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A New Generalization of the Maxwell Distribution.

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Abstract: A new univariate three-parameter distribution, the transmuted exponentiated Maxwell distribution, is proposed and studied. This new univariate distribution can be seen as generalization of the Maxwell distribution and its respective exponentiated and transmuted versions. The new generalization is generate using the families of exponentiated and transmuted distributions. Some probabilistic properties are studied, maximum likelihood estimation discussed, derive the functions to be used in reliability studies and present an application with a real data. We hope that the new distribution proposed will serve as an alternative model to the Maxwell and the respective exponentiated and transmuted versions.

Keywords: Exponentiated distributions, Maxwell distribution, Transmuted distributions, Transmuted exponentiated Maxwell distribution.

1 Introduction

A random variable X follows a Maxwell distribution, denoted $X \sim M(\alpha)$, if its cumulative distribution function (cdf) and probability density function (pdf) are given by

$$F_X(x;\alpha) = \frac{4}{\sqrt{\pi}} \int_0^x \alpha^{\frac{3}{2}} u^2 e^{-\alpha u^2} du = H\left(\alpha x^2; \frac{3}{2}, 1\right) \quad (1)$$

and

$$f_X(x;\alpha) = \frac{4}{\sqrt{\pi}} \alpha^{\frac{3}{2}} x^2 e^{-\alpha x^2}, \quad (2)$$

respectively, where $x > 0$, $\theta > 0$ is a scale parameter and $H(x;\alpha,\beta) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u} du$ is the cdf of a gamma random variable. Tyagi and Bhattacharya [21] obtained the minimum variance unbiased estimator, Bayes estimator and the reliability function of this distribution. Chaturvedi and Rani [7] generalized the Maxwell distribution and they obtained classical and Bayesian estimators for generalized distribution. Bekker and Roux [6] studied empirical Bayes estimation for the Maxwell distribution. Shakil et al. [18] studied the distributions of $|XY|$ and $|X/Y|$ when X and Y are independent random variables having the Maxwell and Rayleigh distributions. Kazmi et al. [11] obtained the Bayesian estimation for two components mixture of the Maxwell distribution assuming type I censored data.

Definition 1. *A random variable X is said to have an exponentiated distribution if its cumulative distribution function (cdf) and probability density function (pdf) are given by*

$$G_X(x;\theta) = F(x)^\theta \quad \text{and} \quad g_X(x;\theta) = \theta f(x) F(x)^{\theta-1}, \quad (3)$$

respectively, where $\theta > 0$ is a shape parameter, $F(x)$ and $f(x) = \frac{d}{dx}F(x)$ are the cdf and pdf of the so called base distribution, respectively.

If $\theta = 1$ we have the distribution of the base random variable. Several distribution with positive support have been introduced using the exponentiated distributions family. For example, exponentiated exponential [8], exponentiated Weibull [15], Burr Type X [20], among others.

Definition 2. (Shaw and Buckley, [19]). *A random variable X is said to have transmuted distribution if its cumulative distribution function (cdf) and probability density function (pdf) are given by*

$$G_X(x;\lambda) = (1+\lambda)F(x) - \lambda F^2(x) \quad (4)$$

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and

$$g_X(x;\lambda) = f(x)\,(1+\lambda-2\lambda F(x)), \qquad (5)$$

respectively, where $|\lambda| \leq 1$ *is a shape parameter and* $F(x)$ *and* $f(x) = \frac{d}{dx}F(x)$ *are the cdf and pdf of the base distribution, respectively.*

For $\lambda = 0$, we have the distribution of the base random variable. Several distribution of positive support has been introduced using the transmuted distributions family, for example; transmuted Gumbel [3], transmuted Weibull [4], transmuted Rayleigh [13], transmuted generalized Rayleigh [14], among other distributions.
Iriarte and Astorga [9] used the family of transmuted distributions to introduce the transmuted Maxwell distribution. A random variable X follows a transmuted Maxwell distribution, denoted $X \sim TM(\alpha,\lambda)$, if its pdf is given by

$$f_X(x;\alpha,\lambda) = \sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}x^2 e^{-\frac{\theta x^2}{2}}\left[1+\lambda-2\lambda G(\frac{\theta x^2}{2};\frac{3}{2},1)\right],$$

where $x > 0$, $\alpha > 0$, $|\lambda| \leq 1$ and G is the cdf of a gamma random variable. If $\lambda = 0$ the Maxwell distribution is obtained.
In this paper, we introduce a new three parameters distribution that can be seen as a generalization of the Maxwell distribution. We use the families of the exponentiated distributions and transmuted distributions, considering as the baseline function a Maxwell cumulative distribution function, to generate the new model. In this way, two shape parameters are added to the Maxwell distribution.
The paper is organized as follows. In Section 2 we derive its density, moments and asymmetry and kurtosis coefficients of the new distribution. In Section 3 we discuss maximum likelihood estimation and calculate the elements of the observed information matrix. In Section 4 the reliability function is derived. In Section 5 we obtain the density function of order statistics. Section 6 presents application to real data sets. The application illustrates the good performance of the model proposed in real applications. Final conclusions are reported in Section 7.

