Applied Mathematics & Information Sciences
*An International Journal*

http://dx.doi.org/10.18576/amis/110401

# Certain Subordination-Preserving Family of Integral Operators Associated with $p$-Valent Functions

*H. M. Srivastava*[1,2,*], *M. K. Aouf*[3], *A. O. Mostafa*[3] *and H. M. Zayed*[4]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada.
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China.
[3] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
[4] Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt.

Received: 6 Mar. 2016, Revised: 24 Feb. 2017, Accepted: 26 Feb. 2017
Published online: 1 Jul. 2017

**Abstract:** In this paper, we obtain a number of subordination, superordination and sandwich-type results related to a certain family of integral operators defined on the space of $p$-valent functions in the open unit disk. Applications of these subordination and superordination theorems to the familiar Gauss hypergeometric function are also considered. The results derived in this paper are shown to generalize several known subordination, superordination and sandwich-type theorems.

**Keywords:** Analytic functions; Univalent functions; Close-to-Convex functions; $p$-Valent functions; Differential subordination; Superordination; Subordination chain; Integral operators; Sandwich-type theorems; Gauss hypergeometric function.

## 1 Introduction, Definitions and Preliminaries

Let $\mathscr{H} = \mathscr{H}(\mathbb{U})$ be the class of functions analytic in open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\} = \overline{\mathbb{U}} \setminus \partial\mathbb{U}.$$

Also let $\mathscr{H}[a,n]$ be the subclass of $\mathscr{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

and set

$$\mathscr{H}_0 = \mathscr{H}[0,1] \qquad \text{and} \qquad \mathscr{H} = \mathscr{H}[1,1].$$

Furthermore, let $\mathscr{A}(p)$ be the class of all functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\}), \quad (1)$$

which are analytic in $\mathbb{U}$. We note that $\mathscr{A}(1) = \mathscr{A}$.

For $f, g \in \mathscr{H}(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $g(z)$ or $g(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

In such a case, we write $f(z) \prec g(z)$. If the function $g$ is univalent, then we have the following equivalence (see [14] and [15]; see also the recent work [20]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and let the function $h(z)$ be univalent in $\mathbb{U}$. If the function $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the following first-order differential subordination:

$$\psi\big(\mathfrak{p}(z), z\mathfrak{p}'(z); z\big) \prec h(z), \qquad (2)$$

then $\mathfrak{p}(z)$ is a solution of the differential subordination (2). A given univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of the differential subordination (2) if

* Corresponding author e-mail: harimsri@math.uvic.ca

$\mathfrak{p}(z) \prec \mathfrak{q}(z)$ for all $\mathfrak{p}(z)$ satisfying (2). A univalent dominant $\widetilde{\mathfrak{q}}(z)$ that satisfies the subordination condition: $\widetilde{\mathfrak{q}}(z) \prec \mathfrak{q}(z)$ for all dominants of (2) is called the best dominant. If the functions $\mathfrak{p}(z)$ and $\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z); z\right)$ are univalent in $\mathbb{U}$ and if $\mathfrak{p}(z)$ satisfies the following first-order differential superordination:

$$h(z) \prec \psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z); z\right), \tag{3}$$

then $\mathfrak{p}(z)$ is a solution of the differential superordination (3). An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of the differential superordination (3) if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for all $\mathfrak{p}(z)$ satisfying (3). A univalent subordinant $\widetilde{\mathfrak{q}}(z)$ that satisfies $\mathfrak{q}(z) \prec \widetilde{\mathfrak{q}}(z)$ for all subordinants of (3) is called the best subordinant (see [14] and [15]).

For functions $f(z) \in \mathscr{A}(p)$ $(p \in \mathbb{N})$ and for the parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$ and $p\beta + \gamma = p\alpha + \delta$, we introduce the integral operator $I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta} : \mathscr{A}(p) \to \mathscr{A}(p)$ as follows:

$$I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z) = \left(\frac{p\beta+\gamma}{z^{\gamma}\Phi(z)} \int_0^z [f(t)]^{\alpha}\, \phi(t) t^{\delta-1}\, dt\right)^{\frac{1}{\beta}} \tag{4}$$

where, and in what follows, all powers are tacitly assumed to be the corresponding principal values.

**Remark 1.** We note special cases of the integral operator defined by (4) above.

(i) For $p = 1$, we obtain

$$I(f)(z) = \left(\frac{\beta+\gamma}{z^{\gamma}\Phi(z)} \int_0^z [f(t)]^{\alpha}\, \phi(t) t^{\delta-1}\, dt\right)^{\frac{1}{\beta}}$$
$$(\alpha+\delta = \beta+\gamma),$$

where the operator $I$ was introduced by Miller *et al.* [16] and studied by (for example) Cho *et al.* [5] (see also [7] and [18]).

