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A Note on the Exponential Stability of Linear Systems with Variable Retardations

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Abstract: We here investigate the exponential stability of a kind of linear systems of first order with variable delay. By means of an auxiliary functional, we discuss exponential stability of solutions of the system considered. During the proof, we also benefit from linear matrix inequalities (LMIs).

Keywords: Linear system, first of order, exponential stability, variable retardation.

1 Introduction

In the literature, stability analysis of first order linear systems with time-varying delays of the form

$$\dot{x}(t) = Ax(t) + Dx(t-h(t))$$

has received considerable attention.

In fact, Sun [9] considered the following system with multiple variable delays described as

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{n} A_i x(t-h_i(t)) + \Delta f(t, x(t), x(t-h_1(t)), ..., x(t-h_p(t)) + Bu(t).$$

The author discussed the stability of solutions of this equation.

In 2007, Phat and Nam [7] dealt with the following the time-varying linear system

$$\dot{x}(t) = \sum_{i=1}^{N} \alpha_i(t) A_i x(t) + \sum_{i=1}^{N} \alpha_i(t) B_i u(t).$$

They constructed sufficient conditions which guarantee the solutions of that system are exponential stable.

Later, 2010, by defining an appropriate Lyapunov-Krasovski functional, Niamsup and Phat [6] obtained specific conditions related the exponential stability the solutions of the system

$$\dot{x}(t) = -A_\sigma x(t) + B_\sigma f_\sigma(x(t)) + C_\sigma g_\sigma(x(t-\tau(t))),$$

where

$$\sigma(.) : R^n \to \{1, 2, ..., N\},$$
$$f_i(x(t)) = (f_{i1}(x_1(t)), ..., f_{in}(x_n(t)))^T$$

and

$$g_i(x(t)) = (g_{i1}(x_1(t)), ..., g_{in}(x_n(t)))^T.$$

As distinguished from this line, the following article is also notable.

Phat et al. [8] took into consideration the retarded differential problem

$$\dot{x}(t) = Ax(t) + Dx(t-h(t)), t \in \Re^+, \Re^+ = [0, \infty),$$
$$x(t) = \phi(t), t \in [-h_2, 0].$$

The authors investigated α-exponential stability of the zero solution of this problem. For a comprehensive review and some recent and related results on the qualitative properties of solutions to various linear and non-linear systems, we refer the reader to see Boyd et al. [1], De Oliveira et al. [2], Gu et al. [3], Hale and Verduyn Lunel [4], Kwon and Park [5], Shao [10], Sun et al. [11], Tunç [12]-[16], Wang et al. [17], Zhang et al. [18] and the references in these works.

In particular, motivated by the paper of Phat et al. [8] and those found in the literature, we consider following

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problem with multiple variable retardations:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} D_i(t)x(t-h_i(t)), \qquad (1)$$
$$x(t) = \varphi(t), t \in [-h_2, 0],$$

where $x(t) \in R^n$ is the state, $A, D_i(t) \in M^{n\times n}(i = 1,2,...,n)$, and $\phi(t) \in C^1([-h_2,0],R^n)$ is the initial function with the norm $\|\phi\| = \sup_{-h_2\le t\le 0}\{\|\phi(t)\|,\|\dot{\phi}(t)\|\}$. The time-varying delay functions $h_i(t)$ satisfies

$$0 \le h_{1i} \le h_i(t) \le h_{2i}, (i = 1,2,...,n), t \in \Re^+.$$

The purpose of this article is to give specific conditions, which guarantee α-exponential stability of the system considered. By the defining a suitable Lyapunov-Krasovskii functional, we proof a new theorem on the topic. Our result generalizes and improves the some results found in the literature.

2 Exponential stability

We first express some auxiliary results.

