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Analytical Evaluation of the Integral of Any Order Polynomials on Tetrahedral Regions

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Abstract: This paper presents an analytical method to set out the integral of any polynomial function $f(x,y,z)$ on a tetrahedral region T by using its four vertexes. The method uses a coordinate transformation which involves the four vertexes of the tetrahedron, whose Jacobian is simple. The last integral is not difficult to solve given that recurrence formula is very simple, furthermore we have developed an algorithm which can evaluate the integral when integrating function is generated by several multiplications of polynomials without necessity of develop the products. This method can be used in finite element method because the most functions involved in this method are polynomial ones. The method here presented is faster than Gauss-Legendre quadrature or n order if the amount of monomials present on $f(x,y,z)$ is least than n^3.

Keywords: general tetrahedron, analytical integration, polynomial functions, finite element methods, algorithm

1 Introduction

The integration on a tetrahedron is very useful in the finite element method (FEM) [2]. To solve this kind of integrals, there are many numerical methods to approximate triple integrals [3], [4]. However, the analytical methods of integration of certain functions can give better results than those obtained with the numerical methods, since the analytical ones give more accurate results. Although the analytical formulas can be larger, it is worth to get the more accurate values when the finite element method involves time, since some of these integrals must be calculated just once. Usually, many problems that use the FEM posess a complex geometry which needs an unstructured grids in which case the tetrahedron cell is perfect because it lets a fully unstructured spatial discretization by the use of a high-order nodal basis[5]. While it is true that analytical formulas can be really long, the variable changes proposed here has a really short Jacobian transformation as you can see in equation (11) where $\mathbf{r}_{jk} \cdot \mathbf{r}_{ki} \times \mathbf{r}_{il} \beta^2$ is just a constant times one variable to second degree. So, it can be easily used with polynomial functions which are commons in the FEM [6] and makes this method useful for this kind of problems.

There are several articles about numerical integration using Gauss quadrature over some plane surfaces, rectangle and square region [7], triangle regions [8] and polygonal ones [9] and for integrals on a standard tetrahedron a study found in [10] shows a method to solve the integrals using a Gauss Legendre-Gauss Jacobi quadrature rules. Although Gauss Legendre quadrature when uses a Legendre polynom of n order gives exact result for any polynomial integrate function of degree less than $2n$ [11], for higher order the results become an approximation. However the method that are presented in this paper gives the exact value for an arbitrary tetrahedral region and for any polynomial integrating function and only needs the coordinate vertexes. This paper includes an algorithm that helps to use the method more easily. This is necessary due to the polynomial function comes from the products of other more simple polynomial terms, therefore the amount of terms can be so huge, which could make that our method be impractical to use.

2 Region generation method

First we present the method to determine the point inside of the triangle. Let us consider a triangle with vertexes

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which are denoted by i, j and k with $\mathbf{r}_i$, $\mathbf{r}_j$ and $\mathbf{r}_k$ being the coordinates of each one respectively. The angle between the edges $\overline{ij}$ and $\overline{ik}$ is denoted by θ_i, between the edges $\overline{kj}$ and $\overline{ki}$ is θ_k, and the angle between the edges $\overline{ij}$ and $\overline{jk}$ by θ_j as is shown in the Figure 1. The vector $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ it is the vector leading from the point i toward point j, the same way for the vectors $\mathbf{r}_{ik}$, $\mathbf{r}_{jk}$.

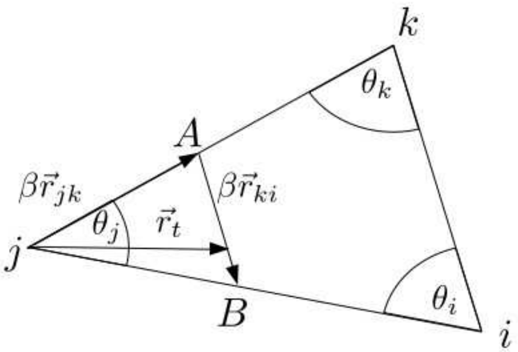


Fig. 1: Points inside of the triangle

Then any point on edge $\overline{jk}$ can be generated by $\mathbf{r}_j + \beta\mathbf{r}_{jk}$ where $0 \le \beta \le 1$, if we draw a straight line from this point to the egde $\overline{ji}$ such that jAB triangle be similar to triangle jik, then one point on the edge $\overline{AB}$ can be generated by $\mathbf{r}_j + \beta\mathbf{r}_{jk} + \lambda\beta\mathbf{r}_{ki}$ where $0 \le \lambda \le 1$. Therefore, the points $\mathbf{r}_t$ inside of the triangular region can be generated by the formula

$$\mathbf{r}_t = \mathbf{r}_j + \beta\mathbf{r}_{jk} + \alpha\mathbf{r}_{ki} \text{ for } 0 \le \beta \le 1,\ 0 \le \alpha \le \beta. \quad (1)$$

