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On the Entropy of Progressive Hybrid Censoring Schemes

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Abstract: In a recent days, progressive hybrid censoring scheme based on combination of Type-I and Type-II progressive censoring schemes has been discussed and studied quite extensively in literature of reliability analysis. In this paper, we use the entropy decomposition in progressive hybrid censoring data to obtain expression in a simple form for the entropy of progressively hybrid Type-I and Type-II censored data. Moreover, We compute the entropy in progressive hybrid Type-1 and Type II censored samples for the scaled exponential distribution to illustrate the existence of the presented method.

Keywords: Entropy; progressive hybrid censoring; Order statistics; Markov chain

1 Introduction

Censoring schemes are found to be the famous and most common to use in the experiment of life-testing due to its advantages in saving cost and time. During last few years, the progressive censoring schemes proved to be an important tool in reliability analysis as compared to other censoring schemes presented in literature so far. The progressive censoring is flexible and more general censoring than Type-I and Type-II censoring mechanism. This permits the removal of live experiment unit in different intermittent times during the experiment in addition to the removal at the termination of the experiment. Balakrishnan and Cramer [1] presented progressively Type-II censored samples as: Let n units be placed in test at time zero. Immediately following the first failure, R_1 surviving units are removed form the test at random. Then immediately following the second failure, R_2 surviving units are removed form the test at random. This process continues until at the time of mth observed failure, the remanding: $R_m = n - R_1 - R_2 - ..... - R_{m-1} - m$, which are all removed from the experiment. So the life testing stops at the mth failure. The observed failure times $X = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ constitute progressive Type-II censored OS. In this scheme, either experiment may last quite a long time or stop too early. Therefore, Kundu and Joarder [2] and Childs et al. [3] suggested a progressive hybrid schemes. According to this scheme, Type-I progressive hybrid censored data follows if we end experiment at a pre planned time T which comes before the mth failure and Type-II progressive hybrid censored data follows if experiment continues until time T even if the mth failure occurs before T. This sampling scheme will likely give more information about the tail of the distribution under consideration because of the progressive censoring and the limit of experimental time which is T. Moreover, progressive hybrid censored data overcomes the disadvantages arise by using censored data of Type-I and Type-II. For example, the time of termination of experiment is uncontrolled in censoring of Type-II, whereas the efficiency level is uncontrolled in Type-I censored data. Many researches have been done to consider the entropy information from random samples in an ordered data, one can see, [4], [5], [6], [7], [8] and references therein. To determine the best possible efficient censoring scheme, the role of the entropy is very important. Abo-Eleneen [9] and [10] studied the entropy and the optimal scheme in progressive censoring of Type-II. Cramer and Bagh [11] explored plans of maximum/minimum entropy For progressive Type-II censoring schemes. Awad [12] discussed the optimal scheme that maximizes the informational efficiency with respect to some different optimality based on entropy measures for Pareto distribution. Morabbi and Razmkhah [13] studied entropy in hybrid censoring data. In this

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article, an exact expressions for entropy information contained in both types of progressively hybrid censored data is derived. We illustrate this expressions based on presented plan for optimal censoring by applying it to life time distribution.

2 Entropy in progressive Type-I hybrid censoring

In this section, we obtain an exact expression for the entropy in progressive hybrid Type-I censoring scheme. First, we present some necessary tools. Suppose $U_1, \cdots, U_n$ is i.i.d. random sample of size n from absolutely continuous distribution with cumulative distribution function (cdf) $G(u;\theta)$ and probability density function (pdf) $g(u;\theta)$ where θ is a real valued parameter and the sample is arranged in ascending order. Morabbi and Razmkhah [13] presented the following result.

Lemma 2.1. Suppose U_r represent $\{min((U_{r:n}, T), I(U_{r:n} \leq T))\}$. Then

1.The entropy in U_1 can be obtained as

$$H_{U_1} = G_{1:n}(1-\log n) - \int_0^T \log h(u) g_{1:n}(u;\theta) du,$$

2.The conditional entropy in U_{r+1} given U_r, can be obtained as

$$H_{U_{r+1}|U_r} = \{(1-\log(n-r+1)\}\{1-C_n^{r-1}\bar{G}^{n-r+1}(T)\} - \int_0^T \log h(u) g_{r:n}(u;\theta) du,$$

where $h(u) = \frac{g(u)}{1-G(u)}$, is the hazard function.