Proposition 2.1 (see Phat et al. [8]). For any symmetric positive definite matrix $N \in M^{n\times n}$ and $a, b \in R^n$, we have

$$\pm a^T b \le a^T Na + b^T N^{-1} b.$$

Proposition 2.2 (see Phat et al. [[8]]. For any symmetric positive definite matrix $M \in R^{n\times n}$, scalar $\gamma > 0$ and vector function $w : [0,\gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left[\int_0^\gamma w(s)ds\right]^T M\left[\int_0^\gamma w(s)ds\right] \le \gamma(\int_0^\gamma w^T(s)Mw(s)ds).$$

Proposition 2.3 (see Phat et al.[8]). Let E, H and F be any constant matrices of appropriate dimensions and $F^T F \le I$. For any $\varepsilon > 0$, we have

$$EFH + H^T F^T E^T \le \varepsilon EE^T + \varepsilon^{-1} H^T H.$$

Proposition 2.4 (Schur complement Lemma-(see Phat et al. [8])). Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \;\; or \;\; \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

Our main result is the following theorem.

Theorem. Let α be a positive constant. If there exist symmetric and positive definite matrices P, Q, R, U and S_i, $i = (1,2,...,5)$, such that the following LMI holds

$$\mu = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & S_4 - S_5\sum_{i=1}^n D_i(t) \\ * & * & * & * & M_{55} \end{bmatrix} < 0, \qquad (2)$$

then the zero solution of system (1) is α-exponential stable, where

$$M_{11} = A^T P + PA + 2\alpha P - \sum_{i=1}^{n}(e^{-2\alpha h_{1i}} + e^{-2\alpha h_{2i}})R$$
$$+ 0.5S_1(I-A) + 0.5(I-A^T)S_1^T + 2Q,$$
$$M_{12} = \sum_{i=1}^{n} e^{-2\alpha h_{1i}} R - S_2 A,$$
$$M_{13} = \sum_{i=1}^{n} e^{-2\alpha h_{2i}} R - S_3 A,$$
$$M_{14} = \sum_{i=1}^{n} PD_i(t) - \sum_{i=1}^{n} S_1 D_i(t) - S_4 A,$$
$$M_{15} = S_1 - S_5 A,$$
$$M_{22} = \sum_{i=1}^{n}[-e^{-2\alpha h_{1i}} Q - e^{-2\alpha h_{2i}} R],$$
$$M_{24} = \sum_{i=1}^{n}[-S_2 D_i(t) + e^{-2\alpha h_{2i}} U],$$
$$M_{33} = -\sum_{i=1}^{n} e^{-2\alpha h_{2i}}[Q + R + U],$$
$$M_{34} = \sum_{i=1}^{n}[-S_3 D_i(t) + e^{-2\alpha h_{2i}} U],$$
$$M_{44} = -\sum_{i=1}^{n}[S_4 D_i(t) + D_i^T(t) S_4^T + e^{-2\alpha h_{2i}} U],$$
$$M_{55} = S_5 + S_5^T + \sum_{i=1}^{n} h_{1i}^2 R + \sum_{i=1}^{n} h_{2i}^2 R + \sum_{i=1}^{n}(h_{2i} - h_{1i})^2 U,$$
$$\lambda_1 = \lambda_{\min}(P),$$
$$\lambda_2 = \lambda_{\max}(P) + 2\sum_{i=1}^{n} h_{2i}\lambda_{\max}(Q) + 2\sum_{i=1}^{n} h_{2i}^2 \lambda_{\max}(R)$$
$$+ \sum_{i=1}^{n}(h_{2i} - h_{1i})^2 \lambda_{\max}(U).$$

Moreover, the solution $x(t,\phi)$ of system (1) satisfies

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_1}{\lambda_2}} e^{-\alpha t}\|\phi\|, \forall t \in R^+.$$

Proof. We define a Lyapunov- Krasovskii functional for system (1) by

$$V(t,x_t) = \sum_{i=1}^{6} V_i,$$

where

$$V_1 = x^T(t)Px(t),$$
$$V_2 = \sum_{i=1}^{n}\int_{t-h_{1i}}^{t} e^{2\alpha(s-t)}x^T(s)Qx(s)ds,$$
$$V_3 = \sum_{i=1}^{n}\int_{t-h_{2i}}^{t} e^{2\alpha(s-t)}x^T(s)Qx(s)ds,$$
$$V_4 = \sum_{i=1}^{n} h_{1i}\int_{-h_{1i}}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)R\dot{x}(\tau)d\tau ds,$$
$$V_5 = \sum_{i=1}^{n} h_{2i}\int_{-h_{2i}}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)R\dot{x}(\tau)d\tau ds,$$
$$V_6 = \sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{-h_{2i}}^{-h_{1i}}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)U\dot{x}(\tau)d\tau ds.$$

It is now easy to verify that

$$\lambda_1\|x(t)\|^2 \le V(t,x_t) \le \lambda_2\|x_t\|^2, \forall t \ge 0. \quad (3)$$

Hence, we omit the details of the calculations.

