. (13b).

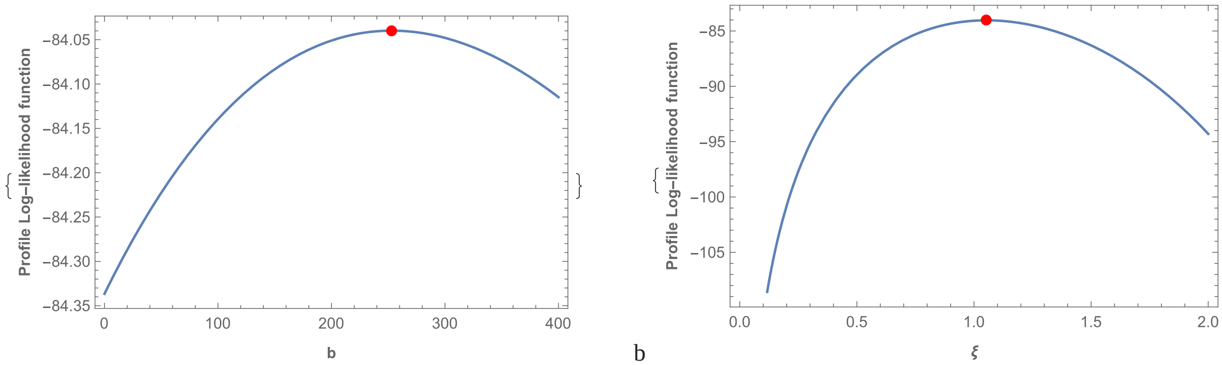

Fig. 6: (a) The profile log-likelihood of the parameter b for the intervals between successive failures data (b) The profile log-likelihood of the parameter ξ for the intervals between successive failures data

Table 2: The MLEs of unknown parameters, the K-S test with the corresponding P-value, the W^* test with the corresponding P-value and A^* test with the corresponding P-value for different models using the burning velocity data

Model	MLEs	K-S test	p-value	(W^*)	p-value	(A^*)	p-value
IGLED	c=0.215, b=0.2464, ξ=2.7587	0.1237	0.3692	0.09881	0.591	0.6141	0.63477
IWD	c=0.4772, ξ=4.1741	0.1322	0.292	0.1183	0.5024	0.7298	0.5345
GIW(c,a,ξ)	c=0.596 , a=0.396, ξ=4.1741	0.1322	0.292	0.1183	0.5024	0.7298	0.5345
IRD	b=0.51	0.26	0.0012	1.2754	0.00056	6.454	0.00059
IED	c=0.524	0.3904	1×10^{-7}	2.764	2.52×10^{-7}	13.36	4.16×10^{-7}
Log-Normal(c,b)	c=-0.591466, b=0.374694	0.157	0.132	0.462	0.0498	2.745	0.0369
IGD(c,b)	c=0.61, b=3.7113	0.1676	0.091	0.589	0.0238	3.321	0.01885

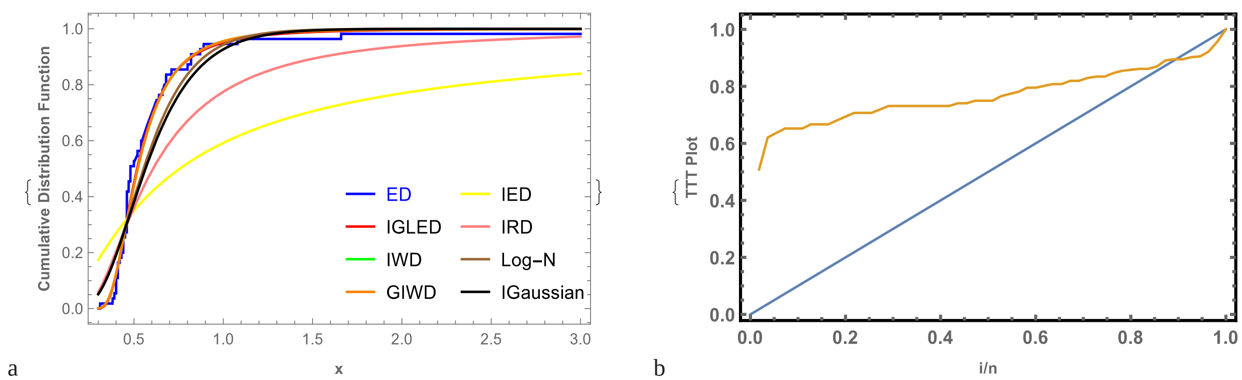

Fig. 7: a) Empirical distribution functions versus distribution functions of modeling distributions based on the burning velocity of different chemical materials data b) Scaled TTT transform of the burning velocity of different chemical materials data.

