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Analysis of the Cylinder Glaciation Models in Seawater

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Abstract: An axisymmetric Stefan-type model is presented for glaciation of a cylindrical gas pipeline in sea-water, which is solved numurically. Two simplified models are formulated, one of which admits exact analytical solution. Comparison of the three models in a particular case shows good agreement and indicates that simple models could be useful in estimating ice thickness on pipelines in sea-water.

Keywords: Stefan problem, glaciation, pipeline, seawater, numerical and analitical solution, calculation examples

1 Introduction

In the northern seas, for example in the Barents Sea, pipeline glaciation is possible.

In the book ”Models of sea gas-pipelines” [1] the mathematical models and algorithms for computing the steady-state gas transportation in gas-pipelines running from Shtokman gas-field in the Barents Sea to the Teriberka village were presented. These models enable to calculate the temperature, the density, the pressure, the velocity profile in the gas flow and the ice thickness upon the surface of pipeline for steady-state gas flow. In the book the characteristic parameters values of the model are presented, in particular the ambient temperature of sea water $T^* = 272$K, the freezing temperature of see-water $T_* = 271$ K. A hot gas under pressure about 23 MPa is delivered to the inlet of pipeline. During the passage of the pipeline route the gas cools down due to heat exchange with the environment and due to gasdynamic effects. For long sea gas-pipelines without compressor substations the gas temperature $T_g(z,t)$ at pipeline end may be lower than the seawater-ice transition temperature $T_* = 271$ K. In these areas upon the outer surface of pipeline glaciation may occur. In [1] the calculation results of the ice thicknesses in steady flow of gas are given. In calculations of unsteady gas flow the ice formation dynamics is important, one depends on the gas temperature, on the heat conductivities of the layers comprising the coat of the gas-pipeline, on the ambient temperature of sea water and on the flow over pipeline. An ice layer affects the heat exchange processes between the gas flow and the environment as well as the pipeline buoyancy. The general mathematical model of the process involves the glaciation model that is inseparable, strictly speaking, from the processes model in the gas flow.

The complete non-stationary gas transportation model is a considerably complicated problem for numerical solution even in the quasi-one-dimensional formulation [1]–[4]. Any computation algorithm for the gas transportation system of equations involves a computation algorithm of heat exchange through the lateral surface, taking into consideration the varying over time ice layer thickness. The ice layer thickness is of interest by itself for pipeline buoyancy estimation.

The possibility of simplifying the glaciation dynamics model is of great importance for creating effective computational algorithms of gas transportation problems via sea gas-pipelines. In the present work a procedure for obtaining such simpler models is proposed.

We consider a glaciation problem of a cylindrical gas pipeline with radius R = 0.67 m. The pipeline is flowed around by a sea-water at temperature $T^* = 272$ K. The temperature distribution $T_0(t)$ of homogeneous pipeline is defined as $T_0(t) = \dfrac{m_1}{m_2+t} + m_3$, where m_1 [K/sec], m_2 [sec], m_3 [K] are dimensional constants. The choice of this distribution is based on characteristic behavior of gas temperature in cross-section distant from the inlet of pipeline. In such cross-section the gas temperature cools down below the water-ice transition temperature. The

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parameter m_3 corresponds to value of gas temperature in this cross-section in steady flow, the parameter m_1 corresponds to the cooling rate of gas temperature. In practice the range of variation of the value m_3 is of interest between 271 K and 265 K. The pipeline temperature behavior $T_0(t)$ for $m_1 = 15120$ K/sec, $m_2 = 5040$ sec, $m_3 = 268$ K is shown in Fig. 1. The glaciation dynamics depends on the heat flux vector from seawater to the glaciation front $y(t)$. The value of the heat flux vector can be obtained from the external flow problem of pipeline. In present paper this problem is not considered. We assume axial symmetry, so that the temperature T depends only on the radial coordinate and on time, T(r,t). It is assumed the radial component of the

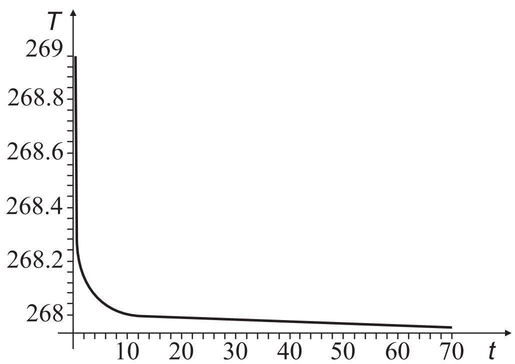

Fig. 1: Behavior of pipeline temperature in a cross-section.

heat flux vector q from seawater to glaciation front is known and unchanged, which is equal to $q = 31\ W/m^2$. The condition $q = \text{const}$ corresponds to the quasi-stationary solution of the heat equation in the water boundary-layer outside the pipeline. q can be represented in the form $q = \gamma(T^* - T_*)$, where γ is the heat transfer coefficient. It depends on the thermal boundary-layer thickness δ_*, which, in turn, depends on the bottom currents and on the convective stirring intensities by the change of water salinity near the phase-transition front [1].