We now calculate the time derivatives of V_i, $(i = 1,2,...,6)$. Then, it can be followed that

$$\begin{aligned}\dot{V}_1 &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)\\
&= [x^T(t)A^T + \sum_{i=1}^{n} x^T(t-h_i(t))D_i^T(t)]Px(t)\\
&\quad + x^T(t)P[Ax(t) + \sum_{i=1}^{n} D_i(t)x(t-h_i(t))]\\
&= x^T(t)A^TPx(t) + \sum_{i=1}^{n} x^T(t-h_i(t))D_i^T(t)Px(t)\\
&\quad + x^T(t)PAx(t) + x^T(t)P\sum_{i=1}^{n} D_i(t)x(t-h_i(t))\\
&= x^T(t)[A^TP + PA]x(t) + 2x^T(t)P\sum_{i=1}^{n} D_i(t)x(t-h_i(t)),\end{aligned}$$

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^{n}[x^T(t)Qx(t) - e^{-2\alpha h_{1i}}x^T(t-h_{1i})Qx(t-h_{1i})\\
&\quad - 2\alpha\int_{t-h_{1i}}^{t} e^{2\alpha(s-t)}x^T(s)Qx(s)ds]\\
&= x^T(t)Qx(t) - \sum_{i=1}^{n} e^{-2\alpha h_{1i}}x^T(t-h_{1i})Qx(t-h_{1i}) - 2\alpha V_2,\end{aligned}$$

$$\begin{aligned}\dot{V}_3 &= \sum_{i=1}^{n}[x^T(t)Qx(t) - e^{-2\alpha h_{2i}}x^T(t-h_{2i})Qx(t-h_{2i})\\
&\quad - 2\alpha\int_{t-h_{2i}}^{t} e^{2\alpha(s-t)}x^T(s)Qx(s)ds]\\
&= x^T(t)Qx(t) - \sum_{i=1}^{n} e^{-2\alpha h_{2i}}x^T(t-h_{2i})Qx(t-h_{2i}) - 2\alpha V_3,\end{aligned}$$

$$\begin{aligned}\dot{V}_4 &= \sum_{i=1}^{n} h_{1i}\int_{-h_{1i}}^{0}\dot{x}^T(t)R\dot{x}(t)ds - \sum_{i=1}^{n} h_{1i}\int_{-h_{1i}}^{0} e^{2\alpha s}\dot{x}^T(t+s)R\dot{x}(t+s)ds\\
&\quad + \sum_{i=1}^{n} h_{1i}\int_{-h_{1i}}^{0}\int_{t+s}^{t} -2\alpha e^{2\alpha(\tau-t)}\dot{x}^T(\tau)R\dot{x}(\tau)d\tau ds\\
&= \sum_{i=1}^{n} h_{1i}^2\dot{x}^T(t)R\dot{x}(t) - \sum_{i=1}^{n} h_{1i}\int_{t-h_{1i}}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s) - 2\alpha V_4.\end{aligned}$$

By using the mean value theorem for integrals, we have

$$\dot{V}_4 \le \sum_{i=1}^{n} h_{1i}^2\dot{x}^T(t)R\dot{x}(t) - \sum_{i=1}^{n} h_{1i}e^{-2\alpha h_{1i}}\int_{t-h_{1i}}^{t}\dot{x}^T(s)R\dot{x}(s) - 2\alpha V_4.$$