From Table 4, based on the p-value associated with the k-s distance value, one can show that

1. The IRD and IED must be rejected at $\alpha \geq 0.001$.
2. The IGD and log-normal distribution must be rejected at $\alpha \geq 0.21$.
3. The IGLED and IWD must not be reject at any considerable α.
4. The IGLED fits data better than another distributions because it has the highest p-value.

Furthermore, the IGLED is the best distribution fits the data based on (W^*) and (A^*).

8 Conclusion

This paper deals with a new lifetime distribution known as IGLED. The unimodality property is studied for the pdf and HR function of IGLED. From Section (6), one can show that the IGLED is very good model for the

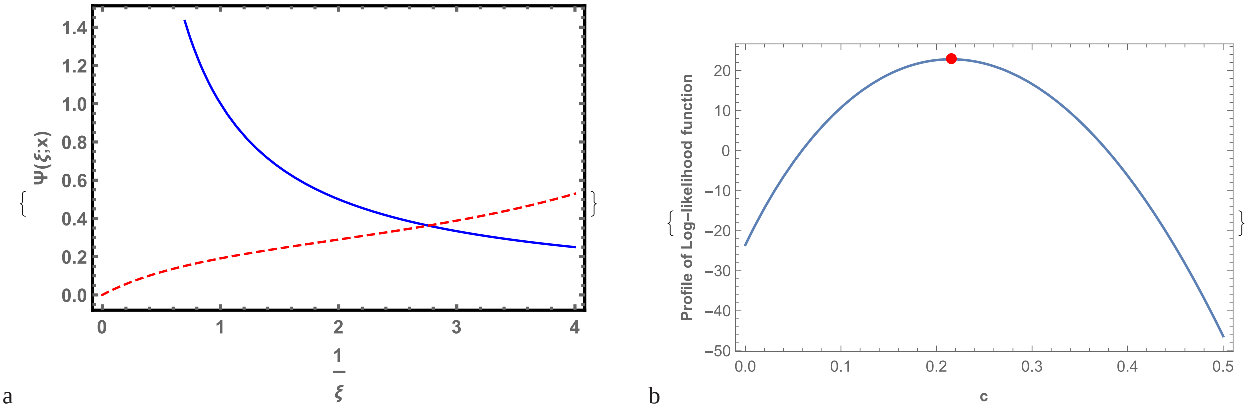

Fig. 8: (a) Plot of the $\frac{1}{\xi}$ and $\Psi(\xi;x)$ functions for the burning velocity of different chemical materials data. (b) The profile log-likelihood of the parameter c for the burning velocity of different chemical materials data

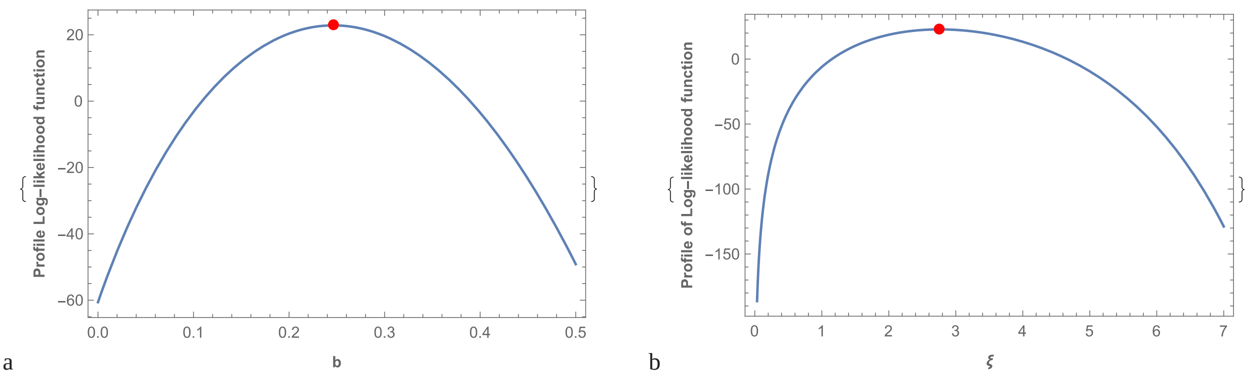

Fig. 9: (a) The profile log-likelihood of the parameter b for the burning velocity of different chemical materials data (b) The profile log-likelihood of the parameter ξ for the burning velocity of different chemical materials data

