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Non-Gaussian White Noise Functional Solutions of χ-Wick-Type Stochastic KdV Equations

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Abstract: In this paper, KdV equations with variable coefficients and Wick-type stochastic KdV equations are investigated. White noise functional solutions are shown by Hermite transform, homogeneous balance principle and F-expansion method. By means of the direct connection between the theory of hypercomplex systems and white noise analysis, we setup a full framework to study the stochastic partial differential equations with non-Gaussian parameters. Using this framework and F-expansion method, we present multiple families of exact and stochastic travelling wave solutions for the variable coefficients KdV equations and the stochastic KdV equations with non-Gaussian parameters, respectively. These solutions include functional solutions of Jacobi elliptic functions (JEFs), trigonometric and hyperbolic types.

Keywords: KdV equations; F-expansion method; Non-Gaussian space; Wick product; Hermite transform; White noise analysis.

1 Introduction

Let ρ be a non-Gaussian probability measure on a locally compact space Q. Consider the quasinuclear rigging [3]

$$H_{-q}^{\chi} \supseteq L_2(Q, d\rho(x)) \supseteq H_q^{\chi}$$

where the zero space $L_2(Q, d\rho(x))$ is the space of square integrable functions defined on a commutative normal hypercomplex system $L_1(Q, dm(x))$ with basis Q and multiplicative measure m [2,36]. The spaces H_q^{χ} and H_{-q}^{χ} are the spaces of test and generalized functions respectively, which are constructed via the Delsarte characters $\chi_n \in C(Q)$.

This paper is mainly concerned with Wick-type stochastic KdV equations with non-Gaussian parameters:

$$U_t + \Psi_1(t)\Diamond_\chi U \Diamond_\chi U_x + \Psi_2(t)\Diamond_\chi U_{xxx} = 0, \quad (1)$$

where "$\Diamond_\chi$" is the χ-Wick product on H_{-q}^{χ}, which is defined in the next section, and Ψ_1 and Ψ_2 are non-Gaussian H_{-q}^{χ}- 1-valued functions. Moreover, when the χ-Wick product "$\Diamond_\chi$" is replaced by the ordinary product in Eq.(1), we obtain the variable coefficients KdV equations [4,5]:

$$u_t + \psi_1(t)uu_x + \psi_2(t)u_{xxx} = 0, \quad (2)$$

where ψ_1 and ψ_2 are bounded measurable or integrable functions on $\mathbb{R}_+$.

The KdV equation is one of the essential nonlinear equations in mathematics and physics [4,5,10]. Therefore, it is important to find solutions for this equation. The KdV has many applications in many branches of nonlinear science[4]. This equation is an important mathematical model arising in many different physical contexts to describe many phenomena which are simultaneously involved in nonlinearity, dissipation, dispersion, and instability, especially at the description of turbulence processes.

Since the solutions of KdV equation possess their actual physical application, which is the reason why so many methods, such as Exp-function method proposed by He and Wu,[21] As is well known, solitons are universal phenomenon, appearing in a great array of contexts such as, for example, nonlinear optics, plasma physics, fluid dynamics, semiconductors and many other systems [31, 34,28]. Studying of nonlinear evolution equations modeling various physical phenomena has played a significant role in many scientific applications such as water waves, nonlinear optics, plasma physics and solid state physics [33].

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The KdV equations (2) describes the propagation of nonlinear wave. Moreover, if the problem is considered in a non-Gaussian stochastic environment, we can get non-Gaussian stochastic KdV equations. In order to give the exact stochastic solutions of the non-Gaussian stochastic KdV equations, we only consider this problem in a non-Gaussian white noise environment, that is, we will study the variable coefficients stochastic KdV equations (1).

It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in a stochastic environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. Wadati [30] first answered the interesting question, “How does external noise affect the motion of solitons?” and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. The Cauchy problems associated with stochastic partial differential equations (SPDEs) was discussed by many authors, e.g., de Bouard and Debussche [25,4,5], Debussche and Printems [6,7] and Ghany and Hyder [15]. By means of white noise functional analysis [23], Ghany et al. [11,12,13,14,16,17,18,19] studied more intensely the white noise functional solutions for some nonlinear SPDEs. Furthermore, Okb El Bab, Ghany, Hyder and Zakarya [20,1], studied some important subjects related to a construction of non-Gaussian white noise analysis using the theory of hypercomplex systems.

The goal of this paper is to explore exact and stochastic travelling wave solutions for the KdV equations (2) and the stochastic KdV equation (1), respectively. Firstly, we develop a non-Gaussian Wick calculus based on the theory of hypecomplex systems $L_1(Q,dm(x))$. That is, we use the Delsarte characters $\chi_n(x)$ to introduce a χ-Wick product and χ-Hermite transform on the space of generalized functions H^{χ}_{-q} (with the zero space $L_2(Q,dm(x))$) and discuss their properties. Secondly, by means of the usual properties of complex analytic functions, we setup a framework to study the SPDEs with non-Gaussian parameters. Finally, we apply this framework and the F-expansion method [18] to give a multiple families of exact and stochastic travelling wave solutions for the KdV equation (2) and the stochastic KdV equations (1), respectively. The resultant solutions include functional solutions of JEFs, trigonometric and hyperbolic types. Moreover, we support our results by detailed example.

This paper is organized as follows: Section 2 is devoted to study the SPDEs with non-Gaussian parameters. In Section 3, we apply the results obtained in Section 2 and F-expansion method to give the exact and stochastic travelling wave solutions for the KdV equations (2). In Section 4, we obtain exact non-Gaussian white noise functional solutions for Wick-type stochastic KdV equations (1). The last section is devoted to a summary and discussion.

2 SPDEs with Non-Gaussian Parameters

In the Gaussian case, if the objects of a differential equation are regarded as $(\mathscr{S})_{-1}$-valued ($(\mathscr{S})_{-1}$ is the Kondratiev space of stochastic distributions constructed upon Gaussian measure), we often obtain a more realistic mathematical model of the situation. This model called a Wick-type stochastic differential equation (see [23] for more details). Generally, we can introduce a non-Gaussian Wick-type stochastic model by the replacement of $(\mathscr{S})_{-1}$ on H^{χ}_{-q} and the Wick product associated with the Gaussian measure on the χ-Wick product.