(ii) For $p = 1$ and $\Phi(z) = \phi(z) = 1$, we obtain

$$I_{\beta,\gamma}(f)(z) = \left(\frac{\beta+\gamma}{z^{\gamma}} \int_0^z [f(t)]^{\beta}\, t^{\gamma-1}\, dt\right)^{\frac{1}{\beta}},$$

where the operator $I_{\beta,\gamma}$ was introduced by Miller and Mocanu [14] and studied by (for example) Bulboacă (see [1], [2] and [3]).

In order to prove our results, we need the following definitions and lemmas.

**Definition 1** (see [14]). We denote by $\mathscr{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{\zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z\to\zeta} q(z) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$. We also denote by $\mathscr{Q}(a)$ the subclass of $\mathscr{Q}$ for which $q(0) = a$.

**Definition 2** (see [14]). A function

$$L(z,t) \qquad (z \in \mathbb{U};\ t \geqq 0)$$

is said to be a subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for all $0 \leqq s \leqq t$.

**Lemma 1** (see [19]). *The function*

$$L(z,t) : \mathbb{U} \times [0,\infty) \longrightarrow \mathbb{C}$$

*of the form*:

$$L(z,t) = a_1(t) z + a_2(t) z^2 + \cdots$$
$$\left(a_1(t) \neq 0;\ t \geqq 0;\ \lim_{t\to\infty} |a_1(t)| = \infty\right)$$

*is said to be a subordination chain if and only if*

$$\Re\left(\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right) > 0 \qquad (z \in \mathbb{U};\ t \geqq 0)$$

*and*

$$|L(z,t)| \leqq K_0 |a_1(t)| \qquad (|z| < r_0 < 1;\ t \geqq 0)$$

*for some positive constants* $K_0$ *and* $r_0$.

**Lemma 2** (see [10]). *Suppose that the function*

$$\mathfrak{H} : \mathbb{C}^2 \to \mathbb{C}$$

*satisfies the condition*:

$$\Re\{\mathfrak{H}(is;t)\} \leqq 0$$

*for all real s and for all*

$$t \leqq -\frac{n}{2}\left(1+s^2\right) \qquad (n \in \mathbb{N}).$$

*If the function* $\mathfrak{p}(z)$ *given by*

$$\mathfrak{p}(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

*is analytic in* $\mathbb{U}$ *and*

$$\Re\left\{\mathfrak{H}\left(\mathfrak{p}(z); z\mathfrak{p}'(z)\right)\right\} > 0 \qquad (z \in \mathbb{U}),$$

*then*

$$\Re\{\mathfrak{p}(z)\} > 0 \qquad (z \in \mathbb{U}).$$

**Lemma 3** (see [11]). *Let* $\kappa, \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$. *Also let* $h \in \mathscr{H}(\mathbb{U})$ *with* $h(0) = c$. *If*

$$\Re\{\kappa h(z) + \gamma\} > 0 \qquad (z \in \mathbb{U}),$$

*then the solution of the following differential equation*:

$$\mathfrak{q}(z)+\frac{z\mathfrak{q}'(z)}{\kappa\mathfrak{q}(z)+\gamma}=h(z)\qquad(z\in\mathbb{U};\ \mathfrak{q}(0)=c)$$

*is analytic in* $\mathbb{U}$ *and satisfies the inequality given by*

$$\Re\{\kappa\mathfrak{q}(z)+\gamma\}>0\qquad(z\in\mathbb{U}).$$

**Lemma 4** (see [14]). *Let* $\mathfrak{p}\in\mathscr{Q}(a)$ *and let the function* $q(z)$ *given by*

$$q(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots$$

*be analytic in* $\mathbb{U}$ *with* $q(z)\neq a$ *and* $n\in\mathbb{N}$. *If the function* $q(z)$ *is not subordinate to the function* $\mathfrak{p}(z)$, *then there exist two points*

$$z_0=r_0e^{i\theta}\in\mathbb{U}\qquad\text{and}\qquad\zeta_0\in\partial\mathbb{U}\setminus E(q)$$