Table 3: MLEs of unknown parameters, the K-S test with the corresponding P-value, the W^* test with the corresponding P-value and A^* test with the corresponding P-value for different models using the fatigue lives data

Model	MLEs	K-S test	p-value	(W^*)	p-value	(A^*)	p-value
IGLED	c=4.435, b=52.017, ξ=3.785	0.177	0.9134	0.031	0.9731	0.2399	0.9755
IWD	c=7.804, ξ=5.294	0.179	0.9062	0.0324	0.9676	0.2588	0.9655
GIW(c,a,ξ)	c=4.3208, a=22.879, ξ=5.294	0.179	0.9062	0.0324	0.9676	0.2588	0.9655
IRD	b=136.776	0.335	0.211	0.3009	0.1344	1.528	0.17
IED	c=8.495	0.4399	0.0417	0.5654	0.02727	2.726	0.0378

Lorentz curve and B_c. This property make the new model has important role in income inequality. Figures (16a,b) and (17a,b) show the contours of the log-likelihood for the various data and the red points indicate the values of the MLEs of the parameters. The applications of the IGLED to real data sets are given to show that it may engage wider in reliability analysis, engineering chemistry and economic.

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Appendix.

The following integrals must be calculated for constructing the explicit forms of MRL time, MWT, VRL time and VRRL time.

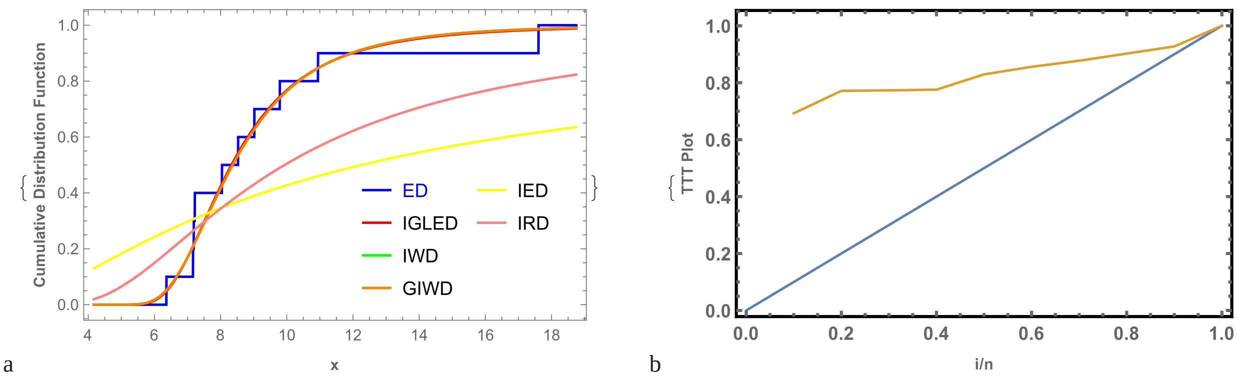


Fig. 10: a) Empirical distribution functions versus distribution functions of modeling distributions based on fatigue lives data b) Scaled TTT transform of the Fatigue lives data.