Takin into account Proposition 2.2 and the Newton-Leibniz formula, it follows that

$$\begin{aligned}-\sum_{i=1}^{n} h_{1i}\int_{t-h_{1i}}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\le -\sum_{i=1}^{n}[\int_{t-h_{1i}}^{t}\dot{x}(s)ds]^T R[\int_{t-h_{1i}}^{t}\dot{x}(s)ds]\\
&= -\sum_{i=1}^{n}[x(t)-x(t-h_{1i})]^T R[x(t)-x(t-h_{1i})].\end{aligned}$$

Thus

$$\begin{aligned}\dot{V}_4 &\le \sum_{i=1}^{n} h_{1i}^2\dot{x}^T(t)R\dot{x}(t)\\
&\quad - \sum_{i=1}^{n} e^{-2\alpha h_{1i}}[x(t)-x(t-h_{1i})]^T R[x(t)-x(t-h_{1i})] - 2\alpha V_4\end{aligned}$$

and

$$\begin{aligned}\dot{V}_5 &= \sum_{i=1}^{n} h_{2i}\int_{-h_{2i}}^{0}\dot{x}^T(t)R\dot{x}(t)ds\\
&\quad - \sum_{i=1}^{n} h_{2i}\int_{-h_{2i}}^{0} e^{2\alpha s}\dot{x}^T(t+s)R\dot{x}(t+s)ds\\
&\quad + \sum_{i=1}^{n} h_{2i}\int_{-h_{2i}}^{0}\int_{t+s}^{t} -2\alpha e^{2\alpha(\tau-t)}\dot{x}^T(\tau)R\dot{x}(\tau)d\tau ds\\
&= \sum_{i=1}^{n} h_{2i}^2\dot{x}^T(t)R\dot{x}(t) - \sum_{i=1}^{n} h_{2i}\int_{t-h_{2i}}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)R\dot{x}(s)ds - 2\alpha V_5.\end{aligned}$$

By using mean value theorem for integrals, it is clear that

$$\dot{V}_5 \le \textstyle\sum_{i=1}^{n} h_{2i}^2\dot{x}^T(t)R\dot{x}(t) - \sum_{i=1}^{n} h_{2i}e^{-2\alpha h_{2i}}\int_{t-h_{2i}}^{t}\dot{x}^T(s)R\dot{x}(s)ds - 2\alpha V_5.$$

By means of Proposition 2.2 and the Newton-Leibniz formula, we find

$$\begin{aligned}-\sum_{i=1}^{n} h_{2i}\int_{t-h_{2i}}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\le -\sum_{i=1}^{n}[\int_{t-h_{2i}}^{t}\dot{x}(s)ds]^T R[\int_{t-h_{2i}}^{t}\dot{x}(s)ds]\\
&= -\sum_{i=1}^{n}[x(t)-x(t-h_{2i})]^T R[x(t)-x(t-h_{2i})].\end{aligned}$$

Then, we can see that

$$\begin{aligned}\dot{V}_5 &\le \sum_{i=1}^{n} h_{2i}^2\dot{x}^T(t)R\dot{x}(t)\\
&\quad - \sum_{i=1}^{n} e^{-2\alpha h_{2i}}[x(t)-x(t-h_{2i})]^T R[x(t)-x(t-h_{2i})] - 2\alpha V_5\end{aligned}$$

and

$$\dot V_6 = \sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{-h_{2i}}^{-h_{1i}}\dot x^T(t)U\dot x(t)ds$$
$$-\sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{-h_{2i}}^{-h_{1i}}e^{2\alpha s}\dot x^T(t+s)U\dot x(t+s)ds$$
$$+\sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{-h_{2i}}^{-h_{1i}}\int_{t+s}^{t}$$
$$-2\alpha e^{2\alpha(\tau-t)}\dot x^T(\tau)U\dot x(\tau)d\tau ds$$
$$=\sum_{i=1}^{n}(h_{2i}-h_{1i})^2\dot x^T(t)U\dot x(t)$$
$$-\sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{t-h_{2i}}^{t-h_{1i}}e^{2\alpha(s-t)}\dot x^T(s)U\dot x(s)ds-2\alpha V_6.$$