Let $\mathbf{U} = (U_{1:m:n}, \cdots, U_{m:m:n})$ constitute Type-II progressive censored OS from a sample of size n. Then likelihood function presented by[15] as

$$g_{1\cdots m:m:n}(\mathbf{U};\theta) = \prod_{r=1}^{m} c_{r-1} g(u_{r:m:n};\theta)[1-G(u_{r:m:n};\theta)]^{R_r}, \tag{1}$$

where $c_r = n - R_1 - R_2 - R_3 \cdots - R_r - r)$, $f_{1\cdots m:m:n}(\mathbf{U};\theta)$ is joint density function of $(U_{1:m:n}, U_{2:m:n} \cdots, U_{m:m:n})$. Balakrishnan et al. [14] and Abo-Eleneen [9] established the following expression for the entropy of progressive Type-II censored order statistics.

Lemma 2.2. The entropy of progressive Type-II censored order statistics, $H_{1\cdots m:m:n}(\mathbf{U})$, in terms of the hazard function $h(u) = g(u)/G(u)$ is given as a summation of single integrals as

$$H_{1\cdots m:m:n}(\mathbf{U};\theta) = m - \log c - \int_{-\infty}^{\infty} \log h(u) \sum_{r=1}^{m} g_{r:m:n}(u;\theta) du, \tag{2}$$

where, $c = \prod_{r=1}^{m} c_{r-1}$ and $g_{r:m:n}(u)$ is the pdf of the r-th progressively Type-II censored order statistic. The marginal density $g_{r:m:n}$ is given by [16], as

$$g_{r:m:n}(u;\theta) = \sum_{j=1}^{r} c_{j,r} g(u_j;\theta)(1-G(u_j;\theta))^{c_{j-1}-1}, \tag{3}$$

where

$$c_{j,r} = \prod_{k=1,k\neq j}^{r} \frac{c_{k-1}}{c_{k-1}-c_{j-1}}, \text{and} \sum_{j=1}^{r} c_{j,r} = 1.$$

The joint pdf $g_{1\cdots m:m:n}(\mathbf{U})$ presented by [15] can be written as

$$g_{1\cdots m:m:n}(\mathbf{U};\theta) = g_{1:m:n}(u;\theta) \prod_{r=1}^{m-1} g_{r+1|r:m:n}(u_{r+1}|u_r;\theta), \tag{4}$$

where $g_{r+1|r:m:n}(u_{r+1}|u_r;\theta)$ is the conditional density of $U_{r+1:m:n}$ given $U_{r:m:n}$. Note that this conditional density function is the density of the first order statistics $Y_{1:c_{r-1}}$, where Y has the pdf, $\frac{g(y;\theta)}{1-G(u_r;\theta)}$ with sample size $(n - R_1 - \cdots R_r - r)$. Then we have

$$g_{r+1|r:m:n}(u_{r+1}|u_r;\theta) = c_{r-1} \frac{g(u_r;\theta)}{1-G(u_{r-1};\theta)} \{\frac{1-G(u_r;\theta)}{1-G(u_{r-1};\theta)}\}^{c_{r-1}-1}, u_r > u_{r-1}. \tag{5}$$

Following [2] and [3], the experiment is terminated at $min(U_{m:m:n}, T)$ in a Type-I progressive hybrid censoring scheme. Hence, progressive hybrid censored Type-I data may be recorded as $\mathbf{T} = (T_{1:m:n}, \cdots, T_{m:m:n})$, with $T_{i:m:n} = (X_i, \delta_i)$, for $i = 1, \cdots, m$ and $X_i = min(U_{i:m:n}, T)$, $\delta_i = I(U_{i:m:n} \leq T)$. To obtain the joint density of $\mathbf{T}$ denoted by $g_{1:m:n,\cdots,m:m:n}(t_1, \cdots, t_m)$, we can use the decomposition

$$g_{1\cdots m:m:n}(t_1, \cdots, t_m) = g_{1:m:n}(t_1) \prod_{r=1}^{m-1} g_{r+1:m:n|i:m:n}(t_{r+1}|t_r), \tag{6}$$

where $g_{r+1|r:m:n}(t_{r+1}|t_r)$ can be obtained, if we consider only $U_{i:m:n} \leq T$.