Let us write down the one-dimensional non-stationary model of the homogeneous cylindrical gas pipeline glaciation dynamic under the assumptions made.

2 Model 1

$$\frac{\partial T}{\partial t} = \frac{\tilde{a}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad r \in (R, R+y), \quad t > t_0; \tag{1}$$

$$T(r,t_0) = T^0(r), \quad r \in [R, R+y_0]; \tag{2}$$

$$T(R,t) = T_0(t), \quad t > t_0; \tag{3}$$

$$T(R+y,t) = T_*, \quad t > t_0; \tag{4}$$

$$\lambda \frac{\partial T}{\partial r}\bigg|_{R+y} - q = Q\frac{\mathrm{d}y}{\mathrm{d}t}, \quad t > t_0; \tag{5}$$

$$y\big|_{t_0} = y_0. \tag{6}$$

We use the following designations: R is the radius of the cylinder [m]; ρ, λ, $\tilde{c}$, $\tilde{a} = \frac{\lambda}{\rho\tilde{c}}$ are the density [kg/m^3], the thermal conductivity [W/m.K], the specific heat [kJ/kg.K] and the thermal diffusivity of sea ice [m^2/sec]; $Q = \tilde{Q}\rho$, $\tilde{Q}$ is the latent heat (of fusion) for sea ice [kJ/kg]; r is the radial coordinate in cylindrical coordinate system; $T = T(r,t)$ is the temperature distribution in ice layer [K]; $y = y(t)$ is the ice thickness [m] upon the surface of pipeline at time moment t; q is the radial component of the heat flux vector from seawater to glaciation front; $r = R + y(t)$ is the coordinate of glaciation front; y_0 is the ice thickness at the initial time; $T_0(t)$ is the temperature distribution in ice layer at the initial time. By $y_0 = 0$ following equalities hold $T(r,t_0) = T(R,t_0) = T_0(t_0)$.

Condition (4) expresses that the front is at the freezing temperature. Eq. (5) is the Stefan condition, which expresses the heat balance between the heat fluxes vectors at the ice-water interface and an amount of heat released during ice formation. Eq. (1) is the heat equation in ice layer; Eqs. (2), (6) are the initial conditions; Eq. (3) is a Derichlet boundary condition at the outer surfaces of pipeline.

In Model 1 the parameters λ, $\tilde{c}$, ρ, T_*, $\tilde{Q}$ are considered to be constant. In real problems it may need to take into consideration dependency of these thermodynamic properties on the seawater salinity and on other factors [5].

Despite its simplifying assumptions the task still remains a challenging problem for numeric solution. One is related to the nonlinear Stefan-type problems with a moving boundary. There are various analytical and numerical approaches to the solution of similar problems. The reviews of such methods can be found, for example, in [6]–[14]. Under condition $q = \text{const}$ Model 1 has no self-similar solution.

As it is known, a method with the explicit tracking of moving surface can be used for the numerical solution one-dimensional Stefan problem [15]–[18]. We use the method of Douglas and Gallie [15], [16], which is an iterative finite difference method, with variable time steps. In this approach the time step size τ_{n+1} is variable and at the $(n+1)$-th temporal level it is determined so that the ice thickness increased on constant value h during this time step. The value h is the space grid step size. The convergence of this numerical method was proved for a Stefan-type problem in [15], [16].

3 Numerical algorithm

We calculate indicated problem in a dimensional form. A uniform spatial grid with the step size h (in meters), $r_i = R + ih$, $i = 0, 1, \ldots, N$ is introduced. We denote τ^s and τ_{n+1} time steps size at the $(n+1)$-th temporal level in the s-th iteration and after iteration process end, respectively; $t_{n+1} = \tau_1 + \tau_2 + \ldots + \tau_{n+1}$ is the overall process time (all time values are expressed in seconds); T_i^s, T_i^{n+1} are the ice temperature in the i-th node at the $(n+1)$-th temporal level in the s-th iteration and after iteration process end, respectively. The values T_i^n and τ_n are assumed known. New values T_i^{n+1} and τ_{n+1} are calculated as follows.