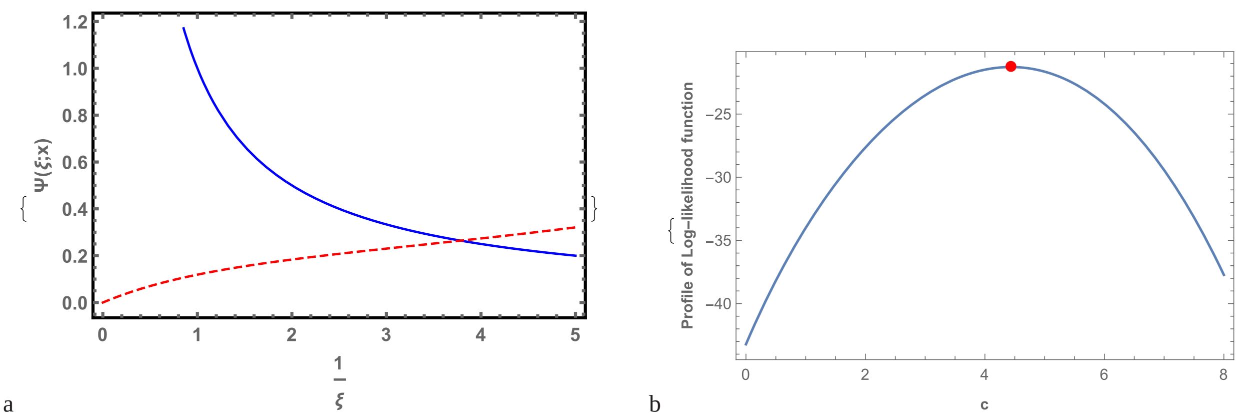


Fig. 11: (a) Plot of the $\frac{1}{\xi}$ and $\Psi(\xi;x)$ functions for the fatigue lives data. (b) The profile log-likelihood of the parameter c for the fatigue lives data

Table 4: MLEs of unknown parameters, the K-S test with the corresponding P-value, the W^* test with the corresponding P-value and A^* test with the corresponding P-value for different models using the annual wage data

Model	MLEs	K-S test	p-value	(W^*)	p-value	(A^*)	p-value
IGLED	c=66.6216, b=10593.9, ξ=6.31625	0.1209	0.773	0.08716	0.652	0.658	0.595
IWD	c=113.489, ξ=8.7882	0.1219	0.764	0.0934	0.618	0.698	0.561
GIW(c,a,ξ)	c=230.56, ξ=8.79, a=0.002	0.1219	0.764	0.0934	0.618	0.698	0.561
IRD	b=28372.1	0.4002	0.000134	1.447	0.00023	7.128	0.00029
IED	c=120.45	0.5001	6×10^{-7}	2.137	6.295×10^{-7}	10.037	0.0000132
Log-Normal(c,b)	c=4.80399, b=0.164704	0.1933	0.212	0.229	0.2171	1.34	0.219
IGD(c,b)	c=123.767, b=4495.2	0.195	0.2045	0.233	0.211	1.35	0.216

–For calculating the following integral $I1=\int_t^{\infty} x\, f(x;\Theta)\, dx$
Making use of $v=(\frac{c}{x}+\frac{d}{2\,x^2})^{\xi}$, yields

$$I1=\int_0^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}(\frac{c\,+\,\sqrt{c^2\,+\,2\,b\,v^{\frac{1}{\xi}}}}{2\,v^{\frac{1}{\xi}}})\,e^{-v}\,dv.$$

One can show that

$$(\frac{c+\sqrt{c^2+2\,b\,v^{\frac{1}{\xi}}}}{2\,v^{\frac{1}{\xi}}})=\frac{c}{2}\,v^{\frac{-1}{\xi}}+\frac{1}{2}\,v^{\frac{-1}{\xi}}\,(c^2+2\,b\,v^{\frac{1}{\xi}})^{\frac{1}{2}}.$$

Also, it is easy to show that $\frac{2\,b\,v^{\frac{1}{\xi}}}{c^2}<1$ when $v<(\frac{c^2}{2\,b})^{\xi}$ and $\frac{c^2}{2\,b\,v^{\frac{1}{\xi}}}<1$ when $v>(\frac{c^2}{2\,b})^{\xi}$. Then the integral $I1$