2 The transmuted exponentiated Maxwell distribution

Proposition 1. *A random variable* X *follows a transmuted exponentiated Maxwell (TEM) distribution if its cumulative distribution function (cdf) is given by*

$$G_X(x;\alpha,\theta,\lambda) = (1+\lambda)F^{\theta}(\alpha x^2;\frac{3}{2},1) - \lambda F^{2\theta}(\alpha x^2;\frac{3}{2},1),$$

and the respective probability density function (pdf) is

$$g_X(x;\alpha,\theta,\lambda) = \frac{4}{\sqrt{\pi}}\theta\alpha^{\frac{3}{2}}x^2 e^{-\alpha x^2}F^{\theta-1}(\alpha x^2;\frac{3}{2},1)\left(1+\lambda-2\lambda F^{\theta}(\alpha x^2;\frac{3}{2},1)\right),$$

where $x > 0$, $\alpha > 0$ *is a scale parameter,* $\theta > 0$ *is a shape parameter,* $|\lambda| \leq 1$ *is a parameter that makes the asymmetry more flexible and*

$$F(x;\alpha,\beta) = \int_0^x \frac{\beta^{\alpha}}{\Gamma(\alpha)}u^{\alpha-1}e^{-\beta u}\,du$$

is the cumulative distribution function of the gamma distribution. We denote this as $X \sim TEM(\alpha,\theta,\lambda)$.

Proof. Replacing the expressions shown in (3) into (4) and (5), we obtained

$$G_X(x;\theta,\lambda) = (1+\lambda)F(x)^{\theta} - \lambda F(x)^{2\theta}$$

and

$$g_X(x;\theta,\lambda) = \theta f(x)F(x)^{\theta-1}\left[1+\lambda-2\lambda F(x)^{\theta}\right],$$

and replacing (1) and (2) into this expressions, the result is obtained. □

Next, we present two transformations related to the TEM distributions.

Proposition 2. *Let* $X \sim TEM(\alpha,\theta,\lambda)$. *Then,*

(a) $W = aX \sim TEM(\alpha/a^2,\theta,\lambda)$ *for all* $a > 0$;

(b) *The pdf of* $W = \log(T)$ *is given by*

$$f_W(w;\alpha,\theta,\lambda) = \frac{4}{\sqrt{\pi}}\theta\alpha^{3/2}e^{2w-\alpha e^w}F^{\theta-1}(\alpha e^{2w};\frac{3}{2},1)\left[1+\lambda-2\lambda F^{\theta}(\alpha e^{2w};\frac{3}{2},1)\right],\ w \in \mathbb{R}$$

Proof. Parts (a) and (b) are directly obtained from the change-of-variables method. □

Remark. Part (a) of Proposition 2 indicates that the TEM distributions belong to the scale family, Part (b) can be used to study regression models in same lines as in the context of regression models for positive random variables; see McDonald and Butler [12]. In addition, Part (a) allows us to obtain a two parameter TEM distribution. That is, if $X \sim TEM(\alpha,\theta,\lambda)$, then $\sqrt{\alpha}X \sim TEM(1,\theta,\lambda)$. It is known that the Maxwell distribution arises as the model of the magnitude of the velocity of a randomly chosen molecule of a gas in a closed container under the assumption that the gas is not flowing and that the pressure in the gas is the same in all directions, see Johnson et al. [10]. Then the TEM distribution has the same motivation with the characteristic of being more flexible.

Figure 1 depicts some of the shapes that the density function of the transmuted exponentiated Maxwell distribution can take for different values of the parameters α, θ and λ.

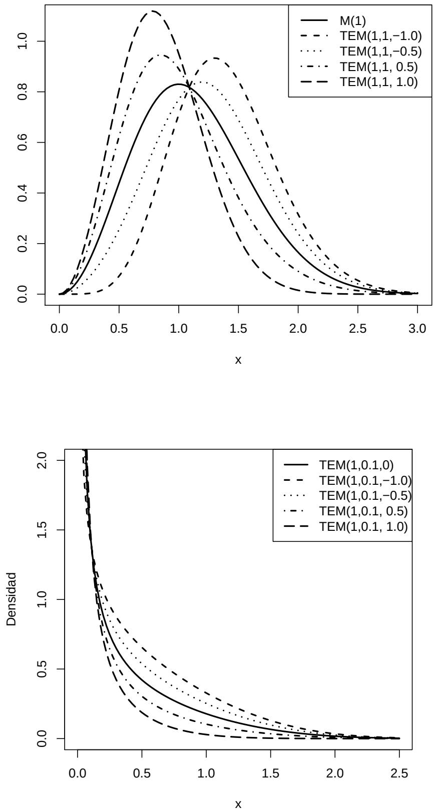


Fig. 1: Plot of the transmuted exponentiated Maxwell density, $TEM(\alpha,\theta,\lambda)$.

2.1 Properties

In this subsection some basic properties of the transmuted exponentiated Maxwell distribution are considered.
Let $X \sim TEM(\alpha,\theta,\lambda)$, then

1. For $\lambda = 0$ we obtain

$$g_X(x;\alpha,\theta) = \frac{4}{\sqrt{\pi}}\theta\alpha^{\frac{3}{2}}x^2e^{-\alpha x^2}\left(F\left(\alpha x^2;\frac{3}{2},1\right)\right)^{\theta-1},$$

namely, the density function of the exponentiated Maxwell distribution.

2. For $\alpha = \frac{\beta}{2}$ and $\theta = 1$ we obtain

$$g_X(x;\beta,\lambda) = \sqrt{\frac{2}{\pi}}\beta^{\frac{3}{2}}x^2e^{-\frac{\beta}{2}x^2}\left[1+\lambda-2\lambda G(\frac{\beta x^2}{2};\frac{3}{2},1)\right],$$

namely, the density function of the transmuted Maxwell distribution. (Iriarte and Astorga,[9]).