The time step size τ^s is set. Eq. (1) is approximated by an implicit finite difference method and it is solved by sweep method at the region $r \in (R, R + (n+1)h)$ using boundary condition (3), (4). Using found sequence T_i^s, $i = 0, \ldots, n+1$ the heat flux vector q_2 is calculated

$$q_2 = \lambda (T_{n+1}^s - T_n^s)/h, \quad T_{n+1}^s = T_*.$$

The heat flux vector q_1 is calculated using the sequence T_i^n

$$q_1 = \lambda (T_1^n - T_0^n)/h.$$

The time new step size τ^{s+1} in the $(s+1)$-th iteration is defined using the value τ^s and heat fluxes q_1, q_2, q as follows

$$\tau^{s+1} = \tau^s + Q(h - h^s)/(q_1 - q), \qquad h^s = \tau^s (q_2 - q)/Q.$$

Here q is a given heat flux from seawater to glaciation front. The iterative process is terminated if the inequality $\left|\tau^{s+1} - \tau^s\right| \le \varepsilon$ holds for a specified small quantity ε. The first time step size in the zeroth approximation can be found, for example, using the known analytical Stefan problem solution [19]. Further the step value is refined according to the presented algorithm. For the following steps τ as a zeroth approximation can choose previous step size τ_n. The ice layer thickness at time moment t_n is equal to nh.

Using the Douglas-Gallie scheme, we calculated a particular test case using the boundary value $T_0(t) = m_1/(m_2 + t) + m_3$, with $q = 31$ W/m^2, and parameter values:

$$\begin{aligned} &R = 0.67\,\text{m}, \;\; \lambda = 2.3\,\text{W/m.K}, \;\; \rho = 928\,\text{kg/m}^3, \\ &\tilde{Q} = 335\,\text{kJ/kg}, \;\; \tilde{c} = 2.1\,\text{kJ/kg.K}, \;\; T_* = 271\,\text{K}, \\ &m_1 = 15120\,\text{K/sec}, \;\; m_2 = 5040\,\text{sec}, \;\; m_3 = 268\,\text{K}, \\ &h = 0.001\,\text{m}, \qquad \varepsilon = 0.1\,\text{sec}, \\ &t_0 = 0\,\text{sec}, \qquad y_0 = 0\,\text{m}. \end{aligned} \tag{7}$$

As an example, in Table 1 we present the numerical solution of the pipeline glaciation dynamics in seawater with parameter values (7). In the table first line the values of ice thickness y_n (in centimetres) at the time moment t_n are presented, t_n is shown in hours. The bottom line contains the time step sizes τ_n (in minutes).

Table 1: Numerical solution of the pipeline glaciation dynamics.

y_n, cm.	0.1	1	2	3	4	5	6
t_n, h.	0.14	1.89	4.98	9.52	15.74	23.93	34.36
τ_n, min.	8.6	14.3	20.2	31.5	42.4	54.9	69.5

In this example the ice layer thickness increases to 6 cm over about 35 hours. The calculations show Model 1 is sensitive to change of the value q. In calculations we applied the dimensional quantities, so ε was equal to 0.1 that corresponded to 0.1 sec. With this precision the number of iterations at every time moment was not more than 4. The obtained results show the ice temperature distributions $T(r,t)$ are close to linear distributions. This conclusion is in qualitative agreement with the measured data on temperature distribution in growing sea ice [5].

The calculation of all pipeline glaciation process up to achieve steady-state regime with a unchanged temperature and a unchanged ice thickness advisable to perform in two stages. At the first stage Eqs. (1)(6) are solved using the Douglas-Gallie scheme described above. Here, we define the moment of time t_0, after which the pipeline temperature can be assumed constant equal to m_3 and the temperature distribution in ice layer can be considered close to quasi-stationary logarithmic distribution (which under small y and major R is almost linear).

From this moment t_0 the solution of non stationary problem (1)(6) is approximated by the solution stationary problem (Model 1$'$).

Model 1$'$

$$T(r,t) = A(t) + B(t)\ln r, \quad r \in [R, R+y], \quad t \ge t_0;$$

$$T(R,t) = m_3, \quad t \ge t_0;$$

$$T(R+y,t) = T_*, \quad t \ge t_0;$$

$$\lambda \frac{B(t)}{R+y} - q = Q\frac{\mathrm{d}y}{\mathrm{d}t}, \quad t > t_0;$$

$$y\big|_{t_0} = y_0.$$

The values t_0, y_0 are defined using Model 1. The solution of Model 1$'$ convenient to search in non-dimensional form. We introduce the dimensionless variables:

$$y' = y/r_x, \quad t' = t/t_x, \quad R' = R/R_x,$$

accepted characteristic values r_x, t_x are equal $r_x = 1$ cm, $t_x = 1$ hour.