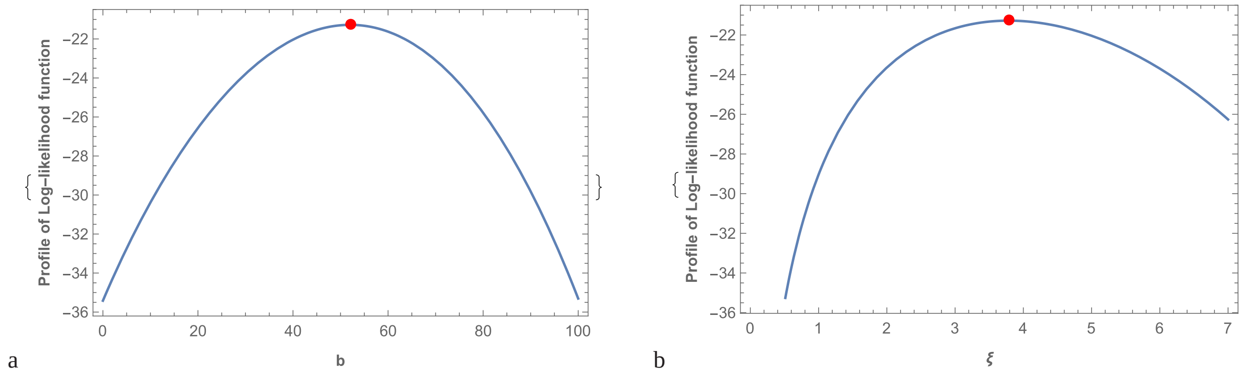


Fig. 12: (a) The profile log-likelihood of the parameter b for the fatigue lives data (b) The profile log-likelihood of the parameter ξ for the fatigue lives data

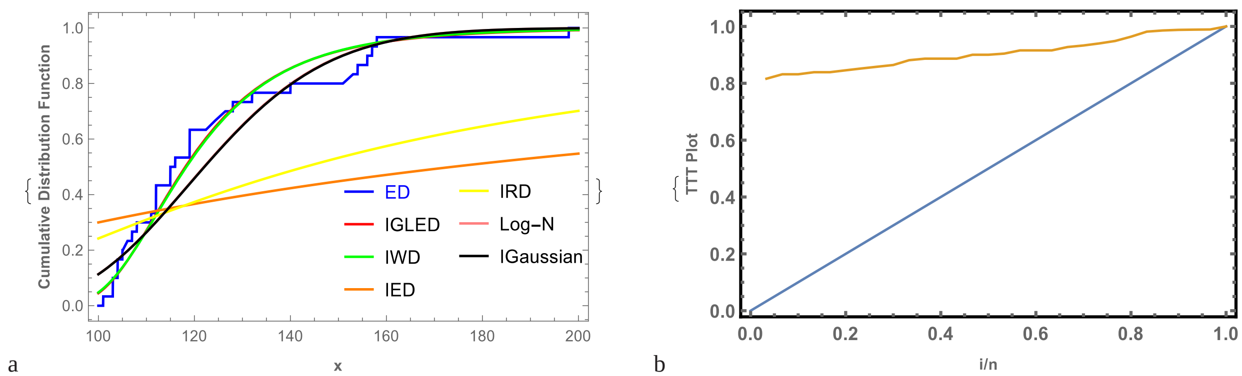


Fig. 13: a) Empirical distribution functions versus distribution functions of modeling distributions based on the annual wage data b) Scaled TTT transform of the annual wage data.

can be written as:

$$I1=\begin{cases}\frac{c}{2}\int_0^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}} v^{\frac{-1}{\xi}}\, e^{-v}\, dv+\frac{c}{2}\,\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2\,b}{c^2}\right)^i \int_0^{(\frac{c^2}{2\,b})^{\xi}} v^{\frac{i-1}{\xi}}\, dv+\frac{1}{2}\sqrt{2\,b}\,\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{c^2}{2\,b}\right)^i \int_{(\frac{c^2}{2\,b})^{\xi}}^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}} v^{\frac{-2\,i-1}{2\,\xi}}\, dv, & (\frac{c^2}{2\,b})^{\xi}<(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi};\\ \frac{c}{2}\int_0^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}} v^{\frac{-1}{\xi}}\, e^{-v}\, dv+\frac{c}{2}\,\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2\,b}{c^2}\right)^i \int_0^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}} v^{\frac{i-1}{\xi}}\, dv, & (\frac{c^2}{2\,b})^{\xi}>(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}.\end{cases} \tag{48}$$

–For calculating the following integral $I2=\int_t^{\infty} x^2\, f(x;\Theta)\, dx$

Making use of $v=(\frac{c}{x}+\frac{d}{2\,x^2})^{\xi}$, yields

$$I2=\int_0^{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}} \left(\frac{c\,+\,\sqrt{c^2\,+\,2\,b\,(v)^{\frac{1}{\xi}}}}{2\,(v)^{\frac{1}{\xi}}}\right)^2 e^{-v}\, dv.$$

It is clear that

$$\left(\frac{c+\sqrt{c^2+2\,b\,(v)^{\frac{1}{\xi}}}}{2\,(v)^{\frac{1}{\xi}}}\right)^2 =\frac{c^2}{2}\,v^{\frac{-2}{\xi}}+\frac{b}{2}\,v^{\frac{-1}{\xi}}+\frac{c}{2}v^{\frac{-2}{\xi}}\,\left(c^2+2\,b\,v^{\frac{1}{\xi}}\right)^{\frac{1}{2}}.$$