3. For $\lambda = 0$ and $\theta = 1$ we obtain

$$g_X(x;\alpha) = \frac{4}{\sqrt{\pi}}\alpha^{\frac{3}{2}}x^2e^{-\alpha x^2},$$

that is, the density function of the Maxwell distribution.

2.2 Moments

Proposition 3. *Let $X \sim TEM(\alpha,\theta,\lambda)$. Then, for $r = 1,2,\ldots$ it follows that the r-th moment is given by*

$$\mu_r = \alpha^{-r/2}a_r\,,$$

where a_r is defined as

$$a_r = \int_0^1 (1+\lambda-2\lambda w)\left(F^{-1}\left(w^{1/\theta};\frac{3}{2},1\right)\right)^{\frac{r}{2}}dw,$$

where F^{-1} is the quantile function of the gamma distribution.

Proof. Using the definition of moments, the r-th moment is given by

$$\mu_r = \frac{4}{\sqrt{\pi}}\theta\alpha^{\frac{3}{2}} \int_0^\infty x^{r+2}e^{-\alpha x^2}F^{\theta-1}(\alpha x^2;\tfrac{3}{2},1)\left(1+\lambda-2\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right)dx,$$

and considering the changing of variable $w = F^{\theta}(\alpha x^2;\frac{3}{2},1)$, the result follows. □

Corollary 1. *If $X \sim TEM(\alpha,\theta,\lambda)$, then*

$$E(X) = \frac{a_1}{\sqrt{\alpha}} \quad \text{and} \quad Var(X) = \frac{a_2-a_1^2}{\alpha}.$$

Corollary 2. *If $X \sim TEM(\alpha,\theta,\lambda)$, then the coefficients of asymmetry ($\sqrt{\beta_1}$) and kurtosis (β_2) are, respectively,*

$$\sqrt{\beta_1} = \frac{a_3-3a_1a_2+2a_1^3}{(a_2-a_1^2)^{3/2}}$$

and

$$\beta_2 = \frac{a_4-4a_1a_3+6a_1^2a_2-3a_1^4}{(a_2-a_1^2)^2}.$$

Remark. As $\lambda = 0$ the asymmetry and kurtosis coefficients take the values

$$\sqrt{\beta_1}_{EM} = \frac{b_3-3b_1b_2+2b_1^3}{(b_2-b_1^2)^{3/2}}$$

and

$$\beta_{2EM} = \frac{b_4-4b_1b_3+6b_1^2b_2-3b_1^4}{(b_2-b_1^2)^2},$$

respectively, which correspond to those for the exponentiated Maxwell distribution, where $b_r = \int_0^1\left(F^{-1}\left(w^{1/\theta};\frac{3}{2},1\right)\right)^{\frac{r}{2}}dw$, $r = 1,2,3,4$ and F^{-1} is the quantile function of the gamma distribution. As $\theta = 1$ and $\lambda = 0$ the asymmetry and kurtosis coefficients take the values 0.485692804 and 3.108164, respectively, which correspond to those for the classical Maxwell distribution. Figures 2 depict plots for the asymmetry and kurtosis coefficients of the TEM distribution for different values of its parameters θ and λ. In the 2D graphics it can be seen that the asymmetry and kurtosis coefficients of the TEM distribution can take the values of skewness and kurtosis of the EM and M distributions. The graphs were developed using the MATLAB software, see A1 and A2 in Appendix .

3 Inference

In this section we discuss maximum likelihood estimation for parameters α, θ and λ for the transmuted exponentiated Maxwell distribution.

3.1 Maximum Likelihood estimation

For a random sample $X_1,\ldots,X_n$ from the distribution $TEM(\alpha,\theta,\lambda)$, the log likelihood function can be written as

$$l(\alpha,\theta,\lambda) = n\log\left(\frac{4}{\sqrt{\pi}}\right) + \frac{3n}{2}\log(\alpha) + n\log(\theta) +2\sum_{i=1}^{n}\log(x_i) + (\theta-1)\sum_{i=1}^{n}\log F(x_i) - \alpha\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}\log H(x_i),$$

where $F(x_i) = F\left(\alpha x^2;\frac{3}{2},1\right)$ is the cdf of the gamma distribution and $H(x_i) = 1+\lambda-2\lambda F^{\theta}(x_i)$. The maximum likelihood equations are given by

$$\frac{3n}{2\alpha} - \sum_{i=1}^{n}x_i^2 + (\theta-1)\sum_{i=1}^{n}\frac{F_1(x_i)}{F(x_i)} + \sum_{i=1}^{n}\frac{H_1(x_i)}{H(x_i)} = 0, \qquad (6)$$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\log F(x_i) + \sum_{i=1}^{n}\frac{H_2(x_i)}{H(x_i)} = 0, \qquad (7)$$

$$\sum_{i=1}^{n}\frac{H_3(x_i)}{H(x_i)} = 0, \qquad (8)$$

where $F_1(x_i) = \frac{\partial F(x_i)}{\partial \alpha} = \frac{2}{\sqrt{\pi}}\alpha^{1/2}x_i^3 e^{-\alpha x_i^2}$; $H_1(x_i) = \frac{\partial H(x_i)}{\partial \alpha} = -\frac{4}{\sqrt{\pi}}\lambda\theta\alpha^{1/2}x_i^3 e^{-\alpha x_i^2}F^{\theta-1}(x_i)$; $H_2(x_i) = \frac{\partial H(x_i)}{\partial \theta} = -2\lambda F^{\theta}(x_i)\log F(x_i)$ and $H_3(x_i) = \frac{\partial H(x_i)}{\partial \lambda} = 1-2F^{\theta}(x_i)$.