The Wick product was first introduced by Wick [32] and used as a tool to renormalize certain infinite quantities in quantum field theory. Later on, the Wick product was considered, in a stochastic setting, by Hida and Ikeda [22]. In [8], Dobroshin and Minlos were comprehensively treated this subject both in mathematical physics and probability theory. Currently, the Wick product provides a useful concept for various applications, for example, it is important in the study of stochastic ordinary and partial differential equations (see, e.g., [23]).

In this section, we define a new Wick product, called χ-Wick product, on the space H^{χ}_{-q} with respect to a non-Gaussian probability measure ρ. Then, we give the definition of the χ-Hermite transform and apply it to establish a characterization theorem for the space H^{χ}_{-q} (for more details see [20,1]).

Definition 2.1. [20] Let $\xi = \sum_{m=0}^{\infty} \xi_m q^{\chi}_m$, $\eta = \sum_{n=0}^{\infty} \eta_n q^{\chi}_n \in H^{\chi}_{-q}$ with $\xi_m, \eta_n \in \mathbb{C}$. The χ-Wick product of ξ, η, denoted by $\xi \diamond_\chi \eta$, is defined by the formula

$$\xi \diamond_\chi \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q^{\chi}_{m+n}. \tag{3}$$

It is important to show that the spaces $H^{\chi}_{-q}, H^{\chi}_{q}$ are closed under χ-Wick product.

Lemma 2.1. [20] *If* $\xi, \eta \in H^{\chi}_{-q}$ and $\varphi, \psi \in H^{\chi}_{q}$, *we have*

1. $\xi \diamond_\chi \eta \in H^{\chi}_{-q}$,
2. $\varphi \diamond_\chi \psi \in H^{\chi}_{q}$.

The following important algebraic properties of the χ-Wick product follow directly from definition 2.1.

Lemma 2.2. [20] *For each* $\xi, \eta, \zeta \in H^{\chi}_{-q}$, *we get*

1. $\xi \diamond_\chi \eta = \eta \diamond_\chi \xi$ *(Commutative law),*
2. $\xi \diamond_\chi (\eta \diamond_\chi \zeta) = (\xi \diamond_\chi \eta) \diamond_\chi \zeta$ *(Associative law),*
3. $\xi \diamond_\chi (\eta + \zeta) = \xi \diamond_\chi \eta + \xi \diamond_\chi \zeta$ *(Distributive law).*

Remark. According to Lemmas 2.1 and 2.2, we can conclude that the spaces H_{-q}^{χ} and H_{q}^{χ} form topological algebras with respect to the χ-Wick product.

As shown in Lemmas 2.1 and 2.2, the χ-Wick product satisfies all the ordinary algebraic rules for multiplication. Therefore, one can carry out calculations in much the same way as with usual products. But, there are some problems when limit operations are involved. To treat these situations it is convenient to apply a transformation, called the χ-Hermite transform, which converts χ-Wick products into ordinary (complex) products and convergence in H_{-q}^{χ} into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}$. The original Hermite transform, which first appeared in Lindstrøm et al. [25], has been applied by the authors in many different connections. Now, we give the definition of the χ-Hermite transform and discuss its basic properties.

Definition 2.2. [20] Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$ with $\xi_n \in \mathbb{C}$. Then, the χ-Hermite transform of ξ, denoted by $\mathscr{H}_{\chi}\xi$ or $\widehat{\xi}$, is defined by

$$\mathscr{H}_{\chi}\xi(z) = \widehat{\xi}(z) = \sum_{n=0}^{\infty} \xi_n z^n \in \mathbb{C} \quad \text{(when convergent)}. \tag{4}$$

In the following, we define for $0 < M, q < \infty$ the neighborhoods $\mathbb{O}_q(M)$ of zero in $\mathbb{C}$ by

$$\mathbb{O}_q(M) = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |z^n|^2 K^{qn} < M^2 \right\}. \tag{5}$$

It is easy to see that

$$q \leq p,\ N \leq M \Rightarrow \mathbb{O}_p(N) \subseteq \mathbb{O}_q(M).$$

The conclusion above can be stated as follows:

Proposition 2.1. [20] *If $\xi \in H_{-q}^{\chi}$, then $\mathscr{H}_{\chi}\xi$ converges for all $z \in \mathbb{O}_q(M)$ for all $q, M < \infty$.*

A useful property of the χ-Hermite transform is that it converts the χ-Wick product into ordinary (complex) product.

Proposition 2.2. [20] *If $\xi, \eta \in H_{-q}^{\chi}$, then*

$$\mathscr{H}_{\chi}(\xi \diamond_{\chi} \eta)(z) = \mathscr{H}_{\chi}\xi(z).\mathscr{H}_{\chi}\eta(z). \tag{6}$$

for all z such that $\mathscr{H}_{\chi}\xi$ and $\mathscr{H}_{\chi}\eta$ exist.

For χ-Brownian motion, we see that

$$W_{\chi}(t) = \frac{d}{dt} B_{\chi}(t) \quad \text{in } H_{-q}^{\chi}. \tag{7}$$

Therefore, one advantage of working in the general space H_{-q}^{χ} of stochastic distributions is that it contains the solutions of many non-Gaussian stochastic differential equations, both ordinary and partial and in arbitrary dimension. Moreover, if the objects of such equations are regarded as H_{-q}^{χ}-valued, then differentiation can be interpreted in the usual strong sense in H_{-q}^{χ}.

Theorem 2.1. [23,20] *Suppose $u(x,t,z)$ is a solution (in the usual strong, pointwise sense) of the equation*

$$\widehat{A}(x,t,\partial_t,\nabla_x,u,z) = 0, \tag{8}$$

for (x,t) in some bounded open set $D \subset \mathbb{R}^n \times \mathbb{R}_+$, and for all $z \in \mathbb{O}_q(M)$, for some q, M. Moreover, suppose that $u(x,t,z)$ and all its partial derivatives, which are involved in Eq.(8) are (uniformly) bounded for $(x,t,z) \in D \times \mathbb{O}_q(M)$, continuous with respect to $(x,t) \in D$ for each $z \in \mathbb{O}_q(M)$ and analytic with respect to $z \in \mathbb{O}_q(M)$, for all $(x,t) \in D$. Then there exists $U(x,t) \in H_{-q}^{\chi}$ such that $u(x,t,z) = \mathscr{H}_{\chi}U(x,t,z)$ for all $(x,t,z) \in D \times \mathbb{O}_q(M)$ and $U(x,t)$ solves in the strong sense in H_{-q}^{χ} the equation

$$A^{\diamond\chi}(x,t,\partial_t,\nabla_x,U,s) = 0 \quad \text{in } H_{-q}^{\chi}. \tag{9}$$

3 Travelling Solitary Wave Solutions of Eq.(1)

In this section, first we reduce Eq.(1) into a deterministic partial differential equations (PDEs) by applying χ-Hermite transform. Further, by applying proper transformation, the obtained PDEs can be converted into a nonlinear ordinary differential equations (ODEs). Then, by employing the proposed F-expansion method, we obtain a family of exact solutions for Eq.(2).