Now, we obtain a simple expression for the entropy in progressive hybrid Type-I censored data as the following.

Theorem 2.1. The entropy in the progressive hybrid Type-I censoring data $H^I_{T\wedge m:m:n}$ is

$$H^I_{T\wedge m:m:n} = \sum_{r=1}^{m} \{(1-\log_{c_{r-1}})(1-C_{c_{r-1}}^{r-1}\bar{G}^{c_{r-1}+1}(T)) - \int_0^T \log h(u) g_{r:m:n}(u;\theta) du\}.$$

Proof. Using the Markov chain property of the order statistics from progressive Type II censored samples ([15]), we have the following decomposition

$$g_{1\cdots m:m:n}(t_1, \cdots, t_m) = g_{1:m:n}(t_1) \prod_{r=1}^{m-1} g_{r+1|r:m:n}(t_{r+1}|t_r). \tag{7}$$

Therefore, the entropy of Type-I hybrid censored sample can be written as

$$H_{1\cdots m:m:n} = H_{1:m:n} + H_{2:m:n|1:m:n} + \cdots + H_{m:m:n|m-1:m:n}, \tag{8}$$

where $H_{r+1:m:n|r:m:n}$ is the expected entropy in $T_{r+1:m:n}$ given $T_{r:m:n} = u_r$. Since $g_{r|r-1}(t_r|t_r - 1)$, for $U_{r:m:n} \leq T$, can be interpreted as the pdf of $(min\{Y_{1:c_{r-1}} \wedge T\}, I(Y_{1:c_{r-1}} \leq T)$, where $Y_{1:c_{r-1}}$ is the first order statistic of a random sample of size c_{i-1} from random variable truncated from a left with pdf $\frac{g(y)}{1-G(u_i)}$ for $y \geq u_{r-1}$. Using Lemma 2.1 we can obtain the conditional entropy in $Y_{1:c_{r-1}}$ given $u_{r-1:m:n} = u_{r-1}$ as

$$H_{min(Y_{1:c_{r-1}} \wedge T)} = \{(1-\log_{c_{r-1}})(1-C_{c_{r-1}}^{r-1}\bar{G}^{c_{r-1}+1}(T)) - \int_0^T \log h(t) f_{Y_{1:c_{r-1}}}(t|t_{r-1})dt.$$

Hence, the average of the conditional information can be written in[9] as

$$\begin{aligned} H_{r:m:n|r-1:m:n} &= H_{min(Y_{1:c_{r-1}} \wedge T)} \\ &= (1-\log_{c_{r-1}})(1-C_{c_{r-1}}^{r-1}\bar{G}^{c_{r-1}+1}(T)) - \int_0^T \log h(t) g_{r:m:n}(t)dt. \end{aligned}$$

Then, Theorem 2.1 follows from Equation (8).

3 Entropy in progressive Type-II hybrid censoring

In this section, we obtain the entropy in progressive Type-II hybrid censoring scheme. Following, Childs et al. [3] the progressive Type-II hybrid censoring scheme can be defined in the following two setting, first experiment continues up to time T if $U_{m:m:n} < T$ without any further dragging to the survival units after the failure mth second experiment ends at $U_{m:m:n}$ if $U_{m:m:n} > T$, Hence, we have a number R_m of survival units still under the experiment even after the mth failure has occurred when $U_{m:m:n} < T$. Therefore, the progressively Type-II hybrid censored data can be denoted by $(T_1, ..., T_{m+R_m})$, where $T_i = U_{i:m:n}$, for $i = 1, ..., m$, and $T_{m+i} = (min\{U_{n-R_m+i:n,T}\}, I(U_{n-R_m+i:n} \leq T))$, for $i = 1, ..., R_m$.