Therefore, numerical algorithms are required for solving the score equations. One possibility is to employ the subroutine optim with the R Core Team [16].

It is well known that as the sample size increases, the distribution of the MLE tends (under regularity conditions) to the normal distribution with mean vector (α,θ,λ) and covariance matrix equals to the inverse of the Fisher (expected) information matrix. Due to the complexity of the likelihood function it is not possible to obtain analytical expressions for those quantities. It is possible, however, to work with the observed information

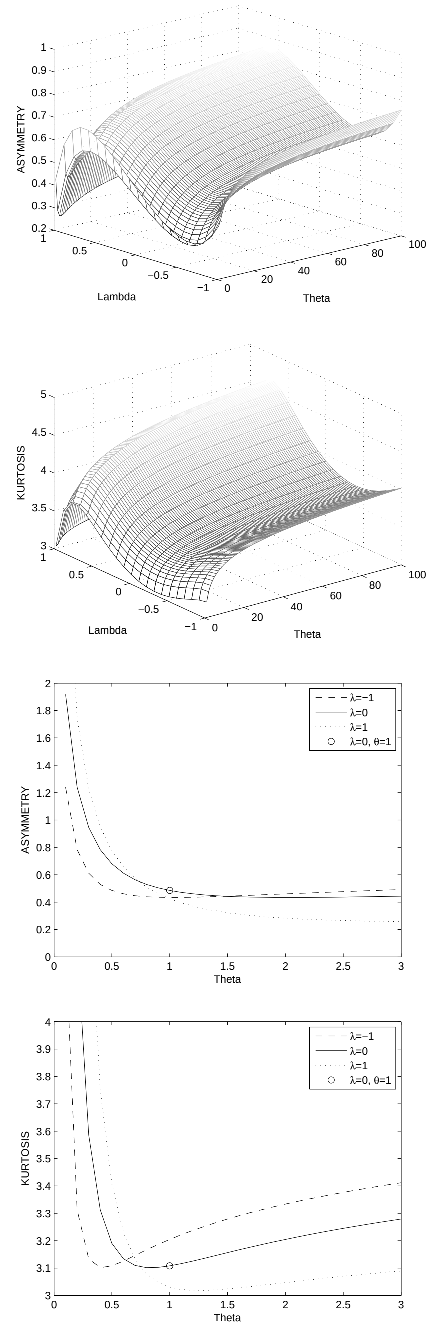


Fig. 2: 2D and 3D graphs of the asymmetry and kurtosis coefficients of the TEM distribution. Skewness and kurtosis of EM (solid line), TEM (dashed and doited lines), M (circle) distributions.

matrix, which is a consistent estimator for the expected information matrix. The observed information matrix follows from the Hessian matrix by replacing unknown parameters by their MLEs. Some algebraic manipulations yield the following Hessian matrix:

$$I_n(\beta) = \begin{pmatrix} \frac{\partial^2 \log L(\beta)}{\partial \alpha^2} & \frac{\partial^2 \log L(\beta)}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L(\beta)}{\partial \lambda \partial \alpha} \\ & \frac{\partial^2 \log L(\beta)}{\partial \theta^2} & \frac{\partial^2 \log L(\beta)}{\partial \lambda \partial \theta} \\ & & \frac{\partial^2 \log L(\beta)}{\partial \lambda^2} \end{pmatrix}$$