*such that*

$$\mathfrak{p}(\mathbb{U}_{r_0})\subset q(\mathbb{U}),\ \ \mathfrak{p}(z_0)=q(\zeta_0)$$

and

$$z_0\mathfrak{p}'(z_0)=m\zeta_0q'(\zeta_0)\qquad(m\geqq n).$$

**Lemma 5** (see [19]). *Let* $q\in\mathscr{H}[a,1]$ *and* $\varphi:\mathbb{C}^2\to\mathbb{C}$. *Suppose also that*

$$\varphi\left(q(z),zq'(z)\right)=h(z).$$

*If* $L(z,t)$ *given by*

$$L(z,t)=\varphi\left(q(z),tzq'(z)\right)$$

*is a subordination chain and* $\mathfrak{q}\in\mathscr{H}[a,1]\cap\mathscr{Q}(a)$, *then*

$$h(z)\prec\varphi\left(q(z),zq'(z)\right)$$

*implies that* $q(z)\prec\mathfrak{p}(z)$. *Furthermore, if*

$$\varphi\left(q(z),zq'(z)\right)=h(z)$$

*has a univalent solution* $q\in\mathscr{Q}(a)$, *then the function* $q$ *is the best subordinant.*

**Definition 3.** Let $c\in\mathbb{C}$ with $\Re(c)>0$ and

$$N=N(c)=\frac{|c|\sqrt{1+2\Re(c)}+\Im(c)}{\Re(c)}.$$

If

$$R=R(z)=\frac{2Nz}{1-z^2}$$

is a univalent function and $b=R^{-1}(c)$, then the open-door function $R_c(z)$ is defined by

$$R_c(z)=R\left(\frac{z+b}{1+\bar{b}z}\right)\qquad(z\in\mathbb{U}).$$

The function $R_c$ is univalent in $\mathbb{U}$, $R_c(0)=c$ and $R_c(\mathbb{U})=R(\mathbb{U})$ is the complex plane slit along the half lines

$$\Re(w)=0\qquad\text{and}\qquad\Im(w)\geqq N$$

and

$$\Re(w)=0\qquad\text{and}\qquad\Im(w)\leqq -N.$$

**Lemma 6 (Integral Existence Theorem)** (see [12] and [13]). *Let* $\phi,\Phi\in\mathscr{H}$ *with*

$$\phi(z)\neq0\qquad\text{and}\qquad\Phi(z)\neq0\qquad(z\in\mathbb{U}).$$

*Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ *with*

$$\beta\neq0,\ \ \alpha+\delta=\beta+\gamma\qquad\text{and}\qquad\Re(\alpha+\delta)>0.$$

*If the function* $g(z)\in\mathscr{A}$ *and*

$$\alpha\frac{zg'(z)}{g(z)}+\frac{z\phi'(z)}{\phi(z)}+\delta\prec R_{\alpha+\delta}(z),$$

*then*

$$G(z)=\left(\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z[g(t)]^{\alpha}\,\phi(t)t^{\delta-1}\,dt\right)^{\frac{1}{\beta}}\in\mathscr{A},$$

$$\frac{G(z)}{z}\neq0\qquad(z\in\mathbb{U})$$

*and*

$$\Re\left(\beta\frac{zG'(z)}{G(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right)>0\qquad(z\in\mathbb{U}).$$

**Lemma 7.** *Let* $p\in\mathbb{N}$. *Also let* $\phi,\Phi\in\mathscr{H}$ *with*

$$\phi(z)\neq0\qquad\text{and}\qquad\Phi(z)\neq0\qquad(z\in\mathbb{U}).$$

*Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ *with*

$$\beta\neq0,\ \ p\alpha+\delta=p\beta+\gamma\qquad\text{and}\qquad\Re(p\alpha+\delta)>0.$$

*If the function* $f(z)\in\mathscr{A}(p)$ *and*

$$\mathscr{A}_{p,\alpha,\delta}=\Bigg\{f:f(z)\in\mathscr{A}(p)\ \ \text{and}\ \ \alpha\frac{zf'(z)}{f(z)}+\frac{z\phi'(z)}{\phi(z)}+\delta\prec R_{p\alpha+\delta}(z)\Bigg\},$$

*then*

$$F(z)=\left(\frac{p\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z[f(t)]^{\alpha}\,\phi(t)t^{\delta-1}\,dt\right)^{\frac{1}{\beta}}=z^p+\cdots\in\mathscr{A}(p),$$

$$\frac{F(z)}{z^p}\neq0\qquad(z\in\mathbb{U})$$

*and*

$$\Re\left(\beta\frac{zF'(z)}{F(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right)>0\qquad(z\in\mathbb{U}).$$

*Proof.* Let $f(z) \in \mathscr{A}(p)$. It is easy to see that the function $g(z)$ given by

$$g(z) = \frac{f(z)}{z^{p-1}} \in \mathscr{A}.$$

A simple computation shows that

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta = \alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} - \alpha(p-1) + \delta,$$

so that

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{p\alpha+\delta}(z).$$

Let

$$\delta + \alpha(p-1) = \delta_1 \qquad \text{and} \qquad \gamma + \alpha(p-1) = \gamma_1.$$