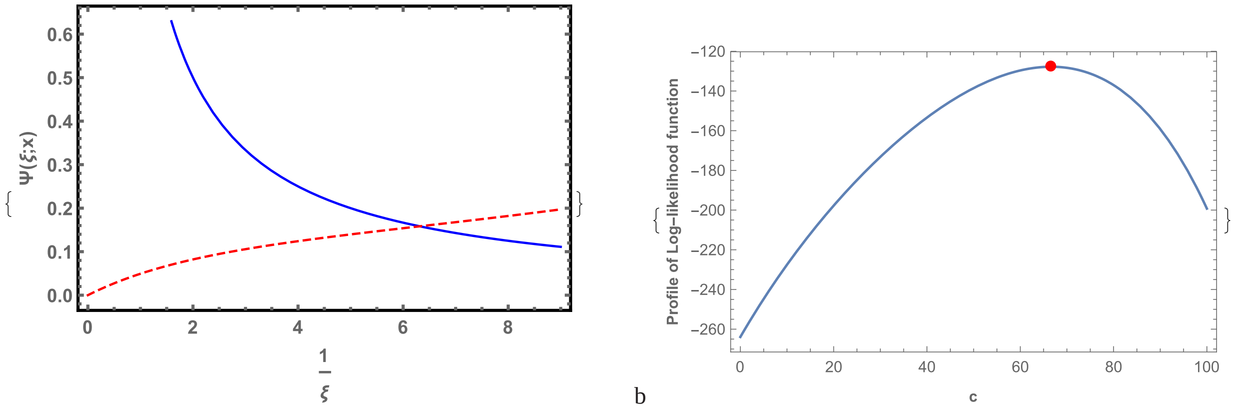


Fig. 14: (a)Plot of the $\frac{1}{\xi}$ and $\Psi(\xi;x)$ functions for the annual wage data. (b) The profile log-likelihood of the parameter c for the annual wage data