Any point within the tetrahedron can be represented by dividing the tetrahedral region in thin triangles ABC as is shown in Figure 2. Then, to generate every point $\mathbf{r}_{tABC}$ inside of the triangle ABC we can start writing the equation (1) as

$$\mathbf{r}_{tABC} = \mathbf{r}_A + \alpha\mathbf{r}_{AB} + \gamma\mathbf{r}_{BC} \text{ for } 0 \le \alpha \le 1,\ 0 \le \gamma \le \alpha \quad (2)$$

where

$$\mathbf{r}_A = \mathbf{r}_j + \beta\mathbf{r}_{jk}, \quad (3)$$
$$\mathbf{r}_C = \mathbf{r}_j + \beta\mathbf{r}_{jk} + \beta\mathbf{r}_{kl} = \mathbf{r}_j + \beta\mathbf{r}_{jl}, \quad (4)$$
$$\mathbf{r}_B = \mathbf{r}_j + \beta\mathbf{r}_{jk} + \beta\mathbf{r}_{ki} = \mathbf{r}_j + \beta\mathbf{r}_{ji}. \quad (5)$$

Therefore

$$\mathbf{r}_{AB} = (\mathbf{r}_B - \mathbf{r}_A) = \beta\mathbf{r}_{ki}, \quad (6)$$
$$\mathbf{r}_{BC} = (\mathbf{r}_C - \mathbf{r}_B) = \beta\mathbf{r}_{il}. \quad (7)$$
$$(8)$$

Hence, for the whole tetrahedron we have:

$$x\hat{x} + y\hat{y} + z\hat{z} = \mathbf{r}_T = \mathbf{r}_j + \beta\mathbf{r}_{jk} + \alpha\beta\mathbf{r}_{ki} + \gamma\beta\mathbf{r}_{il}, \quad (9)$$

where $0 \le \beta \le 1$, $0 \le \alpha \le 1$, $0 \le \gamma \le \alpha$, $\hat{x}$, $\hat{y}$, $\hat{z}$ are the Cartesian unit vectors along x, y and z axes respectively and $\mathbf{r}_T$ represents each point inside of the tetrahedron volume. Now we are going to write the change of variable

$$\begin{aligned} x &= r_{jx} + \beta r_{jkx} + \alpha\beta r_{kix} + \gamma\beta r_{ilx} = x(\beta,\alpha,\gamma), \\ y &= r_{jy} + \beta r_{jky} + \alpha\beta r_{kiy} + \gamma\beta r_{ily} = y(\beta,\alpha,\gamma), \\ z &= r_{jz} + \beta r_{jkz} + \alpha\beta r_{kiz} + \gamma\beta r_{ilz} = z(\beta,\alpha,\gamma), \end{aligned} \quad (10)$$

where the subscripts x, y and z, represent the components of each vector along the x, y and z axes, respectively.

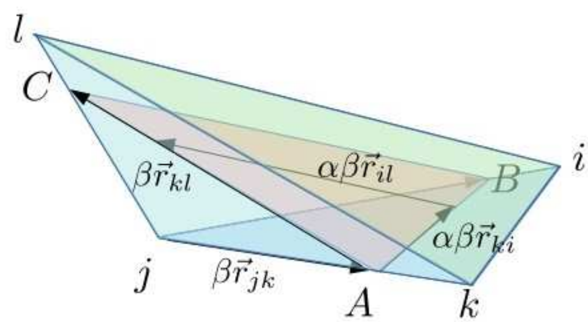


Fig. 2: Tetrahedron region generation.

3 Setting up the integral

The Jacobian of the transformation (10) can be written as follow:

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\beta,\alpha,\gamma)} &= \frac{\partial\mathbf{r}}{\partial\beta} \cdot \frac{\partial\mathbf{r}}{\partial\alpha} \times \frac{\partial\mathbf{r}}{\partial\gamma} \\ &= (\mathbf{r}_{jk} + \alpha\mathbf{r}_{ki} + \gamma\mathbf{r}_{il}) \cdot \beta\mathbf{r}_{ki} \times \beta\mathbf{r}_{il} \\ &= \beta^2\mathbf{r}_{jk} \cdot \mathbf{r}_{ki} \times \mathbf{r}_{il}. \end{aligned} \quad (11)$$