such that

$$\begin{aligned}
\frac{\partial^2 \log L(\beta)}{\partial \alpha^2} &= -\frac{3n}{2\alpha^2} + (\theta-1)\sum_{i=1}^{n}\left[\frac{F_{11}(x_i)}{F(x_i)} - \left(\frac{F_1(x_i)}{F(x_i)}\right)^2\right] \\
&\quad + \sum_{i=1}^{n}\left[\frac{H_{11}(x_i)}{H(x_i)} - \left(\frac{H_{11}(x_i)}{H(x_i)}\right)^2\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \theta \partial \alpha} &= \sum_{i=1}^{n}\left[\frac{F_1(x_i)}{F(x_i)} + \frac{H_{12}(x_i)}{H(x_i)} - \frac{H_1(x_i)H_2(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \lambda \partial \alpha} &= \sum_{i=1}^{n}\left[\frac{H_{13}(x_i)}{H(x_i)} - \frac{H_1(x_i)H_3(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \alpha \partial \theta} &= \sum_{i=1}^{n}\left[\frac{F_1(x_i)}{F(x_i)} + \frac{H_{21}(x_i)}{H(x_i)} - \frac{H_1(x_i)H_2(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \theta^2} &= -\frac{n}{\theta^2} + \sum_{i=1}^{n}\left[\frac{H_{22}(x_i)}{H(x_i)} - \left(\frac{H_2(x_i)}{H(x_i)}\right)^2\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \lambda \partial \theta} &= \sum_{i=1}^{n}\left[\frac{H_{23}(x_i)}{H(x_i)} - \frac{H_2(x_i)H_3(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \alpha \partial \lambda} &= \sum_{i=1}^{n}\left[\frac{H_{31}(x_i)}{H(x_i)} - \frac{H_1(x_i)H_3(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \theta \partial \lambda} &= \sum_{i=1}^{n}\left[\frac{H_{32(x_i)}}{H(x_i)} - \frac{H_2(x_i)H_3(x_i)}{H^2(x_i)}\right], \\
\frac{\partial^2 \log L(\beta)}{\partial \lambda^2} &= -\sum_{i=1}^{n}\left(\frac{H_3(x_i)}{H(x_i)}\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
F_{11}(x_i) &:= \frac{\partial F_1(x_i)}{\partial \alpha} = \frac{\alpha^{-1/2}}{\sqrt{\pi}}\left(1-2\alpha x_i^2\right)x_i^3 e^{-\alpha x_i^2}, \\
H_{11}(x_i) &:= \frac{\partial H_1(x_i)}{\partial \alpha} = -\frac{2\lambda\theta}{\sqrt{\pi\alpha}} x_i^3 e^{-\alpha x_i^2} F^{\theta-2}(x_i) \\
&\qquad \left(\left(1-2\alpha x_i^2\right)F(x_i) + 2\alpha F_1(x_i)\right), \\
H_{12}(x_i) &:= \frac{\partial H_1(x_i)}{\partial \theta} = -\frac{4}{\sqrt{\pi}}\lambda\alpha^{\frac{1}{2}} x_i^3 F^{\theta-1}(x_i)\left(1+\theta\log F(x_i)\right), \\
H_{13}(x_i) &:= \frac{\partial H_1(x_i)}{\partial \lambda} = -\frac{4}{\sqrt{\pi}}\theta\alpha^{1/2} x_i^3 e^{-\alpha x_i^2} F^{\theta-1}(x_i), \\
H_{21}(x_i) &:= \frac{\partial H_2(x_i)}{\partial \alpha} = -2\lambda\frac{F_1(x_i)}{F(x_i)}\left(\theta F^2(x_i)+1\right), \\
H_{22}(x_i) &:= \frac{\partial H_2(x_i)}{\partial \theta} = -2\lambda F^{\theta}(x_i)\log^2 F(x_i), \\
H_{23}(x_i) &:= \frac{\partial H_2(x_i)}{\partial \lambda} = -2F^{\theta}(x_i)\log F(x_i), \\
H_{31}(x_i) &:= \frac{\partial H_3(x_i)}{\partial \alpha} = -2\theta F^{\theta-1}(x_i)F_3(x_i), \\
H_{32}(x_i) &:= \frac{\partial H_3(x_i)}{\partial \alpha} = -2F^{\theta}(x_i)\log F(x_i).
\end{aligned}$$

3.2 Simulation study

In this section, a small scale simulation is performed to illustrate the behavior of the MLEs for parameters α, θ and λ. We generate 1000 random samples of sizes $n=30$, $n=50$ and $n=100$ from the distribution $TEM(\alpha,\theta,\lambda)$ for fixed values of the parameters. Random numbers $X \sim TEM(\alpha,\theta,\lambda)$ can be generated as

$$X = \left[\frac{1}{\alpha}F^{-1}\left(\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda U}}{2\lambda}\right)^{\frac{1}{\theta}}; \frac{3}{2}, 1\right)\right]^{\frac{1}{2}}$$

where $U \sim u(0,1)$, $\alpha > 0$, $\theta > 0$, $|\lambda| \le 1$ y F^{-1} quantile function of the classical gamma distribution. The estimates can be obtained using the optimization method based on Nelder-Mead, quasi-Newton and conjugate-gradient algorithms and implemented in the statistical package R Core Team [16]. Measures and empirical standard deviations are presented in Table 1. Notice that the parameters are well estimated and the estimates are asymptotically unbiased.

4 Reliability Analysis

The reliability function $R(t)$, which is the probability of an item not failing prior to some time t, is defined by $R(t) = 1 - F(t)$. The reliability function of a transmuted exponentiated Maxwell distribution is given by

$$R(t) = 1 - F^{\theta}(\alpha x^2; \frac{3}{2}, 1)\left(1+\lambda-\lambda F^{\theta}(\alpha x^2; \frac{3}{2}, 1)\right).$$

Figure 3 illustrates the reliability function of a transmuted exponentiated Maxwell distribution for different combinations of parameters.

Table 1: Maximum likelihood estimators for parameters α, θ and λ for the TEM distribution.