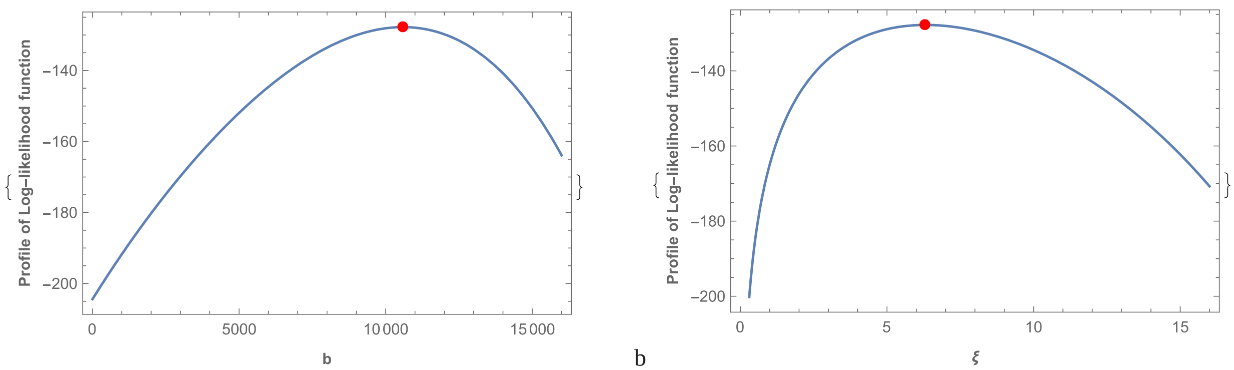


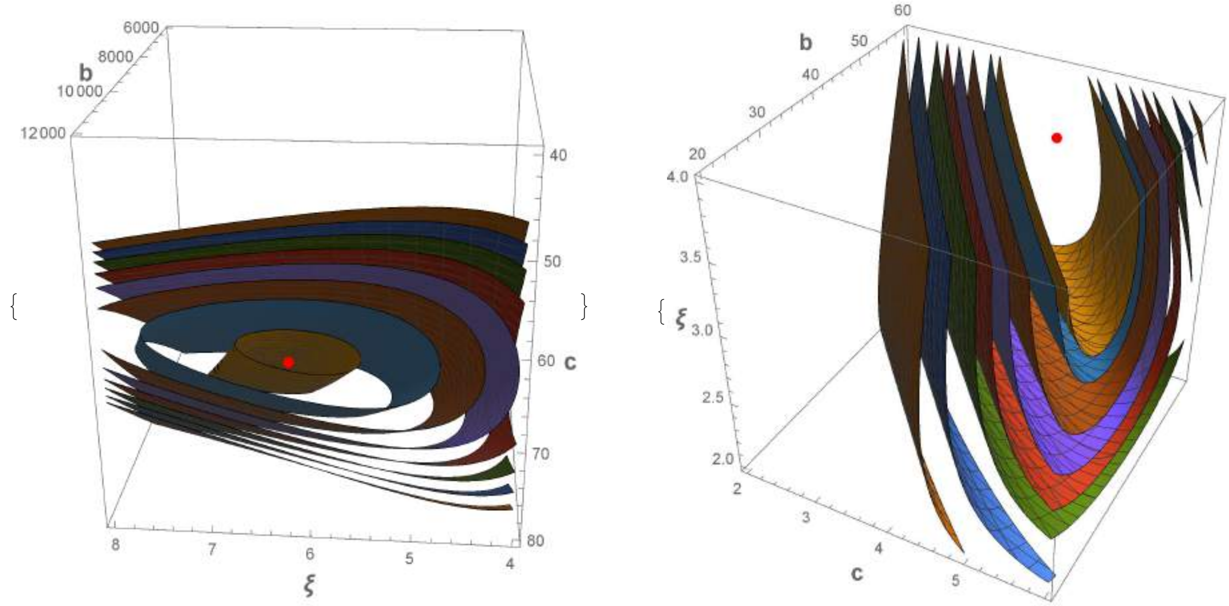
Fig. 15: (a) The profile log-likelihood of the parameter b for the annual wage data (b) The profile log-likelihood of the parameter ξ for the annual wage data

Then the integral $I2$ can be presented as:

$$I2=\begin{cases} \frac{c^2}{2}\int_0^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{-2}{\xi}}e^{-v}\,dv+\frac{b}{2}\int_0^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{-1}{\xi}}e^{-v}\,dv \\ +\frac{c^2}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2b}{c^2}\right)^i\int_0^{(\frac{c^2}{2b})^\xi} v^{\frac{i-2}{\xi}}\,dv+ \\ \frac{c}{2}\sqrt{2b}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{c^2}{2b}\right)^i\int_{(\frac{c^2}{2b})^\xi}^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{-2i-3}{2\xi}}\,dv, \\ \qquad (\frac{c^2}{2b})^\xi<(\frac{c}{t}+\frac{b}{2t^2})^\xi; \\ \\ \frac{c^2}{2}\int_0^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{-2}{\xi}}e^{-v}\,dv+\frac{b}{2}\int_0^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{-1}{\xi}}e^{-v}\,dv \\ +\frac{c^2}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2b}{c^2}\right)^i\int_0^{(\frac{c}{t}+\frac{b}{2t^2})^\xi} v^{\frac{i-2}{\xi}}\,dv, \\ \qquad (\frac{c^2}{2b})^\xi>(\frac{c}{t}+\frac{b}{2t^2})^\xi. \end{cases} \tag{49}$$

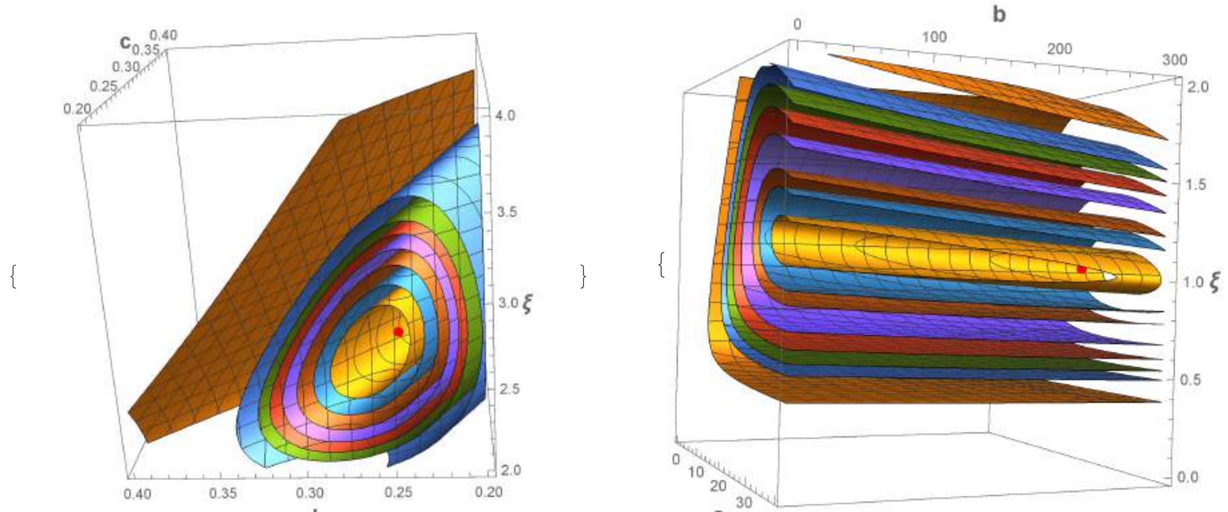
–For calculating the following integral $I3=\int_0^t x\, f(x;\Theta)\,dx$
As in the previous integral $I1$, the integral $I3$ can be given as