By using mean value theorem for integrals, it is clear that

$$\dot V_6 \le \sum_{i=1}^{n}(h_{2i}-h_{1i})^2\dot x^T(t)U\dot x(t)$$
$$-\sum_{i=1}^{n}(h_{2i}-h_{1i})e^{-2\alpha h_{2i}}\int_{t-h_{2i}}^{t-h_{1i}}\dot x^T(s)U\dot x(s)ds-2\alpha V_6.$$

In view of Proposition 2.2 and the Newton-Leibniz formula, it follows that

$$\sum_{i=1}^{n}\int_{t-h_{2i}}^{t-h_{1i}}\dot x^T(s)U\dot x(s)ds$$
$$=\sum_{i=1}^{n}\int_{t-h_{2i}}^{t-h_i(t)}\dot x^T(s)U\dot x(s)ds+\sum_{i=1}^{n}\int_{t-h_i(t)}^{t-h_{1i}}\dot x^T(s)U\dot x(s)ds$$

and

$$\sum_{i=1}^{n}[h_{2i}-h_i(t)]\int_{t-h_{2i}}^{t-h_i(t)}\dot x^T(s)U\dot x(s)ds \ge \sum_{i=1}^{n}\{[\int_{t-h_{2i}}^{t-h_i(t)}\dot x(s)ds]^T$$
$$\times U[\int_{t-h_{2i}}^{t-h_i(t)}\dot x(s)ds]\}$$
$$=\sum_{i=1}^{n}[x(t-h_i(t))-x(t-h_{2i})]^T$$
$$\times U[x(t-h_i(t))-x(t-h_{2i})].$$

Since $0\le h_{1i}\le h_i(t)\le h_{2i}$, $(i=1,2,...,n)$, then we have

$$-\sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{t-h_{2i}}^{t-h_i(t)}\dot x^T(s)U\dot x(s)ds$$
$$\le -\sum_{i=1}^{n}[x(t-h_i(t))-x(t-h_{2i})]^T U[x(t-h_i(t))-x(t-h_{2i})].$$

Similarly, it follows that

$$-\sum_{i=1}^{n}(h_{2i}-h_{1i})\int_{t-h_i(t)}^{t-h_{1i}}\dot x^T(s)U\dot x(s)ds \le -\sum_{i=1}^{n}[x(t-h_{1i})$$
$$-x(t-h_i(t))]^T U[x(t-h_{1i})-x(t-h_i(t))].$$

Then

$$\dot V_6 \le (h_{2i}-h_{1i})^2\dot x^T(t)U\dot x(t)$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}[x(t-h_i(t))-x(t-h_{2i})]^T$$
$$\times U[x(t-h_i(t))-x(t-h_{2i})]$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}[x(t-h_{1i})-x(t-h_i(t))]^T$$
$$\times U[x(t-h_{1i})-x(t-h_i(t))]-2\alpha V_6.$$

Hence, it is obvious that

$$\dot V(.)+2\alpha V(.) \le x^T(t)[A^TP+PA+2\alpha P+2Q]x(t)$$
$$+2x^T(t)P\sum_{i=1}^{n}D_i(t)x(t-h_i(t))$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{1i}}x^T(t-h_{1i})Qx(t-h_{1i})$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_{2i})Qx(t-h_{2i})$$
$$+\sum_{i=1}^{n}\dot x^T(t)[(h_{1i}^2+h_{2i}^2)R+(h_{2i}-h_{1i})^2U]\dot x(t)$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{1i}}[x(t)-x(t-h_{1i})]^TR[x(t)-x(t-h_{1i})]$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}[x(t)-x(t-h_{2i})]^TR[x(t)-x(t-h_{2i}))]$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}[x(t-h_i(t))-x(t-h_{2i})]^T$$
$$\times U[x(t-h_i(t))-x(t-h_{2i}]$$
$$-\sum_{i=1}^{n}e^{-2\alpha h_{2i}}[x(t-h_{1i})-x(t-h_i(t))]^T$$
$$\times U[x(t-h_{1i})-x(t-h_i(t)]. \quad (4)$$