α	θ	λ	$\widehat{\alpha}$ (SD)	$\widehat{\theta}$ (SD)	$\widehat{\lambda}$ (SD)
			$n=30$		
1	0.5	1.0	1.299 (0.454)	0.556 (0.112)	0.898 (0.185)
		0.5	1.245 (0.412)	0.512 (0.135)	0.245 (0.468)
		-1.0	1.095 (0.240)	0.639 (0.242)	-0.888 (0.214)
2	1.0	1.0	2.377 (0.714)	1.128 (0.331)	0.884 (0.239)
		0.5	2.254 (0.570)	1.046 (0.294)	0.337 (0.469)
		-1.0	2.151 (0.419)	1.343 (0.580)	-0.875 (0.240)
3	2.0	1.0	3.566 (0.933)	2.447 (0.939)	0.845 (0.300)
		0.5	3.218 (0.664)	2.152 (0.712)	0.380 (0.476)
		-1.0	3.215 (0.549)	2.848 (1.514)	-0.860 (0.259)
			$n=50$		
α	θ	λ	$\widehat{\alpha}$ (SD)	$\widehat{\theta}$ (SD)	$\widehat{\lambda}$ (SD)
1	0.5	1.0	1.216 (0.351)	0.532 (0.086)	0.907 (0.175)
		0.5	1.199 (0.357)	0.494 (0.101)	0.259 (0.452)
		-1.0	1.060 (0.182)	0.613 (0.190)	-0.880 (0.223)
2	1.0	1.0	2.364 (0.583)	1.100 (0.211)	0.868 (0.227)
		0.5	2.231 (0.510)	1.002 (0.214)	0.304 (0.456)
		-1.0	2.111 (0.286)	1.245 (0.390)	-0.886 (0.196)
3	2.0	1.0	3.461 (0.750)	2.297 (0.568)	0.844 (0.273)
		0.5	3.181 (0.583)	2.038 (0.492)	0.365 (0.467)
		-1.0	3.159 (0.411)	2.551 (0.899)	-0.882 (0.233)
			$n=100$		
α	θ	λ	$\widehat{\alpha}$ (SD)	$\widehat{\theta}$ (SD)	$\widehat{\lambda}$ (SD)
1	0.5	1.0	1.167 (0.303)	0.520 (0.057)	0.912 (0.168)
		0.5	1.109 (0.273)	0.484 (0.077)	0.318 (0.417)
		-1.0	1.037 (0.125)	0.576 (0.137)	-0.904 (0.178)
2	1.0	1.0	2.259 (0.473)	1.060 (0.150)	0.899 (0.200)
		0.5	2.162 (0.431)	0.985 (0.161)	0.333 (0.431)
		-1.0	2.082 (0.216)	1.193 (0.320)	-0.901 (0.181)
3	2.0	1.0	3.328 (0.618)	2.182 (0.406)	0.888 (0.236)
		0.5	3.104 (0.522)	1.978 (0.327)	0.399 (0.426)
		-1.0	3.084 (0.280)	2.383 (0.671)	-0.899 (0.186)

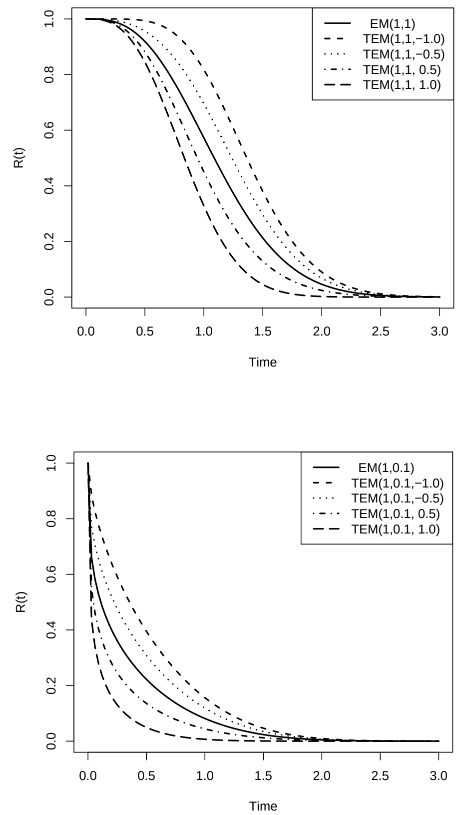


Fig. 3: The reliability function of a TEM distribution.

Another characteristic of interest of a random variable is its hazard rate function defined by

$$h(t)=\frac{f(t)}{1-F(t)},$$

which is an important quantity characterizing the life-time of a certain phenomenon. It can be loosely interpreted as the conditional probability of failure at time t, given it has survived to time t. The hazard rate function for a transmuted exponentiated Maxwell random variable is given by

$$h(t)=\frac{4}{\sqrt{\pi}}\theta\alpha^{\frac{3}{2}}x^2e^{-\alpha x^2}F^{\theta-1}\left(\alpha x^2;\tfrac{3}{2},1\right)$$
$$\frac{1+\lambda-2\lambda F^{\theta}\left(\alpha x^2;\frac{3}{2},1\right)}{1-F^{\theta}\left(\alpha x^2;\frac{3}{2},1\right)\left(1+\lambda-\lambda F^{\theta}\left(\alpha x^2;\frac{3}{2},1\right)\right)}.$$

Figure 4 illustrates the hazard function of a transmuted exponentiated Maxwell distribution for different combinations of parameters.

5 Order Statistics

In statistics, the j^{th} order statistical of a sample is equal to its j^{th}-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the n^{th} order statistics (or, the largest order statistic) is its maximum, that is,

$$X_{(n)}=\max\{X_1,X_2,\ldots,X_n\}.$$

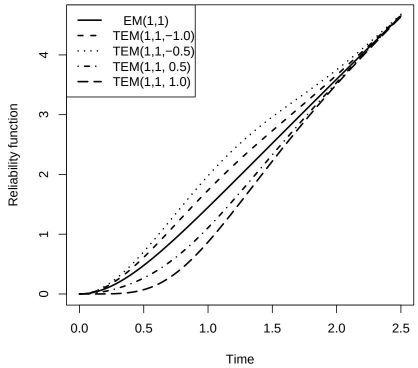
Similarly, $X_{(1)}=\min\{X_1,X_2,\ldots,X_n\}$ is the minimum of the sample.
The sample range is the difference between the maximum and the minimum in the sample. It is clearly a function of the order statistics:

$$\mathrm{Range}\{X_1,X_2,\ldots,X_n\}=X_{(n)}-X_{(1)}.$$

It is well known that if $X_{(1)}\leq X_{(2)}\leq\ldots\leq X_{(n)}$ denotes the order statistics of a random sample $X_1,X_2,\ldots,X_n$ from a continuous population with cdf $F_X(x)$ and pdf f_X then, the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}f_X(x)\left[F_X(x)\right]^{j-1}\left[1-F_X(x)\right]^{n-j},$$

for $j=1,2,\ldots,n$. The pdf of the j^{th} order statistics for transmuted exponentiated Maxwell distribution is given by