Taking the χ-Hermite transform of Eq.(1), we get the following deterministic equations.

$$\widehat{U}_t(x,t,z) + \widehat{\Psi}_1(t,z)\widehat{U}(x,t,z)\widehat{U}_x(x,t,z) + \widehat{\Psi}_2(t,z)\widehat{U}_{xxx}(x,t,z) = 0, \tag{10}$$

where $z \in \mathbb{C}$ is a parameter.

Now, we using F-expansion method to solve Eq.(10). To look for the travelling solitary wave solution of Eq.(10), we use the transformations $\widehat{U}(x,t,z) := u(x,t,z) = u(\zeta(x,t,z)$, $\widehat{\Psi}_1(t,z) := \psi_1(t,z)$ and $\widehat{\Psi}_2(t,z) := \psi_2(t,z)$, with

$$\xi(x,t,z) := kx + \mu \int_0^t \omega(\tau,z)d\tau + c, \tag{11}$$

where k, μ and c are arbitrary constants which are satisfy $k\mu \neq 0$ and $\omega(t,z)$ is a nonzero function of the indicated variables to be determined later. Hence, Eq.(10) can be transformed into the following ODEs.

$$\mu\,\omega(t,z)u' + k\psi_1(t,z)uu' + k^3\psi_2(t,z)u''' = 0, \tag{12}$$

where the prime denote to the differential with respect to ξ. In view of F-expansion method, the solutions of Eq.(12), can be expressed in the forms.

$$u(\xi)=\sum_{i=0}^{N} a_i F^i(\xi), \qquad (13)$$

where a_i are constants to be determined later. considering homogeneous balance between u''' and uu' (the highest order nonlinear terms and the highest order partial derivative of u) in Eq.(12), then we can obtain $N=2$. So, Eq. (13), can be rewritten as following

$$u(x,t,z)=a_0+a_1F(\xi)+a_2F^2(\xi), \qquad (14)$$

where a_0, a_1 and a_2 are constants to be determined later. Substituting (14) with conditions in F-expansion method into (12) and collecting all terms with the same power of $F^i(\xi)[F'(\xi)]^j (i=0,\pm1,\pm2,...,j=0,1)$ as following

$$\begin{cases} \left[a_1\mu\,\omega(t,z)+k\,a_0\,a_1\,\psi_1(t,z)+k^3\,a_1\,Q\,\psi_2(t,z)\right]F' \\ +\left[2\,a_2\,\mu\,\omega(t,z)+k\,\psi_1(t,z)[2\,a_0\,a_2+a_1^2]\right. \\ \left.+8k^3\,a_2\,Q\,\psi_2(t,z)\right]FF' \\ +\left[3\,k\,a_1\,a_2\,\psi_1(t,z)+6\,k^3\,a_1\,P\,\psi_2(t,z)\right]F^2F' \\ +\left[2\,k\,a_2^2\,\psi_1(t,z)+24\,k^3 a_2\,P\,\phi_2(t,z)\right]F^3F'=0. \end{cases} \qquad (15)$$

Setting each coefficients of $F^i(\xi)[F'(\xi)]^j$ to be zero, we get a system of algebraic equations which can be expressed by

$$\begin{cases} a_1\mu\,\omega(t,z)+k\,a_0\,a_1\,\psi_1(t,z)+k^3\,a_1\,Q\,\psi_2(t,z)=0, \\ 2\,a_2\,\mu\,\omega(t,z)+k\,\psi_1(t,z)[2\,a_0\,a_2+a_1^2] \\ +8k^3\,a_2\,Q\,\psi_2(t,z)=0, \\ 3\,k\,a_1\,a_2\,\psi_1(t,z)+6\,k^3\,a_1\,P\,\psi_2(t,z)=0, \\ 2\,k\,a_2^2\,\psi_1(t,z)+24\,k^3 a_2\,P\,\phi_2(t,z)=0, \end{cases} \qquad (16)$$

with solve the system in (16) to get the two sets of coefficients as the following

$$\begin{cases} a_1=0, \\ a_0=\frac{-\mu\,\omega(t,z)-4k^3\,Q\,\psi_2(t,z)}{k\,\psi_1(t,z)}, \\ a_2=\frac{-12\,k^2\,P\,\psi_2(t,z)}{\psi_1(t,z)}, \end{cases} \qquad (17)$$

and

$$\begin{cases} a_1=0,\ a_0=a_0\,(\text{free parameter}), \\ a_2=\frac{-12\,k^2\,P\,\psi_2(t,z)}{\psi_1(t,z)}, \\ \omega=\frac{-k\left[a_0\,\psi_1(t,z)+4\,k^2\,Q\,\psi_2(t,z)\right]}{\mu}. \end{cases} \qquad (18)$$

Substituting by two sets of coefficients in Eqs.(17and 18) into Eq.(14) yields general form solutions of Eq. (1). So by the first set of coefficients we get the following solution

$$u(x,t,z)=\frac{-\mu\,\omega(t,z)-4\,k^3\,Q\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-12\,k^2\,P\,\psi_2(t,z)}{\psi_1(t,z)}\,F^2(\xi(x,t,z)). \qquad (19)$$

By the second set of coefficients we get the other solution

$$u^*(x,t,z)=a_0+\frac{-12\,k^2\,P\,\psi_2(t,z)}{\psi_1(t,z)}\,F^2(\xi^*(x,t,z)), \qquad (20)$$

with

$$\xi^*(x,t,z)=k\Big[x-\int_0^t\left[a_0\,\psi_1(\tau,z) + 4\,k^2\,Q\,\psi_2(\tau,z)\right]\,d\tau\Big]+c.$$