The problem reduces to the ODE for $y'(t)$.

$$\frac{dy'}{dt'} = \frac{a}{(R' + y')\ln(1 + y'/R')} - b \tag{8}$$

with the initial condition

$$y'\big|_{t'_0} = y'_0. \tag{9}$$

The dimensionless quantities a, b and the dimensionless radius R' of the pipeline for parameters (7) are equal to:

$$\begin{aligned} a &= \frac{\lambda t_x (T_* - m_3)}{r_x^2 Q} = 0.79902, \\ b &= \frac{q t_x}{Q r_x} = 0.0359, \quad R' = 67. \end{aligned} \tag{10}$$

The calculations using Model 1 show as the t'_0 can take $t'_0 = 34.36$, then the dimensionless value y is equal to 6 and the temperature distribution in ice layer is logarithmic.

For the numerical solution of the ODE initial value problem (8), (9) we used the Runge-Kutta method. The value of the dimensionless time step size was equal to 0.1. The calculation results of the pipeline glaciation at the second stage are shown in Fig. 2. By the abscissa axis, time in days is indicated, by the ordinate axis the ice thickness in in centimeters is indicated. For this model problem the maximum thickness of the ice layer y_* is equal to 19.628 cm. For a real gas pipeline in steady-state regime the steady ice thickness is less even under lower temperatures. For example, steady ice thickness y_* is equal to 4.4 cm under the gas temperature T_0= 266.5 K [1]. This difference is related to the influence of the heat-insulation layers of a real gas pipeline on the glaciation its surface. The value $y_* = 4.4$ cm is calculated in [1] for pipeline that has three heat-insulation layers.

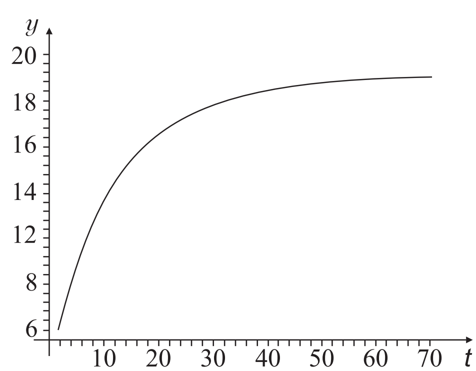

Fig. 2: Dynamic of ice growth during 70 days.

4 Simplified versions of Model 1

As mentioned in the introduction, calculation of glaciation dynamic of pipeline part, in which the gas temperature cools down below the freezing point of sea-water, is inseparable from calculation of temperature, density and gas pressure. The general complete model of the processes is a considerably complicated. In numerical solving the gas temperature is assumed either constant or varying linearly during every time step. Of interest is estimation the admissibility of replacement of non stationary glaciation model by quasi-stationary one, under the linear temperature distribution at every time step. For this purpose we consider two quasi-stationary variants of Model 1 with linear temperature distribution of pipeline $T(R,t) = \alpha + \beta t$, ($\alpha$ [K], β [K/sec]).

Model 2

$$T(r,t) = A(t) + B(t)\ln r, \quad r \in [R, R+y], \quad t \ge t_0;$$
$$T(R,t) = \alpha + \beta t, \quad t \ge t_0;$$
$$T(R+y,t) = T_*, \quad t \ge t_0;$$
$$\lambda \frac{B(t)}{R+y} - q = Q\frac{\mathrm{d}y}{\mathrm{d}t}, \quad t > t_0;$$
$$y\big|_{t_0} = y_0.$$

As well as in Model 1$'$ the problem now reduces to the dimensionless ODE

$$\frac{dy'}{dt'} = \frac{a' - ct'}{(R'+y')\ln(1+y'/R')} - b \tag{11}$$

with the initial condition

$$y'\big|_{t'_0} = y'_0. \tag{12}$$

The dimensionless quantities a', b, c in Model 2 are

$$a' = \frac{\lambda t_x (T_* - \alpha)}{r_x^2 Q}, \quad b = \frac{q t_x}{Q r_x}, \quad c = \frac{\lambda t_x \beta}{r_x^2 Q}. \tag{13}$$

The numerical solution of the ODE (11) with initial condition (12) presents no difficulty. In our calculation we used the Runge-Kutta method. The results of test calculations are presented below (Table 2).