By the change of variable theorem, the integral of any continuos function $f(x,y,z)$ on the tetrahedron region T can be written as:

$$\iiint_T f(x,y,z)dxdydz = \mathbf{r}_{jk} \cdot (\mathbf{r}_{ki} \times \mathbf{r}_{il}) \int_0^1 \int_0^1 \int_0^\alpha H(\beta,\alpha,\gamma)\beta^2 d\gamma d\alpha d\beta, \quad (12)$$

where $H(\beta,\alpha,\gamma) = f(x(\beta,\alpha,\gamma),y(\beta,\alpha,\gamma),z(\beta,\alpha,\gamma))$. Note that if $f(x,y,z) = 1$ then, the volume of the tetrahedron is

$$V_T = \mathbf{r}_{jk} \cdot (\mathbf{r}_{ki} \times \mathbf{r}_{il}) \int_0^1 \int_0^1 \int_0^\alpha \beta^2 d\gamma d\alpha d\beta = \frac{1}{6}\mathbf{r}_{jk} \cdot (\mathbf{r}_{ki} \times \mathbf{r}_{il}), \quad (13)$$

which is the formula to find the volume for a tetrahedron given in [12]. If we set $d\Omega_e = \mathbf{r}_{jk} \cdot (\mathbf{r}_{ki} \times \mathbf{r}_{il})\beta^2 d\gamma d\alpha d\beta$

then we can deduce a really simple formula to find analytically the integral of a monomial function on a tetrahedron:

$$\int_0^1\int_0^1\int_0^\alpha \beta^p\alpha^n\gamma^m d\Omega_e = \frac{6V}{(m+1)(m+n+2)(p+3)}, \tag{14}$$

where $6V = \mathbf{r}_{jk}\cdot(\mathbf{r}_{ki}\times\mathbf{r}_{il})$. Thus, V is tetrahedron's volume measure. The method proposed here is better than one with tetrahedral natural coordinates $(\xi_1,\xi_2,\xi_3,\xi_4)$ because the former has only three coordinates and the monomial integral is:

$$\int_{\Omega_e} \xi_1^i,\xi_2^j,\xi_3^k,\xi_4^l d\Omega_e = 6V\frac{i!j!k!l!}{(i+j+k+l+3)!}, \tag{15}$$

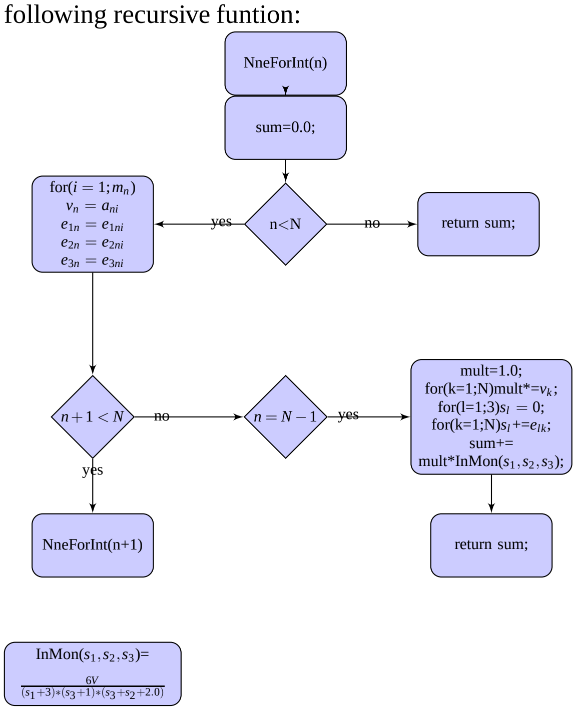
where i, j, k and l are non negative integers, Ω_e is the tetrahedron domain. So, the formula (14) is more simple than (15). It is important to increase the computational performance.

4 Algorithm

The proposed algorithm can solve the following integral:

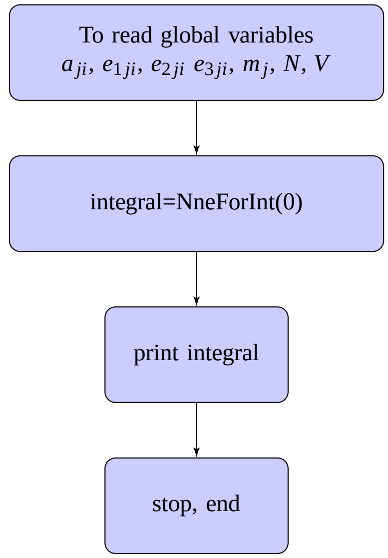
$$\int_0^1\int_0^1\int_0^\alpha \prod_{j=1}^{N}\sum_{i=1}^{m_j} a_{ji}\beta^{e_{1ji}}\gamma^{e_{2ji}}\alpha^{e_{3ji}}d\Omega_e, \tag{16}$$

where a_{ji} represents the coefficient of term i belong to j factor of which exponents of the corresponding variables β, γ and α are e_{1ji}, e_{2ji} and e_{3ji} respectively, m_j is the amount of terms in factor j. In order to evaluate this integral, it is considered an algorithm which uses the following recursive funtion:



Here N is the amount of factors. This recursive function was necessary to keep variable the amount of factors because for each one of this factors, we need to add a nested loop FOR.

Then the main program just need to call this function:



The algorithm for Gauss-Legendre applied to our integral has following form:

$$\int_0^1\int_0^1\int_0^\alpha H(\beta,\alpha,\gamma)\beta^2 d\gamma d\alpha d\beta =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} H\left(\frac{r_{in}+1}{2},\frac{r_{jn}+1}{2},\frac{r_{jn}+1}{4}(r_{kn}+1)\right)\frac{(r_{in}+1)^2}{4}\frac{r_{jn}+1}{16}C_{kn}C_{jn}C_{in} \tag{17}$$

where n is the grade of Legendre polynomial, $c_{in} = \int_{-1}^{1}\prod_{j=1,j\neq i}^{n}\frac{x-r_{jn}}{r_{in}-r_{jn}}dx$ and r_{in} represents i-root of the Legendre polynomial of n order.

5 Results

In this part we present a comparison between Gauss-Legendre quadrature and our method, in order to compare with an exact value we solve integral for this simple case,

$$\int_0^1\int_0^1\int_0^\alpha \beta^m\beta^2 d\gamma d\alpha d\beta = \frac{1}{2(m+3)} \tag{18}$$

Then we are going to show several results for choosing different values of m, results are shown in table 18. We choose $n=5$ for Gauss-Legendre quadrature, then the result will be exact for polynomial under 9 grade.

Then the results shown in table 1 that our method is faster, it can be explain due to our method only has to operate one term while Gauss method has to evaluate the integral function $5^3 = 125$ times (to see equation 17).

Table 1: Gauss-Legendre quadrature vs Our Analytical Method (Average of computing time taken=ACT)

m	Exact Value	GL quadrature		New method	
		% Error	ACT(μs)	% Error	ACT(μs)
5	$\frac{1}{16}$	0	85	0	3
8	$\frac{1}{22}$	0	98	0	3
13	$\frac{1}{32}$	0.3	98	0	3
19	$\frac{1}{32}$	2.8	94	0	3

Then in order to increase number of operations of our method, we are going to calculated this integral:

$$\int_0^1 \int_0^1 \int_0^{\alpha} (\beta+\alpha+\gamma)^p(\beta+2\alpha-\gamma)^q \beta^2 d\gamma d\alpha d\beta \quad (19)$$

Then when the amount of monomial terms starting to

Table 2: Gauss-Legendre quadrature vs Our Analytical Method (2) (Average of computing time taken=ACT)

p	q	Exact Value	GL quadrature		New method	
			% Error	ACT(μs)	% Error	ACT(μs)
2	2	$\frac{1249}{630}$	0	52	0	25
4	5	$\frac{30202147}{388080}$	3.38×10^{-5}	170	0	7856
1	4	$\frac{106591}{25200}$	0	125	0	95

exceed 125 the computing time is also increased. However, even though this number is reached our method can be faster due to the monomial terms are evaluated in a shorter time than integral function as is shown on table 2 in case when p=1 and q=4.

6 Conclusions

This method is simple enough when it works with polynomial functions even when is compared with a numerical method, as it can be, Gauss quadrature rule. This method always provides an exact result of the integral while the Gauss-Legendre method has to increase the order of the Legendre function in order to keep the exactness of results. Although it is necessary to solve as many integrals as terms the polynomial function has got, the given algorithm can do it easily and if amount of terms is less than n^3 where n is the order of Legendre polynomial our method is faster. Furthermore, the way to cover the whole volume of the tetrahedron is also very simple. Then, this method can be useful for a general finite element method when the integral function is an any order polynomial type. Naturally, when the function which is going to be integrated is not of the polynomial type, this method is not applicable, in this case we have to use other numerical method. However the FEM always is going to have two kind of functions to integrate. The first one are polynomial functions and the second ones are functions which can be not polynomial type, this depends on the form of excitement functions or field.

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