From appendix A, we give solutions for some special cases as following

Case 1. If we take $P=1, Q=2m^2-1$ and $R=-m^2(1-m^2)$, we have $F(\xi)\to ds(\xi)$. Hence, we have

$$u_1(x,t,z)=\frac{-\mu\,\omega(t,z)-4\,k^3\,(2m^2-1)\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,ds^2(\xi(x,t,z)), \qquad (21)$$

$$u_1^*(x,t,z)=a_0+\frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,ds^2(\xi_1^*(x,t,z)), \qquad (22)$$

with

$$\xi_1^*(x,t,z)=k\Big[x-\int_0^t\left[a_0\,\psi_1(\tau,z) + 4\,k^2\,(2m^2-1)\,\psi_2(\tau,z)\right]\,d\tau\Big]+c.$$

I. In the limit case when $m\to o$, we have $ds(\xi)\to \csc(\xi)$, thus (21) and (22) become.

$$u_2(x,t,z)=\frac{-\mu\,\omega(t,z)+4\,k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\csc^2(\xi(x,t,z)), \qquad (23)$$

$$u_2^*(x,t,z)=a_0+\frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\csc^2(\xi_2^*(x,t,z)), \qquad (24)$$

with

$$\xi_2^*(x,t,z)=k\Big[x-\int_0^t\left[a_0\,\psi_1(\tau,z)-4\,k^2\,\psi_2(\tau,z)\right]\,d\tau\Big]+c.$$

II. In the limit case when $m\to 1$ we have $ds(\xi)\to csch(\xi)$, thus (21) and (22) become.

$$u_3(x,t,z)=\frac{-\mu\,\omega(t,z)-4\,k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,csch^2(\xi(x,t,z)), \qquad (25)$$

$$u_3^*(x,t,z)=a_0+\frac{-12\,k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,csch^2(\xi_3^*(x,t,z)), \qquad (26)$$

with

$$\xi_3^*(x,t,z)=k\Big[x-\int_0^t\left[a_0\,\psi_1(\tau,z)+4\,k^2\,\psi_2(\tau,z)\right]\,d\tau\Big]+c.$$

Case 2. If we take $P=\frac{1}{4}$, $Q=\frac{m^2+1}{2}$ and $R=\frac{(1-m^2)^2}{4}$, then $F(\xi)\to\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$. Hence, we have

$$u_4(x,t,z) = \frac{-\mu\,\omega(t,z)-2k^3(m^2+1)\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{sn^2(\xi(x,t,z))}{\big[cn\pm dn\big]^2(\xi(x,t,z))}, \tag{27}$$

$$u_4^*(x,t,z)=a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{sn^2(\xi_4^*(x,t,z))}{\big[cn\pm dn\big]^2(\xi_4^*(x,t,z))}, \tag{28}$$

with

$$\xi_4^*(x,t,z)=k\Big[x-\int_0^t\big[a_0\,\psi_1(\tau,z) + 2k^2(m^2+1)\,\psi_2(\tau,z)\big]\,d\tau\Big]+c.$$

I. In the limit case when $m\to o$ we have $\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$ $\to\frac{sin(\xi)}{cos(\xi)\pm 1}$, thus (27) and (28) become.

$$u_5(x,t,z) = \frac{-\mu\,\omega(t,z)-2k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{sin^2(\xi(x,t,z))}{\big[cos(\xi(x,t,z))\pm 1\big]^2}, \tag{29}$$

$$u_5^*(x,t,z)=a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{sin^2(\xi_5^*(x,t,z))}{\big[cos(\xi_5^*(x,t,z))\pm 1\big]^2}, \tag{30}$$

with

$$\xi_5^*(x,t,z)=k\Big[x-\int_0^t\big[a_0\,\psi_1(\tau,z)+2k^2\,\psi_2(\tau,z)\big]\,d\tau\Big]+c.$$

II. In the limit case when $m\to 1$ we have $\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$ $\to\frac{tanh(\xi)}{2sech(\xi)}$, thus (27) and (28) become.

$$u_6(x,t,z) = \frac{-\mu\,\omega(t,z)-4k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{tanh^2(\xi(x,t,z))}{4\,sech^2(\xi(x,t,z))}, \tag{31}$$

$$u_6^*(x,t,z)=a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\frac{tanh^2(\xi_3^*(x,t,z))}{4\,sech^2(\xi_3^*(x,t,z))}. \tag{32}$$

Case 3. If we take $P=\frac{1}{4}$, $Q=\frac{1-2m^2}{2}$ and $R=\frac{1}{4}$, then $F(\xi)\to ns(\xi)\pm cs(\xi)$. Hence, we have

$$u_7(x,t,z) = \frac{-\mu\,\omega(t,z)-2k^3(1-2m^2)\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\big[ns\pm cs\big]^2(\xi(x,t,z)), \tag{33}$$

$$u_7^*(x,t,z)=a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\big[ns\pm cs\big]^2(\xi_6^*(x,t,z)), \tag{34}$$

with

$$\xi_6^*(x,t,z)=k\Big[x-\int_0^t\big[a_0\,\psi_1(\tau,z) + 2k^2(1-2m^2)\,\psi_2(\tau,z)\big]\,d\tau\Big]+c.$$

I. In the limit case when $m\to o$ we have $ns(\xi)\pm cs(\xi)\to csc(\xi)\pm\cot(\xi)$, thus (33) and (34) become.

$$u_8(x,t,z) = \frac{-\mu\,\omega(t,z)-2k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\big[csc\pm\cot\big]^2(\xi(x,t,z)), \tag{35}$$

$$u_8^*(x,t,z)=a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\big[csc\pm\cot\big]^2(\xi_5^*(x,t,z)). \tag{36}$$

II. In the limit case when $m\to 1$ we have $ns(\xi)\pm cs(\xi)\to\coth(\xi)\pm csch(\xi)$, thus (33) and (34) become.

$$u_9(x,t,z) = \frac{-\mu\,\omega(t,z)+2k^3\,\psi_2(t,z)}{k\,\psi_1(t,z)} + \frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)}\,\big[coth\pm csch\big]^2(\xi(x,t,z)), \tag{37}$$

$$u_9^*(x,t,z) = a_0+\frac{-3k^2\,\psi_2(t,z)}{\psi_1(t,z)} \times\big[coth\pm csch\big]^2(\xi_7^*(x,t,z)), \tag{38}$$

with

$$\xi_7^*(x,t,z)=k\Big[x-\int_0^t\big[a_0\,\psi_1(\tau,z)-2k^2\,\psi_2(\tau,z)\big]\,d\tau\Big]+c.$$

Obviously, there are other solutions for Eq.(1). These solutions come from setting different values for the coefficients P,Q and R. (see Appendix A, B and C). The above mentioned cases are just to clarify how far our technique is applicable. For more details see [33,20].