Model 3

For relatively small ice thicknesses the logarithmic temperature distributions in ice layer are close to the linear distributions which are solution of a quasi-stationary problem of plane glaciation process. Model 3 can be written as

$$T(r,t) = A(t) + B(t) r, \quad r \in [R, R+y], \quad t \ge t_0;$$
$$T(R,t) = \alpha + \beta t, \quad t \ge t_0;$$
$$T(R+y,t) = T_*, \quad t \ge t_0;$$
$$\lambda B(t) - q = Q\frac{\mathrm{d}y}{\mathrm{d}t}, \quad t > t_0;$$
$$y\big|_{t_0} = y_0.$$

In non-dimensional variables y', t' the problem reduces to the ODE:

$$\frac{dy'}{dt'} = \frac{a' - ct'}{y'} - b, \tag{14}$$

with the initial condition

$$y'\big|_{t'_0} = y'_0. \tag{15}$$

where the dimensionless quantities a', b, c are defined by Eqs. (13). The Eq. (14) follows from Eq. (11), when $y'/R' \ll 1$. In this case the approximate equality

$$(R' + y')\ln(1 + y'/R') \approx (R' + y')(y'/R') \approx y'.$$

The ODE (14) can be analytically integrated. Under the condition $\triangle = 4c - b^2 < 0$ the solution is written in terms of the new variable $x = \dfrac{a' - ct'}{y'}$ as follows:

$$\ln\left|\frac{t'_0 - d_1}{t' - d_1}\right| = -\frac{1}{2}\ln\left(\left(\frac{x}{x_0}\right)^2\left(\frac{x_0^2 - bx_0 + c}{x^2 - bx + c}\right)\right) - \\ - d_4\ln\left(\left(\frac{2x - d_2}{2x_0 - d_2}\right)\left(\frac{2x_0 - d_3}{2x - d_3}\right)\right), \tag{16}$$

$$x_0 = \frac{a' - ct'_0}{y'_0}, \quad d_1 = a'/c,$$

$$d_2 = b + \sqrt{-\triangle}, \quad d_3 = b - \sqrt{-\triangle}, \quad d_4 = b/(2\sqrt{-\triangle}).$$

The Eq. (16) is converted to the transcendental equation for x:

$$\left(\frac{x}{x_0}\right)\left(\frac{x_0^2 - bx_0 + c}{x^2 - bx + c}\right)^{1/2} \times \\ \times\left(\left(\frac{2x - d_2}{2x_0 - d_2}\right)\left(\frac{2x_0 - d_3}{2x - d_3}\right)\right)^{d_4} = \left|\frac{t' - d_1}{t'_0 - d_1}\right|,$$

which is easy to solve. The ice thickness $y'(t)$ is found from the expression $y'(t') = (a' - ct')/x(t')$.

The found algoritms of glaciation dynamics calculation using presented models show admissibility of using simplified Model 2 and Model 3 for considered class of problem according to the aforesaid suggestions about axial processes symmetry, about constancy of the heat flux vector and about the invariability of thermal characteristics of sea ice.

We now compare simulation results using models considered above for the case of linear-in-time surface temperature:

$$T(R', t') = \alpha + \beta' t'$$

with t' dimensionless time and dimensionless pipeline radius R'. The constants α, β' expressed in units of temperature are taken to be equal to $\alpha = 270.35$ K, $\beta = -0.00331$ K. In calculation the characteristic values r_x, t_x are equal to $r_x = 1$ cm, $t_x = 60$ sec. The other parameters are as (7), exept initial conditions: $t_0 = 1641.6$ sec, $y_0 = 0.003$ m.

Table 2: Comparison of simulation results using simplified models.

t, h.	2.657	6.78	11.364	16.080	20.873	25.719	30.608
y_1, cm.	1	2	3	4	5	6	7
y_2, cm.	1.021	2.018	3.018	4.021	5.027	6.036	7.046
y_3, cm.	1.023	2.028	3.042	4.065	5.095	6.130	7.178

In Table 2, $y_1(t)$, $y_2(t)$, $y_3(t)$ are the dimensional ice layer thicknesses at the corresponding time t on the first line, obtained from Model 1,2,3, respectively with initial values $y_1 = y_2 = y_3 = 0.3$ cm at time $t_0 = 0.456$ h= 1641.6 sec. We see that the three models predict about the same ice thickness, 7 cm, after 30.6 h.

5 Conclusion

In this paper the algorithms to the solution of the homogeneous cylindrical gas pipeline glaciation dynamic problem are presented. The performed calculations prove admissibility of using simplified quasi-stationary models. The considered algorithms allow to estimate the admissibility using of simplified glaciation models for many practical problems. In general model the capability to use instead of the glaciation model 1 its simplified models results in a significant decrease in the computational time when calculating the gas transportation by long sea gas-pipelines in the northern seas. The generalization of presented mathematical models and numerical algorithms of the pipeline with heat insulating layers is not difficult.

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