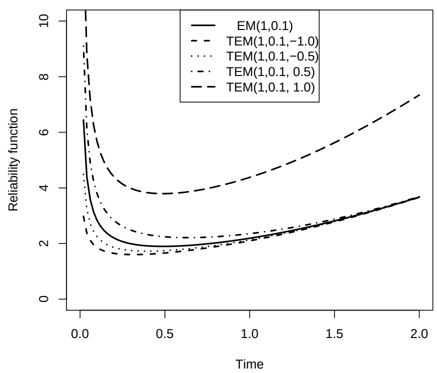


Fig. 4: The hazard function of a TEM distribution.

$$f_{X_{(j)}}(x) = \frac{4\theta\alpha^{\frac{3}{2}}x^2e^{-\alpha x^2}n!F^{\theta j-1}(\alpha x^2;\frac{3}{2},1)}{\sqrt{\pi}(j-1)!(n-j)!}$$

$$\left[1+\lambda-2\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]\cdot\left[1+\lambda-\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]^{j-1}$$

$$\left[1-F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\left[1+\lambda-\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]\right]^{n-j}.$$

Therefore, the pdf the largest order statistics $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \frac{4n}{\sqrt{\pi}}\theta\alpha^{\frac{3}{2}}x^2e^{-\alpha x^2}F^{\theta n-1}(\alpha x^2;\tfrac{3}{2},1)$$

$$\left[1+\lambda-2\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]\left[1+\lambda-\lambda F^{\theta}\left(\alpha x^2;\tfrac{3}{2},1\right)\right]^{n-1}.$$

and the pdf of the smallest order statistics $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = \frac{4n}{\sqrt{\pi}}\theta\alpha^{3/2}x^2e^{-\alpha x^2}F^{\theta-1}(\alpha x^2;\tfrac{3}{2},1)$$

$$\left[1+\lambda-2\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]$$

$$\left[1-F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\left[1+\lambda-\lambda F^{\theta}(\alpha x^2;\tfrac{3}{2},1)\right]\right]^{n-1}.$$

6 Illustrative data set

The fracture behavior of a single edge V-notched Aluminum plate repaired with Kevlar-49/epoxy or e-glass/epoxy pre- preg patches on both sides.

We consider a data set related to a single edge V-notched Aluminun plate repaired with Kevlar 49/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so that complete data with is available with exact times of failure. For previous studies with the data sets see Andrews and Herzberg [2] and Barlow et al. [5]. We calculated the maximum likelihood estimates for the M distribution, which are $\widehat{\alpha} = 0,6547649$. As starting values for the MLEs we took $\theta = 1$ and $\widehat{\lambda} = 0$.

The large sample variance-covariance matrix (the inverse of the observed information matrix) of the parameters for the transmuted exponentiated Maxwell was computed given the estimates $\hat{\alpha} = 0.108$, $\hat{\theta} = 0.252$ and $\hat{\lambda} = 0.883$.

$$I(\widehat{\Theta})^{-1} = \begin{pmatrix} 8,984\cdot10^{-4} & 3,653\cdot10^{-4} & -1,864\cdot10^{-3} \\ 3,653\cdot10^{-4} & 5,845\cdot10^{-4} & -5,037\cdot10^{-6} \\ -1,864\cdot10^{-3} & -5,037\cdot10^{-6} & 1,357\cdot10^{-2} \end{pmatrix}$$

Thus, the large sample estimates for the asymptotic variances of the MLE of θ, α and λ are $Var(\widehat{\alpha}) = 8,984\cdot10^{-4}$; $Var(\widehat{\theta}) = 5,845\cdot10^{-4}$ and $Var(\widehat{\lambda}) = 1,357\cdot10^{-2}$.

The usual Akaike information criterion (AIC) introduced by Akaike [1] and Bayesian information criterion (BIC) proposed by Schwarz [17] to measure of the goodness of fit were also computed. It is known that $AIC = 2k - 2\log lik$ and $BIC = k\log n - 2\log lik$ where k is the number of parameters in the model, n is the sample size and loglik is the maximized value of the likelihood function. For the TEM model, $AIC = 2(2) - 2(-108,5968) = 221,1936$ and $BIC = 2\log(101) - 2(-108,5968) = 226,4238$. Similarly, for the TEM model, $AIC = 2(3) - 2(-103,4609)$ and $BIC = 3\log(101) - 2(-103,4609) = 220,7672$. In both cases, the TEM model has the lowest values of AIC and BIC. Thus, the results show that the TEM model fits better the data set. Figure 5 displays the fitted models using the MLEs. Table 2 shows MLEs to each one of the fitted distributions and the corresponding AIC and BIC values.

Table 2: Estimated parameters of the M, TM, EM and TEM distributions for the life of fatigue fracture data set.