By using the relation

$$\dot x(t)-Ax(t)-\sum_{i=1}^{n}D_i(t)x(t-h_i(t))=0,$$

we have

$$2x^T(t)S_1\dot x(t)-2x^T(t)S_1Ax(t)$$
$$-2x^T(t)S_1\sum_{i=1}^{n}D_i(t)x(t-h_i(t))=0,$$
$$\sum_{i=1}^{n}2x^T(t-h_{1i})S_2\dot x(t)-\sum_{i=1}^{n}2x^T(t-h_{1i})S_2Ax(t)$$
$$-\sum_{i=1}^{n}2x^T(t-h_{1i})S_2D_i(t)\dot x(t-h_i(t))=0,$$

$$\sum_{i=1}^{n} 2x^T(t-h_{2i})S_3\dot{x}(t) - \sum_{i=1}^{n} 2x^T(t-h_{2i})S_3Ax(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_{2i})S_3D_i(t)x(t-h_i(t)) = 0,$$
$$\sum_{i=1}^{n} 2x^T(t-h_i(t))S_4\dot{x}(t) - \sum_{i=1}^{n} 2x^T(t-h_i(t))S_4Ax(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_i(t))S_4D_i(t)x(t-h_i(t)) = 0,$$
$$2\dot{x}^T(t)S_5\dot{x}(t) - 2\dot{x}^T(t)S_5Ax(t)$$
$$-2x^T(t)S_5\sum_{i=1}^{n} D_i(t)x(t-h_i(t)) = 0. \quad (5)$$

In view of (4) and (5), we can obtain

$$\dot{V}(.) + 2\alpha V(.) \leq x^T(t)[A^TP + PA + 2\alpha P + 2Q]x(t)$$
$$+2x^T(t)P\sum_{i=1}^{n} D_i(t)x(t-h_i(t))$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{1i}} x^T(t-h_{1i})Qx(t-h_{1i})$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{2i}} x^T(t-h_{2i})Qx(t-h_{2i})$$
$$+\sum_{i=1}^{n} \dot{x}^T(t)[(h_{1i}^2 + h_{2i}^2)R + (h_{2i}-h_{1i})^2U]\dot{x}(t)$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{1i}} [x(t) - x(t-h_{1i})]^T$$
$$\times R[x(t) - x(t-h_{1i})]$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t) - x(t-h_{2i})]^T$$
$$\times R[x(t) - x(t-h_{2i})]$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t-h_i(t)) - x(t-h_{2i})]^T$$
$$\times U[x(t-h_i(t)) - x(t-h_{2i})]$$
$$-\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t-h_{1i}) - x(t-h_i(t))]^T$$
$$\times U[x(t-h_{1i}) - x(t-h_i(t))]$$
$$+2x^T(t)S_1\dot{x}(t) - 2x^T(t)S_1Ax(t)$$
$$-2x^T(t)S_1\sum_{i=1}^{n} D_i(t)x(t-h_i(t))$$
$$+\sum_{i=1}^{n} 2x^T(t-h_{1i})S_2\dot{x}(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_{1i})S_2Ax(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_{1i})S_2D_i(t)x(t-h_i(t))$$
$$+\sum_{i=1}^{n} 2x^T(t-h_{2i})S_3\dot{x}(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_{2i})S_3Ax(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_{2i})S_3D_i(t)x(t-h_i(t))$$
$$+\sum_{i=1}^{n} 2x^T(t-h_i(t))S_4\dot{x}(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_i(t))S_4Ax(t)$$
$$-\sum_{i=1}^{n} 2x^T(t-h_i(t))S_4D_i(t)x(t-h_i(t))$$
$$+2\dot{x}^T(t)S_5\dot{x}(t) - 2\dot{x}^T(t)S_5Ax(t)$$
$$-2\dot{x}^T(t)S_5\sum_{i=1}^{n} D_i(t)x(t-h_i(t)).$$