4 Non-Gaussian White Noise Functional Solutions

In this section, we employ the results of the Sections 2 and 3 respectively, by using χ-Hermite transform to obtain exact non-Gaussian white noise functional solutions for Wick-type stochastic KdV equations (1). The properties of exponential, trigonometric and hyperbolic functions yield that there exists a bounded open set $D\in\mathbb{R}\times\mathbb{R}_+$, $q<\infty$, $M>0$ such that the solutions $u(x,t,z)$ of Eq.(10) and all its partial derivatives which are involved in Eq.(10) are (uniformly) bounded for $(x,t,z)\in D\times\mathbb{O}_q(M)$, continuous with respect to

$(x,t) \in D$ for each $z \in \mathbb{O}_q(M)$ and analytic with respect to $z \in \mathbb{O}_q(M)$, for all $(x,t) \in D$. Therefore, Theorem 2.1 gives that there exists $U(x,t) \in H^{\chi}_{-q}$ such that $U(x,t) = \mathscr{H}_{\chi}^{-1} u(x,t,z)$. Also, $U(x,t)$ are solves of Eq.(1) in H^{χ}_{-q}. Hence, by applying the inverse χ-Hermite transform to the results of Section 3, we get exact non-Gaussian white noise functional solutions of Eq. (1) as follows.

–Non-Gaussian White Noise Functional Solutions of JEFs Type:

$$U_1(x,t) = \frac{-\mu\,\Omega(t) - 4k^3(2m^2-1)\Psi_2(t)}{k\Psi_1(t)} + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi ds^{\lozenge_\chi 2}(\Xi(x,t)), \quad (39)$$

$$U_2(x,t) = \frac{-\mu\,\Omega(t) - 2k^3(m^2+1)\Psi_2(t)}{k\Psi_1(t)} + \frac{-3k^2\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \frac{sn^{\lozenge_\chi 2}(\Xi(x,t))}{\left[cn^{\lozenge_\chi}(\Xi(x,t)) \pm dn^{\lozenge_\chi}(\Xi(x,t))\right]^{\lozenge_\chi 2}}, \quad (40)$$

$$U_3(x,t) = \frac{-\mu\,\Omega(t) - 2k^3(1-2m^2)\Psi_2(t)}{k\Psi_1(t)} + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \left[ns^{\lozenge_\chi}(\Xi(x,t)) \pm cs^{\lozenge_\chi}(\Xi(x,t))\right]^{\lozenge_\chi 2}, \quad (41)$$

$$U_1^*(x,t) = a_0 + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi ds^{\lozenge_\chi 2}(\Xi_1^*(x,t)), \quad (42)$$

$$U_2^*(x,t) = a_0 + \frac{-3k^2\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \frac{sn^{\lozenge_\chi 2}(\Xi_2^*(x,t))}{\left[cn^{\lozenge_\chi}(\Xi_2^*(x,t)) \pm dn^{\lozenge_\chi}(\Xi_2^*(x,t))\right]^{\lozenge_\chi 2}}, \quad (43)$$

$$U_3^*(x,t) = a_0 + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \left[ns^{\lozenge_\chi}(\Xi_3^*(x,t)) \pm cs^{\lozenge_\chi}(\Xi_3^*(x,t))\right]^{\lozenge_\chi 2}, \quad (44)$$

with

$$\Xi(x,t) = kx + \mu\int_0^t \Omega(\tau)d\tau + c,$$

$$\Xi_1^*(x,t) = k\Big\{x - \int_0^t \left[a_0\Psi_1(\tau) + 4k^2(2m^2-1)\Psi_2(\tau)\right] d\tau\Big\} + c,$$

$$\Xi_2^*(x,t) = k\Big\{x - \int_0^t \left[a_0\Psi_1(\tau) + 2k^2(m^2+1)\Psi_2(\tau)\right] d\tau\Big\} + c,$$

$$\Xi_3^*(x,t) = k\Big\{x - \int_0^t \left[a_0\Psi_1(\tau) + 2k^2(1-2m^2)\Psi_2(\tau)\right] d\tau\Big\} + c.$$

–Non-Gaussian White Noise Functional Solutions of Trigonometric Type:

$$U_4(x,t) = \frac{-\mu\,\Omega(t) + 4k^3\,\Psi_2(t)}{k\Psi_1(t)} + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \csc^{\lozenge_\chi 2}(\Xi(x,t)), \quad (45)$$

$$U_5(x,t) = \frac{-\mu\,\Omega(t) - 2k^3\,\Psi_2(t)}{k\Psi_1(t)} + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \frac{sin^{\lozenge_\chi 2}(\Xi(x,t))}{\left[cos^{\lozenge_\chi}(\Xi(x,t)) \pm 1\right]^{\lozenge_\chi 2}}, \quad (46)$$

$$U_6(x,t) = \frac{-\mu\,\Omega(t) - 2k^3\,\Psi_2(t)}{k\Psi_1(t)} + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \left[csc^{\lozenge_\chi}(\Xi(x,t)) \pm \cot^{\lozenge_\chi}(\Xi(x,t))\right]^{\lozenge_\chi 2}, \quad (47)$$

$$U_4^*(x,t) = a_0 + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \csc^{\lozenge_\chi 2}(\Xi_4^*(x,t)), \quad (48)$$

$$U_5^*(x,t) = a_0 + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \frac{sin^{\lozenge_\chi 2}(\Xi_5^*(x,t))}{\left[cos^{\lozenge_\chi}(\Xi_5^*(x,t)) \pm 1\right]^{\lozenge_\chi 2}}, \quad (49)$$

$$U_6^*(x,t) = a_0 + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi \left[csc^{\lozenge_\chi}(\Xi_5^*(x,t)) \pm \cot^{\lozenge_\chi}(\Xi_5^*(x,t))\right]^{\lozenge_\chi 2}, \quad (50)$$

with

$$\Xi_4^*(x,t) = k\Big\{x - \int_0^t \left[a_0\Psi_1(\tau) - 4k^2\,\Psi_2(\tau)\right] d\tau\Big\} + c,$$

$$\Xi_5^*(x,t) = k\Big\{x - \int_0^t \left[a_0\Psi_1(\tau) + 2k^2\,\Psi_2(\tau)\right] d\tau\Big\} + c.$$