Theorem 3.1. The entropy in the progressive Type-II hybrid censoring data $H^{II}_{T\vee m:m:n}$ is

$$H^{II}_{T\vee m:m:n} = H_{1\cdots m:m:n} + H_c - H_{(n-R_m)\wedge T:n},$$

where $H_{1,\cdots,m:m:n}$, H_c and $H_{(n-R_m)\wedge T:n}$ represent entropy for Type-II progressive censoring, entropy for Type-I censoring and Type-II hybrid censoring, respectively.

Proof. Using the Markov chain property of the order statistics from progressive Type II censored samples we have the following decomposition

$$g_{1\cdots m+R_m:m:n}(t_1,\cdots,t_m) = g_{1\cdots m:m:n}(t_1\cdots,t_m) g_{m+1\cdots m+R_m|m}(t_{m+1},\ldots,t_{m+R_m}|t_m), \quad (9)$$

where $g_{1\ldots m:m:n}$ is the joint density function of the progressively Type-II censored data and $g_{m+1\cdots m+R_m|m:m:n}$ is the conditional joint density of the Type-I hybrid censored data, given $T_m = t_m$. Thus $T_m = U_{n-R_m:n}$.

Using Equation (9), the following decomposition follows from the strong additivity of the entropy

$$H^{II}_{T\vee m:m:n} = H_{1\cdots m:m:n} + H_{m+1,\cdots,m+R_m|m:m:n},$$

where, $H_{1,\cdots,m:m:n}$ and $H_{m+1m+R_m|m:m:n}$ are the entropy in the progressively Type-II censored data $(U_{1:m:n},\cdots,U_{m:m:n})$ and the average of the conditional entropy in the Type-I hybrid censored data $(T_{m+1},\cdots,T_{n+R_m})$ given $T_{n-R_m:n}$ respectively.

Now, we consider the decomposition of $g_{m+1\cdots m+R_m|m}(t_{m+1},\ldots,t_{m+R_m}|u_m)$ as

$$g_{m+1\cdots m+R_m|m}(t_{m+1},\ldots,t_{m+R_m}|u_m) = g_{m+1|m:m:n}(t_{m+1}|t_m)\prod_{r=m+1}^{m+R_m} g_{r+1|r:m:n}(t_{r+1}|t_r). \quad (10)$$

Since $g_{m+r|m+r-1:m:n}(t_{m+r}|t_{m+r-1})$ is the conditional density of min $(U_{n-R_m+r:n}, T)$, given min $(U_{n-R_m+r-1:n}, T)$. Using Lemma 2.1, we can obtain the average of the conditional entropy $H_{m+1\cdots m+R_m|m:m:n}$ as

$$\begin{aligned} H_{m+1\cdots m+R_m|m:m:n} &= \sum_{r=n-R_m+1}^{n} \{(1-\log(n-r+1)\}\{1-C_{n-r+1}^{r-1}\bar{G}^{n-r+1}(T)\} \\ &\quad - \int_0^T \log h(u) g_{r:n}(u)du\}. \end{aligned}$$

We can re-express $H_{m+1\cdots m+R_m|m:m:n}$ as

$$H_{m+1\cdots m+R_m|m:m:n} =$$

$$\sum_{r=1}^{n}\left[\{1-\log(n-r+1)\}\{1-C_{n-r+1}^{r-1}\bar{G}^{n-r+1}(T)\} - \int_0^T \log h(x) g_{r:n}(u)du\right]$$
$$-\sum_{r=1}^{n-R_m}\left[(1-\log(n-r+1)\}\{1-C_{n-r+1}^{r-1}\bar{F}^{n-r+1}(T)\} \int_0^T \log h(x) g_{r:n}(u)du\right]$$

$$= H_c - H_{(n-R_m)\wedge T:n},$$

this completes the proof.