If we use the following estimates

$$\sum_{i=1}^{n} e^{-2\alpha h_{1i}} [x(t) - x(t-h_{1i})]^T R[x(t) - x(t-h_{1i})]$$
$$= \sum_{i=1}^{n} e^{-2\alpha h_{1i}} [x^T(t)Rx(t) - 2x^T(t)Rx(t-h_{1i})$$
$$+x^T(t-h_{1i})Rx(t-h_{1i})],$$
$$\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t) - x(t-h_{2i})]^T R[x(t) - x(t-h_{2i})]$$
$$= \sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x^T(t)Rx(t) - 2x^T(t)Rx(t-h_{2i})$$
$$+x^T(t-h_{2i})Rx(t-h_{2i})],$$
$$\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t-h_i(t)) - x(t-h_{2i})]^T$$
$$\times U[x(t-h_i(t)) - x(t-h_{2i})]$$
$$= \sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x^T(t-h_i(t))Ux(t-h_i(t))$$
$$-2x^T(t-h_i(t))Ux(t-h_{2i}) + x^T(t-h_{2i})Ux(t-h_{2i})]$$

and

$$\sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x(t-h_{1i}) - x(t-h_i(t))]^T$$
$$\times U[x(t-h_{1i}) - x(t-h_i(t))]$$
$$= \sum_{i=1}^{n} e^{-2\alpha h_{2i}} [x^T(t-h_{1i})Ux(t-h_{1i})$$
$$-2x^T(t-h_{1i})Ux(t-h_i(t)) + x^T(t-h_i(t))Ux(t-h_i(t))],$$

then we get

$$
\begin{aligned}
\dot{V}(.)+2\alpha V(.) \le\, & x^T(t)[A^TP+PA+2\alpha P+2Q]x(t) \\
& +2x^T(t)P\sum_{i=1}^{n}D_i(t)x(t-h_i(t)) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{1i}}x^T(t-h_{1i})Qx(t-h_{1i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_{2i})Qx(t-h_{2i}) \\
& +\sum_{i=1}^{n}\dot{x}^T(t)[(h_{1i}^2+h_{2i}^2)R+(h_{2i}-h_{1i})^2U]\dot{x}(t) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{1i}}x^T(t)Rx(t) \\
& +\sum_{i=1}^{n}2e^{-2\alpha h_{1i}}x^T(t)Rx(t-h_{1i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{1i}}x^T(t-h_{1i})Rx(t-h_{1i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t)Rx(t) \\
& +\sum_{i=1}^{n}2e^{-2\alpha h_{2i}}x^T(t)Rx(t-h_{2i}) \\
& -\sum_{i=1}^{n}2e^{-2\alpha h_{2i}}x^T(t-h_{2i})Rx(t-h_{2i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_i(t))Ux(t-h_i(t)) \\
& +\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_i(t))Ux(t-h_{2i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_{2i})Ux(t-h_{2i}) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_{1i})Ux(t-h_{1i}) \\
& +\sum_{i=1}^{n}2e^{-2\alpha h_{2i}}x^T(t-h_{1i})Ux(t-h_i(t)) \\
& -\sum_{i=1}^{n}e^{-2\alpha h_{2i}}x^T(t-h_i(t))Ux(t-h_i(t)) \\
& +2x^T(t)S_1\dot{x}(t)-2x^T(t)S_1Ax(t) \\
& -2x^T(t)S_1\sum_{i=1}^{n}D_i(t)x(t-h_i(t)) \\
& +\sum_{i=1}^{n}2x^T(t-h_{1i})S_2\dot{x}(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_{1i})S_2Ax(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_{1i})S_2D_i(t)x(t-h_i(t)) \\
& +\sum_{i=1}^{n}2x^T(t-h_{2i})S_3\dot{x}(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_{2i})S_3Ax(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_{2i})S_3D_i(t)x(t-h_i(t)) \\
& +\sum_{i=1}^{n}2x^T(t-h_i(t))S_4\dot{x}(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_i(t))S_4Ax(t) \\
& -\sum_{i=1}^{n}2x^T(t-h_i(t))S_4D_i(t)x(t-h_i(t)) \\
& +2\dot{x}^T(t)S_5\dot{x}(t)-2\dot{x}^T(t)S_5Ax(t) \\
& -2\dot{x}^T(t)S_5\sum_{i=1}^{n}D_i(t)x(t-h_i(t)).
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\dot{V}(.)+2\alpha V(.) \le\, & x^T(t)[A^TP+PA+2\alpha P \\
& -\sum_{i=1}^{n}(e^{-2\alpha h_{1i}}+e^{-2\alpha h_{2i}})R+0.5S_1(I-A) \\
& +0.5(I-A^T)S_1^T+2Q]x(t) \\
& +2x^T(t)[\sum_{i=1}^{n}e^{-2\alpha h_{1i}}R-S_2A]\sum_{i=1}^{n}x(t-h_{1i}) \\
& +2x^T(t)[\sum_{i=1}^{n}e^{-2\alpha h_{2i}}R-S_3A]\sum_{i=1}^{n}x(t-h_{2i}) \\
& +2x^T(t)[\sum_{i=1}^{n}PD_i(t)-\sum_{i=1}^{n}S_1D_i(t)-S_4A] \\
& \times\sum_{i=1}^{n}x(t-h_i(t)) \\
& +2\sum_{i=1}^{n}x^T(t)[S_1-S_5A]\dot{x}(t) \\
& +\sum_{i=1}^{n}x^T(t-h_{1i})[-e^{-2\alpha h_{1i}}Q-e^{-2\alpha h_{2i}}R]x(t-h_{1i}) \\
& +2\sum_{i=1}^{n}x^T(t-h_{1i})[-\sum_{i=1}^{n}S_2D_i(t)+e^{-2\alpha h_{2i}}U] \\
& \times x(t-h_i(t)) \\
& +2\sum_{i=1}^{n}x^T(t-h_{1i})S_2\dot{x}(t) \\
& +\sum_{i=1}^{n}x^T(t-h_{2i})[-e^{-2\alpha h_{2i}}Q-e^{-2\alpha h_{2i}}R-e^{-2\alpha h_{2i}}U] \\
& \times x(t-h_{2i}) \\
& +2\sum_{i=1}^{n}x^T(t-h_{2i}))[-\sum_{i=1}^{n}S_3D_i(t)+e^{-2\alpha h_{2i}}U]
\end{aligned}
$$