–Non-Gaussian White Noise Functional Solutions of Hyperbolic Type:

$$U_7(x,t) = \frac{-\mu\,\Omega(t) - 4k^3\,\Psi_2(t)}{k\Psi_1(t)} + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \lozenge_\chi csch^{\lozenge_\chi 2}(\Xi(x,t)), \quad (51)$$

$$U_8(x,t) = \frac{-\mu\,\Omega(t) - 4k^3\,\Psi_2(t)}{k\,\Psi_1(t)} + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \Diamond_\chi \frac{tanh^{\Diamond_\chi 2}(\Xi(x,t))}{4\,sech^{\Diamond_\chi 2}(\Xi(x,t))}, \quad (52)$$

$$U_9(x,t) = \frac{-\mu\,\Omega(t) + 2k^3\,\Psi_2(t)}{k\,\Psi_1(t)} + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \Diamond_\chi \left[coth^{\Diamond_\chi}(\Xi(x,t)) \pm csch^{\Diamond_\chi}(\Xi(x,t))\right]^{\Diamond_\chi 2}, \quad (53)$$

$$U_7^*(x,t) = a_0 + \frac{-12k^2\,\Psi_2(t)}{\Psi_1(t)} \Diamond_\chi\, csch^{\Diamond_\chi 2}(\Xi_6^*(x,t)), \quad (54)$$

$$U_8^*(x,t) = a_0 + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \Diamond_\chi \frac{tanh^{\Diamond_\chi 2}(\Xi_6^*(x,t))}{4\,sech^{\Diamond_\chi 2}(\Xi_6^*(x,t))}, \quad (55)$$

$$U_9^*(x,t) = a_0 + \frac{-3k^2\,\Psi_2(t)}{\Psi_1(t)} \Diamond_\chi \left[coth^{\Diamond_\chi}(\Xi_7^*(x,t)) \pm csch^{\Diamond_\chi}(\Xi_7^*(x,t))\right]^{\Diamond_\chi 2}, \quad (56)$$

with

$$\Xi_6^*(x,t) = k\left\{x - \int_0^t \left[a_0\,\Psi_1(\tau) + 4k^2\,\Psi_2(\tau)\right]\,d\tau\right\} + c,$$

$$\Xi_7^*(x,t) = k\left\{x - \int_0^t \left[a_0\,\Psi_1(\tau) - 2k^2\,\Psi_2(\tau)\right]\,d\tau\right\} + c.$$

We observe that, for different forms of Ψ_1 and Ψ_2, we can get different solutions of Eq.(1) from formulas in above section. We promote this by the following example:

Example. Suppose that. $\Omega(t) = \lambda_1\Psi_1(t)$, $\Psi_1(t) = \lambda_2\Psi_2(t)$, and $\Psi_2(t) = \Pi(t) + \lambda_3 W_\chi(t)$, where $\Psi_1\Psi_2 \neq 0$, $\lambda_1, \lambda_2, \lambda_3$ are arbitrary constants and $\lambda_1\lambda_2\lambda_3 \neq 0$, $\Pi(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$ and $W_\chi(t) = \dot{B}_\chi(t)$ is the 1-parameter non-Gaussian χ-white noise and $B_\chi(t)$ is the 1-parameter χ-Brownian motion. We have the χ-Hermite transform $\widehat{W}_\chi(t,z) = \sum_{n=0}^{n} \chi_n(t) z_n$. Since $\exp^{\Diamond_\chi}(B_t) = \exp\left[B_t - \frac{t^2}{2}\right]$, we have

$$\begin{cases} \sin^{\Diamond_\chi}(B_t) = \sin\left[B_\chi(t) - \frac{t^2}{2}\right], \\ \cos^{\Diamond_\chi}(B_t) = \cos\left[B_\chi(t) - \frac{t^2}{2}\right], \\ \cot^{\Diamond_\chi}(B_t) = \cot\left[B_\chi(t) - \frac{t^2}{2}\right], \\ \csc^{\Diamond}(B_t) = \csc\left[B_\chi(t) - \frac{t^2}{2}\right], \\ \tanh^{\Diamond_\chi}(B_t) = \tanh\left[B_\chi(t) - \frac{t^2}{2}\right], \\ \coth^{\Diamond_\chi}(B_t) = \coth\left[B_\chi(t) - \frac{t^2}{2}\right], \\ sech^{\Diamond_\chi}(B_t) = sech\left[B_\chi(t) - \frac{t^2}{2}\right], \\ csch^{\Diamond_\chi}(B_t) = csch\left[B_\chi(t) - \frac{t^2}{2}\right]. \end{cases} \quad (57)$$

We have a non χ-Wick version of non-Gaussian white noise travelling wave solutions of Eq.(1.1) as follows.

$$U_{10}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 + 4k^3}{k\,\lambda_2} + \frac{-12k^2}{\lambda_2}\ \csc^2(\Theta_1(x,t)), \quad (58)$$

$$U_{11}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 - 2k^3}{k\,\lambda_2} + \frac{-3k^2}{\lambda_2} \times \frac{sin^2(\Theta_1(x,t))}{\left[cos(\Theta_1(x,t)) \pm 1\right]^2}, \quad (59)$$

$$U_{12}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 - 2k^3}{k\,\lambda_2} + \frac{-3k^2}{\lambda_2} \times \left[csc(\Theta_1(x,t)) \pm \cot(\Theta_1(x,t))\right]^2, \quad (60)$$

$$U_{13}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 - 4k^3}{k\,\lambda_2} + \frac{-12k^2}{\lambda_2}\ csch^2(\Theta_1(x,t)), \quad (61)$$