Model	Parameter estimates (SD)	AIC	BIC
TEM	$\widehat{\alpha} = 0,108$ (0,029)	212,921	220,767
	$\widehat{\theta} = 0,252$ (0,024)		
	$\widehat{\lambda} = 0,883$ (0,116)		
EM	$\widehat{\alpha} = 0,194$ (0,037)	221,193	226,423
	$\widehat{\theta} = 0,204$ (0,022)		
TM	$\widehat{\alpha} = 1,050$ (0,094)	496,039	501,269
	$\widehat{\lambda} = 0,793$ (0,087)		
M	$\widehat{\alpha} = 0,654$ (0,053)	530,957	533,572

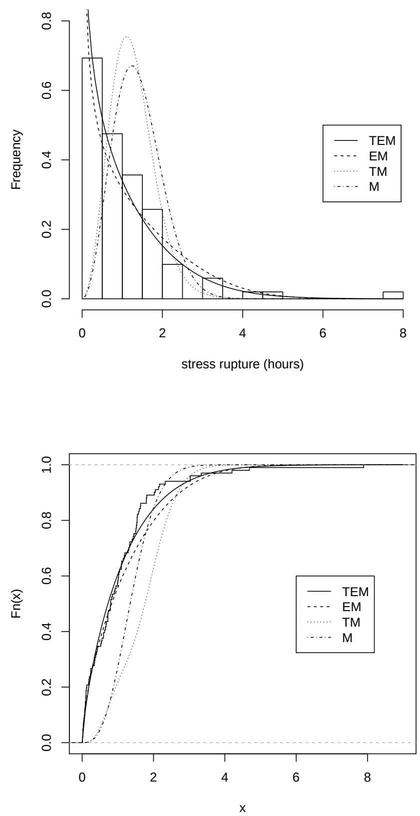


Fig. 5: Models fitted by the maximum likelihood method for the data under study.

7 Concluding Remarks

In this paper, we introduce an generalization of the Maxwell distribution. Exponentiated and transmuted versions of the Maxwell distribution can be considered as special cases of the generalization. The new model is originated using the exponentiated and transmuted distributions families. We derive analytical expressions for the distributional moments and use these results to calculate the expected value, variance and asymmetry and kurtosis coefficients. We calculate the likelihood equations and the elements of the observed information matrix. An instance of simulation is performed to illustrate the behaviour of the parameters in the maximum likelihood estimates, for the new model, finding that the estimates are consistent with our development. We obtain the reliability and hazard functions and density functions to the order statistics. Consequently, we conclude that the new model can be used as an alternative model to the M, EM and TM distributions.

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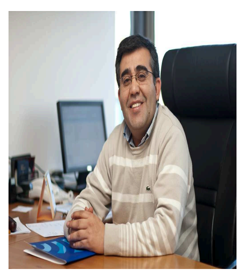
Appendix

A1. Codes for 3D graphs

```
lambda=[-1:0.1:1]
theta=[0.1:0.1:5]
n1=length(lambda)
n2=length(theta)
raizbeta1=zeros(n1,n2)
beta2=zeros(n1,n2)
a=zeros(4,n1,n2)

for r=1:4
for i=1:n1
for j=1:n2
pp=@(uu)(gaminv((uu.^(1./teta(j)))...
,1.5,1)).^(r/2)
a1(r,i,j)=quadgk(pp,0,1)
qq=@(uu)(uu).*((gaminv((uu.^(1./...
teta(j))),1.5,1)).^(r/2))
a2(r,i,j)=quadgk(qq, 0, 1)
a(r,i,j)=a1(r,i,j)+lambda(i).*...
a1(r,i,j)-2*lambda(i).*a2(r,i,j)
end
end
end

for i=1:n1
for j=1:n2
raizbeta1(i,j)=((a(3,i,j)-3*a(1...
,i,j)*a(2,i,j)+2*(a(1,i,j)^3))/
((a(2,i,j)-a(1,i,j)*a(1,i,j))^...
(1.5)));
numbeta2=(a(4,i,j)-4*a(1,i,j)*...
a(3,i,j)+6*a(1,i,j)*a(1,i,j)*a(2,...
i,j)-3*(a(1,i,j)^4));
denbeta2=(a(2,i,j)-a(1,i,j)*a(1,...
i,j))^2;
beta2(i,j)=numbeta2/denbeta2;
end
end

mesh(theta,lambda,raizbeta1)
xlabel('Theta')
ylabel('Lambda')
title('ASYMMETRY')

mesh(theta,lambda,beta2)
xlabel('Theta')
ylabel('Lambda')
title('KURTOSIS')
```

A2. Codes for 2D graphs

```
rb1lme1=raizbeta1(1,:)
plot(theta,rb1lme1,'k--')
hold on
rb1l0=raizbeta1(11,:)
plot(theta,rb1l0,'k-')
rb1l1=raizbeta1(21,:)
plot(theta,rb1l1,'k:')
plot(raizbeta1(11,10),'ko')
axis([0 3 0 2])
legend('\lambda=-1','\lambda=0',...
'\lambda=1','\lambda=0, \theta=1')
xlabel('Theta')
ylabel('ASYMMETRY')

b2lme1=beta2(1,:)
plot(theta,b2lme1,'k--')
hold on
b2l0=beta2(11,:)
plot(theta,b2l0,'k-')
b2l1=beta2(21,:)
plot(theta,b2l1,'k:')
plot(beta2(11,10),'ko')
axis([0 3 3 4])
xlabel('Theta')
ylabel('KURTOSIS')
legend('\lambda=-1','\lambda=0',...
'\lambda=1','\lambda=0, \theta=1');
```



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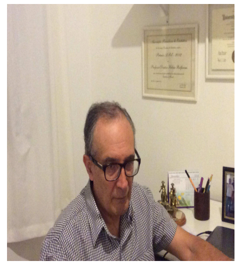

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