$$
\begin{aligned}
& \times x(t-h_i(t)) \\
& +2\sum_{i=1}^{n} x^T(t-h_{2i})S_3\dot{x}(t) \\
& +\sum_{i=1}^{n} x^T(t-h_i(t))[-\sum_{i=1}^{n} 2S_4D_i(t)-e^{-2\alpha h_{2i}}U] \\
& \times x(t-h_i(t)) \\
& +2\sum_{i=1}^{n} x^T(t-h_i(t))[S_4-\sum_{i=1}^{n} S_5D_i(t)]\dot{x}(t) \\
& \dot{x}^T(t)[S_5+S_5^T+\sum_{i=1}^{n} h_{1i}^2R+\sum_{i=1}^{n} h_{2i}^2R \\
& +\sum_{i=1}^{n}(h_{2i}-h_{1i})^2U]\dot{x}(t) \\
& =\xi^T\mu\xi(t),
\end{aligned}
$$

where

$$\xi^T(t)=[x(t),\sum_{i=1}^{n} x(t-h_{1i}),\sum_{i=1}^{n} x(t-h_{2i}),\sum_{i=1}^{n} x(t-h_i(t)),\dot{x}(t)].$$

In view of condition (2), we obtain

$$\dot{V}(t,x_t)\leq -2\alpha V(t,x_t), \forall t\in R^+.$$

Integrating both sides of (6) from 0 to t, we have

$$V(t,x_t)\leq V(\phi)e^{-2\alpha t}, \forall t\in R^+.$$

Furthermore, taking into account estimates (3), we obtain

$$\lambda_1\|x(t,\phi)\|^2\leq V(x_t)\leq V(\phi)e^{-2\alpha t}\leq \lambda_2 e^{-2\alpha t}\|\phi\|^2$$

so that

$$\|x(t,\phi)\|\leq \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|, t\in R^+,$$

which completes the proof of the theorem

3 Conclusion

A kind of linear systems of differential equations of the first order with variable time-lag was considered. The exponential stability of zero solution of the system considered was discussed by a suitable Lyapunov-Krasovskii functional and linear matrix inequalities (LMIs). By means of the result obtained, it was improved and extended some results found in the literature. Our aim is to do a contribution to the relevant literature.

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