$$U_{14}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 - 4k^3}{k\,\lambda_2} + \frac{-3k^2}{\lambda_2}\ \frac{tanh^2(\Theta_1(x,t))}{4\,sech^2(\Theta_1(x,t))}, \quad (62)$$

$$U_{15}(x,t) = \frac{-\mu\,\lambda_1\lambda_2 + 2k^3}{k\,\lambda_2} + \frac{-3k^2}{\lambda_2} \times \left[coth(\Theta_1(x,t)) \pm csch(\Theta_1(x,t))\right]^2, \quad (63)$$

$$U_{10}^*(x,t) = a_0 + \frac{-12k^2}{\lambda_2}\ \csc^2(\Theta_2(x,t)), \quad (64)$$

$$U_{11}^*(x,t) = a_0 + \frac{-3k^2}{\lambda_2}\ \frac{sin^2(\Theta_3(x,t))}{\left[cos(\Theta_3(x,t)) \pm 1\right]^2}, \quad (65)$$

$$U_{12}^*(x,t) = a_0 + \frac{-3k^2}{\lambda_2}\ \left[csc(\Theta_3(x,t)) \pm \cot(\Theta_3(x,t))\right]^2, \quad (66)$$

$$U_{13}^*(x,t) = a_0 + \frac{-12k^2}{\lambda_2}\ csch^2(\Theta_4(x,t)), \quad (67)$$

$$U_{14}^*(x,t) = a_0 + \frac{-3k^2\,\lambda_2}{\lambda_2}\ \frac{tanh^2(\Theta_4(x,t))}{4\,sech^2(\Theta_4(x,t))}, \quad (68)$$

$$U_{15}^*(x,t) = a_0 + \frac{-3k^2}{\lambda_2}\ \left[coth(\Theta_5(x,t)) \pm csch(\Theta_5(x,t))\right]^2, \quad (69)$$

with

$$\Theta_1(x,t) = kx + \mu\lambda_1\lambda_2 \int_0^t A + c,$$

$$\Theta_2(x,t) = k\left\{x - \left[a_0\,\lambda_2 - 4k^2\right] \int_0^t A\right\} + c,$$

$$\Theta_3(x,t) = k\left\{x - \left[a_0\,\lambda_2 + 2k^2\right] \int_0^t A\right\} + c,$$

$$\Theta_4(x,t) = k\left\{x - \left[a_0\,\lambda_2 + 4k^2\right] \int_0^t A\right\} + c,$$

$$\Theta_5(x,t) = k\left\{x - \left[a_0\,\lambda_2 - 2k^2\right] \int_0^t A\right\} + c,$$

where $A = \left(\Pi(\tau)d\tau + \lambda_3[B_\chi(t) - \frac{t^2}{2}]\right)$.

5 Summary and Discussion

The propagation of nonlinear wave in systems with polarity symmetry can be described by the KdV equations (1.2). If the problem is considered in a non-Gaussian stochastic environment, we can get non-Gaussian stochastic KdV equations. In order to give the exact stochastic solutions of the non-Gaussian stochastic KdV equations, we only consider this problem in a non-Gaussian white noise environment, that is, we investigate the variable coefficients stochastic KdV equations (1.1). For this aim, we develop a non-Gaussian Wick calculus based on the theory of hypecomplex systems $L_1(Q,dm(x))$. Precisely, we use the direct connection between the hypecomplex systems and the white noise analysis [3,1] and the Delsarte characters $\chi_n(x)$ to introduce a χ-Wick product and χ-Hermite transform on the space of generalized functions H^{χ}_{-q} (with the zero space $L_2(Q,dm(x))$) and discuss their properties. By means of the usual properties of complex analytic functions, we proved a characterization theorem for H^{χ}_{-q}, and setup a framework to study the SPDEs with non-Gaussian parameters (for more details see[20]). Finally, we employ this framework and F-expansion method to give a multiple families of exact travelling wave solutions of KdV equations (2) and non-Gaussian white noise functional solutions of Wick-type stochastic KdV equations in (1), respectively. The obtained solutions include functional solutions of JEFs, trigonometric and hyperbolic types. Obviously, the planner which we have proposed in this paper can be also applied to other non-linear PDEs in mathematical physics such as KdV-Burgers, modified KdV-Burgers, KdV-Burgers-Kuramoto [35,26,9], generalized Hirota-Satsuma coupled KdV system [27,29], Zhiber-Shabat and Benjamin-Bona-Mahony equations. We observe that the used F-expansion Method has many other particular solutions, depending on the parameters P, Q and R, this in turn gives many other exact solutions for the considered stochastic KdV equations. Also, we have discussed the solutions of SPDEs driven by non-Gaussian white noise, this discussion is less detailed than the Gaussian discussion but more general, because it deals with the dual pairing generated by integration with respect to a non-Gaussian measure.

Appendices

–Appendix A.

The ODE and JEFs: Relation between values of (P, Q, R) and corresponding $F(\xi)$ in ODE.

$$(F')^2(\xi) = PF^4(\xi) + QF^2(\xi) + R,$$

P	Q	R	$F(\xi)$
1	$-1-m^2$	m^2	$\mathrm{ns}\xi = \frac{1}{\mathrm{sn}\xi}, \mathrm{dc}\xi = \frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\xi = \frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\xi = \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\frac{1}{4}$	$\frac{m^2+1}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi \pm \mathrm{dn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi \pm \mathrm{cs}\xi$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}\xi \pm \mathrm{ds}\xi$

–Appendix B.

The jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$.

$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1, sc\xi \to \tan\xi,$
$sd\xi \to \sin\xi, cd\xi \to \cos\xi, ns\xi \to \csc\xi, nc\xi \to \sec\xi,$
$nd\xi \to 1, cs\xi \to \cot\xi, ds\xi \to \csc\xi, dc\xi \to \sec\xi.$

–Appendix C.

The jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$.

$sn\xi \to \tanh\xi, cn\xi \to sech\xi, dn\xi \to sech\xi, sc\xi \to \sinh\xi,$
$sd\xi \to \sinh\xi, cd\xi \to 1, ns\xi \to \coth\xi, nc\xi \to \cosh\xi,$
$nd\xi \to \cosh, cs\xi \to csch\xi, ds\xi \to csch\xi, dc\xi \to 1.$

For more details (See Appendices A, B and C.)[33,20].