$$I3=\begin{cases} \frac{c}{2}\int_{(\frac{c}{t}+\frac{b}{2t^2})^\xi}^{\infty} v^{\frac{-1}{\xi}}e^{-v}\,dv+\frac{c}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2b}{c^2}\right)^i \\ \int_{(\frac{c}{t}+\frac{b}{2t^2})^\xi}^{(\frac{c^2}{2b})^\xi} v^{\frac{i-1}{\xi}}\,dv+\frac{1}{2}\sqrt{2b}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{c^2}{2b}\right)^i \\ \int_{(\frac{c^2}{2b})^\xi}^{\infty} dv, \qquad (\frac{c^2}{2b})^\xi>(\frac{c}{t}+\frac{b}{2t^2})^\xi; \\ \\ \frac{c}{2}\int_{(\frac{c}{t}+\frac{b}{2t^2})^\xi}^{\infty} v^{\frac{-1}{\xi}}e^{-v}\,dv+\frac{1}{2}\sqrt{2b}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i} \\ \left(\frac{c^2}{2b}\right)^i\int_{(\frac{c}{t}+\frac{b}{2t^2})^\xi}^{\infty} v^{\frac{-2i-1}{2\xi}}\,dv,\ (\frac{c^2}{2b})^\xi<(\frac{c}{t}+\frac{b}{2t^2})^\xi. \end{cases} \tag{50}$$



a b

Fig. 16: (a) The contour of log-likelihood for the annual wage data (b) The contour of log-likelihood for the fatigue lives data



a b

Fig. 17: (a) The contour of log-likelihood for the burning velocity of different chemical materials data (b) The contour of log-likelihood for the intervals between successive failures data

– For calculating the following integral $I4 = \int_0^t x^2 \, f(x;\Theta)\, dx$
As in the previous integral $I2$, the integral $I4$ can be obtained as:

$$I4=\begin{cases}\frac{c^2}{2}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{\infty} v^{\frac{-2}{\xi}}\,e^{-v}\,dv+\frac{b}{2}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{\infty} v^{\frac{-1}{\xi}}\,e^{-v}\,dv\\ +\frac{c^2}{2}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{2\,b}{c^2}\right)^{i}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{(\frac{c^2}{2\,b})^{\xi}} v^{\frac{i-2}{\xi}}dv\\ +\frac{c}{2}\sqrt{2\,b}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{c^2}{2\,b}\right)^{i}\int_{(\frac{c^2}{2\,b})^{\xi}}^{\infty} v^{\frac{-2\,i-3}{2\,\xi}}dv,\\ \qquad (\frac{c^2}{2\,b})^{\xi}>(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi};\\ \\ \frac{c^2}{2}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{\infty} v^{\frac{-2}{\xi}}\,e^{-v}\,dv+\frac{b}{2}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{\infty} v^{\frac{-1}{\xi}}\,e^{-v}\,dv\\ +\frac{c}{2}\sqrt{2\,b}\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{c^2}{2\,b}\right)^{i}\int_{(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}}^{\infty} v^{\frac{-2\,i-3}{2\,\xi}}dv,\\ \qquad (\frac{c^2}{2\,b})^{\xi}<(\frac{c}{t}+\frac{b}{2\,t^2})^{\xi}.\end{cases}\tag{51}$$

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