4 An application

EXAMPLE 2.1. For the exponential distribution with pdf $g(x;\lambda) = \frac{1}{\lambda}\exp(-x/\lambda), x > 0, \lambda > 0$ and cdf $G(x;\lambda) = 1-\exp(-x/\lambda)$ the hazard function is $1/\lambda$. Using Theorem 2.1, the entropy in Type-I hybrid censored data with a censoring time T can be readily obtained as

$$H^{I}_{T\wedge m:m:n} = \sum_{r=1}^{m}\{(1-\log_{c_{r-1}})(1-C_{c_{r-1}}^{r-1}\bar{G}^{c_{r-1}+1}(T/\lambda)) - \log(T/\lambda)G_{r:m:n}(T/\lambda)\}. \quad (11)$$

Also, in Theorem 3.1 using Lemma 2.2 and Lemma 2.1 for computing the entropy in Type-II progressive censored data and the entropy in Type-II censored data respectively, we may obtain the entropy in progressively Type-II hybrid censored data as follows

$$\begin{aligned} H^{II}_{T\vee m:m:n} &= m - \log c - \log\frac{1}{\lambda}\sum_{r=1}^{m} G_{r:m:n}(T/\lambda)dx \\ &\quad + \sum_{r=1}^{n-R_m+1}[(1-\log(n-r+1)\}\{1-C_{n-r+1}^{r-1}\bar{G}^{n-r+1}(T/\lambda)\}] \\ &\quad - \log(1/\lambda)\sum_{r=1}^{n-R_m+1} G_{r:n}(T/\lambda), \end{aligned} \quad (12)$$

Table 1: Best ten censoring schemes of Type-I and Type-II progressive hybrid from scaled exponential distribution for $(n = 10, m = 5)$

n	m	Censoring scheme	$H^{I}_{T\wedge m:m:n}$	Censoring scheme	$H^{II}_{T\vee m:m:n}$
10	5	(0,0,1,4,0)	5.373	(0,0,1,4,0)	4.985
10	5	(0,0,2,3,0)	3.850	(0,0,2,3,0)	3.159
10	5	(5,0,0,0,0)	3.015	(5,0,0,0,0)	2.881
10	5	(4,1,0,0,0)	2.917	(4,1,0,0,0)	2.784
10	5	(3,2,0,0,0)	2,750	(3,2,0,0,0)	2.637
10	5	(0,0,3,2,0)	2.665	(2,3,0,0,0)	2.510
10	5	(0,0,2,3,0)	2.600	(0,0,3,2,0)	2.429
10	5	(2,2,1,0,0)	2.338	(0,0,0,0,5)	2.170
10	5	(0,1,2,2,0)	2.226	(0,1,2,2,0)	1.952
10	5	(0,2,1,2,0)	2.052	(0,2,1,2,0)	1.748

where

$$G_{r:m:n}(x) = 1 - \sum_{j=1}^{r} c_{j,r} g(x_j)(1 - G(x_j)^{c_{j-1}-1},$$

and

$$G_{r:n}(x) = \sum_{j=r}^{n} C_j^n [G(x)]^j [1 - G(x)]^{n-j}.$$

Since progressive censoring is a diverse censoring plan specifically there is a flexibility in the choice of $(R_1, \cdots, R_m)$, for example, if we take $n = 10$ and $m = 5$, we obtain a total of censoring schemes $\binom{n-1}{m}$, which gives a value of 126. We consider some different values for $R_i s$ in the censoring scheme $(R_1, \cdots R_m)$, when $n = 10$ and $m = 5$. For the special case $\theta = 2$, and the median of the scaled exponential distribution as the pre-fixed censoring time i.e. $(T = \theta \log 2,)$. Table 1 provides the values of $H^{I}_{T\wedge m:m:n}$ and $H^{II}_{T\vee m:m:n}$. The entries were calculated by Equations (11), (12), and MATHEMATICA. We may specified the optimal progressive hybrid censoring schemes out of 126 possible schemes for which the maximum entropy is attained. In Table 1, we record the ten best progressive hybrid censoring schemes based on maximizing the entropy. Table 1 shows that the entropy in progressive Type-II hybrid censoring and the entropy in progressive Type-I hybrid censoring are maximized at censoring scheme $(0, 0, 1, 4, 0)$, for all censoring schemes and for the chosen censoring Time T.

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