References

[1] A. S. Okb El Bab, H. A. Ghany and M. Zakarya, A construction of non-Gaussian white noise analysis using the theory of hypercomplex systems, *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences*, **16** (2016) 11-25.

[2] A. S. Okb El Bab, A. M. Zabel, H. A. Ghany, Harmonic analysis in Hypercomplex systems, *International Journal of Pure and Applied Mathematics*, **80** No.5 (2012), 739-750.

[3] Yu. M. Berezansky, A connection between the theory of hypergroups and white noise analysis, *Reports on Mathematical Physics* **36** (1995), 215-234.

[4] A. de Bouard and A. Debussche, On the stochastic Korteweg-de Vries equation, *J. Funct. Anal.* **154** (1998), 215-251.

[5] A. de Bouard and A. Debussche, White noise driven Korteweg-de Vries equation, *J. Funct. Anal.* **169** (1999), 532-558.

[6] A. Debussche and J. Printems, Numerical simulation of the stochastic Korteweg-de Vries equation, *Physica D: Nonlinear Phenomena* **134** (1999), 200-226.

[7] A. Debussche and J. Printems, Effect of a localized random forcing term on the Korteweg-de Vries equation, *J. Comput. Anal. Appl.* **3** (2001), 183-206.

[8] R. L. Dobrushin and R. L. Minlos, Polynomials in linear random functions, *Russian Math. Surveys* **32** (1977), 71-127.

[9] Z.T. Fu, S.K. Liu, S.D. Liu, New exact solutions to the KdV-Burgers-Kuramoto equation. Chaos Solitons Fractals, 23 (2005): 609-616.

[10] C. S. Gardner, J. M. Greene, M. D. Kruskal, Method for solving the Korteweg-de Vries equation, *Physics Review Letters* **19** (1967), 1095-1097.
[11] H. A. Ghany, Exact solutions for stochastic generalized Hirota-Satsuma coupled KdV equations, *Chin. J. Phys.* **49** (2011), 926-940.
[12] H. A. Ghany, Exact solutions for stochastic fractional Zakharov-Kuznetsov equations, *Chin. J. Phys.* **51** (2013), 875-881.
[13] H. A. Ghany and A. Hyder, White noise functional solutions for the Wick-type two-dimensional stochastic Zakharov-Kuznetsov equations, *International Review of Physics* **6** (2012), 153-157.
[14] H. A. Ghany and M. Zakarya, Generalized solutions of Wick-type stochastic KdV-Burgers equations using Exp-function method, *International Review of Physics* **8** (2014), 38-46.
[15] H. A. Ghany and A. Hyder, Local and global well-posedness of stochastic Zakharov-Kuznetsov equation, *J. Comput. Anal. Appl.* **15** (2013), 1332-1343.
[16] H. A. Ghany, A. S. Okb El Bab, A. M. Zabal, and A. Hyder, The fractional coupled KdV equations: Exact solutions and white noise functional approach, *Chin. Phys. B* **22** (2013), 080501.
[17] H. A. Ghany and M. Zakarya, Exact solutions for Wick-type stochastic coupled KdV equations, *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences* **14** (2014), 57-71.
[18] H. A. Ghany and M. Zakarya, Exact travelling wave solutions for Wick-type stochastic Schamel-KdV equations using F-expansion method, *Physics Research International* **1** (2014), 1-9.
[19] H. A. Ghany, S. K. Elagan and A. Hyder, Exact travelling wave solutions for stochastic fractional Hirota-Satsuma coupled KdV equations, *Chin. J. Phys.* **53** (2015), 1-14.
[20] A. Hyder and M. Zakarya, Non-Gaussian Wick calculus based on hypercomplex systems, *International Journal of Pure and Applied Mathematics*, **109** No. 3 (2016), 539-556.
[21] J.-H. He and X.-H.Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons and Fractals*, **30**, no. 3, (2006), 700708,
[22] T. Hida and N. Ikeda, Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. *Proc. Fifth Berkeley Symp.Math. Stat. Probab. II* (1965), 117-143. Univ. California Press, Berkeley.
[23] H. Holden, B. Øsendal, J. Ubøe and T. Zhang, *Stochastic partial differential equations*, Springer Science+Business Media, LLC, (2010).
[24] W. Junmin, Y. Xiao, Riemann theta functions solutions to the KdV-Burgers-Kuramoto equation, International Journal of Nonlinear Science, **11** No.3 (2011), 284-289.
[25] T. Lindstrøm, B. Øksendal and J. Ubøe, Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): *Stochastic Analysis*. Cambridge University Press: Cambridge, MA, (1991), 261-303.
[26] S.D. Liu, S.K. Liu, Z.H. Huang, Q. Zhao, New exact solutions to the KdV-Burgers-Kuramoto equation, *Prog. Nat. Sci.*, **9** (1999), 912-918.
[27] H. Eleuch, N. Ben Nessib and R. Bennaceur, Quantum Model of emission in weakly non ideal plasma, *Eur. Phys. J. D* **29** (2004), 391-395.
[28] E. G. Fan, Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation, *Phys. Lett. A.* **282**(2001), 18-22.
[29] Z. Yan, The extended Jacobi elliptic function expansion method and its application in the generalized Hirota-Satsuma coupled KdV system, *Chaos, Solitons and Fractals*, **15** (2003), 575-583
[30] M. Wadati, Stochastic Korteweg-de Vries equation, *J. Phys. Soc. Jap.* **52** (1983), 2642-2648.
[31] A. M. Wazwaz, *Partial differential equations and solitary waves theory*, Springer-Verlag: Beijing (2009).
[32] G. C. Wick, The evaluation of the collinear matrix, *Phys. Rev.* **80** (1950), 268-272.
[33] Yu. Zhou, M. Wang, Yu. Wang, Periodic wave solutions to a coupled KdV equations with variable coefficients, *Physics Letters A*, **308** (2003), 31-36.
[34] H. Eleuch and R. Bennaceur, An optical soliton pair among absorbing three-level atoms, *J. Opt. A : Pure Appl. Opt.* **5** (2003),528-533.
[35] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progress of Theoretical Physics* **55** no. 2,(1967), 356369.
[36] A. M. Zabel and Buthinah A. Bin Dehaish, Levy Khinchin Formula on Commutative Hypercomplex System, *KYUNGPOOK Math. Journal* **48** (2008), 559-575.

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