Since

$$p\alpha + \delta = p\beta + \gamma \qquad \text{and} \qquad \Re(p\alpha + \delta) > 0,$$

we obtain

$$\beta + \gamma_1 = \alpha + \delta_1 \qquad \text{and} \qquad \Re(\alpha + \delta_1) > 0.$$

Thus, clearly, it follows from Lemma 6 that

$$G(z) = \left( \frac{\beta + \gamma_1}{z^{\gamma}\Phi(z)} \int_0^z [g(t)]^{\alpha} \phi(t) t^{\delta_1 - 1} dt \right)^{\frac{1}{\beta}} = z + \cdots \in \mathscr{A},$$

$$\frac{G(z)}{z} \neq 0 \qquad (z \in \mathbb{U}$$

and

$$\Re\left( \beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma_1 \right) > 0 \qquad (z \in \mathbb{U}),$$

which imply that

$$F(z) = \left( \frac{p\beta + \gamma}{z^{\gamma}\Phi(z)} \int_0^z [f(t)]^{\alpha} \phi(t) t^{\delta - 1} dt \right)^{\frac{1}{\beta}} = z^p + \cdots \in \mathscr{A}(p),$$

$$\frac{F(z)}{z^p} = \frac{G(z)}{z} \neq 0 \qquad (z \in \mathbb{U})$$

and

$$\Re\left( \beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma_1 \right) = \Re\left( \beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right) > 0 \qquad (z \in \mathbb{U}).$$

This completes the proof of Lemma 7.

## 2 Main Results

Unless otherwise mentioned, we assume throughout this paper that the parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with

$$\beta \neq 0 \qquad \text{and} \qquad p\beta + \gamma = p\alpha + \delta$$

such that

$$\Re(p\alpha + \delta) > 0,$$

and (as already mentioned in Section 1) all powers are tacitly assumed to be the corresponding principal values.

**Theorem 1.** *Let $f, g \in \mathscr{A}_{p,\alpha,\delta}$ and*

$$\Re\left(1 + \frac{zv''(z)}{v'(z)}\right) > -\delta \tag{5}$$

$$\left( v(z) = z \left( \frac{g(z)}{z^p} \right)^{\alpha} \phi(z) \right),$$

*where $\delta$ is given by*

$$\delta = \frac{1 + |a|^2 - \left|1 - a^2\right|}{4\Re(a)} \tag{6}$$

$$\left( a = p\beta + \gamma - 1;\ \Re(a) > 0 \right).$$

*Then the following subordination condition*:

$$u(z) = z \left( \frac{f(z)}{z^p} \right)^{\alpha} \phi(z) \prec v(z) \tag{7}$$

*implies that*

$$z \left( \frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p} \right)^{\beta} \Phi(z) \prec z \left( \frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z^p} \right)^{\beta} \Phi(z) \tag{8}$$

*and the function*

$$z \left( \frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z^p} \right)^{\beta} \Phi(z)$$

*is the best dominant.*

*Proof.* Define the functions $U(z)$ and $V(z)$ in $\mathbb{U}$ by

$$U(z) = z \left( \frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p} \right)^{\beta} \Phi(z) \qquad (z \in \mathbb{U}) \tag{9}$$

and

$$V(z) = z \left( \frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z^p} \right)^{\beta} \Phi(z) \qquad (z \in \mathbb{U}). \tag{10}$$

Then, in view Lemma 7, it these two functions $U(z)$ and $V(z)$ are well defined by (9) and (10), respectively. We first show that, if

$$q(z)=1+\frac{zV''(z)}{V'(z)} \qquad (z\in\mathbb{U}), \tag{11}$$

then

$$\Re\{q(z)\}>0 \qquad (z\in\mathbb{U}).$$

From (4) and the definitions of the functions $v(z)$ and $V(z)$, we obtain

$$(p\beta+\gamma)v(z)=zV'(z)+(p\beta+\gamma-1)V(z). \tag{12}$$

Hence we have

$$\begin{aligned}1+\frac{zv''(z)}{v'(z)}&=q(z)+\frac{zq'(z)}{q(z)+p\beta+\gamma-1}\\&=h(z) \qquad (z\in\mathbb{U}).\end{aligned} \tag{13}$$

It follows from (5) and (13) that

$$\Re\{h(z)+p\beta+\gamma-1\}>0 \qquad (z\in\mathbb{U}). \tag{14}$$

Moreover, by using Lemma 3, we conclude that the differential equation (13) has a solution $q(z)\in\mathscr{H}(\mathbb{U})$ with

$$h(0)=q(0)=1.$$

We now let

$$\mathfrak{H}(u_1;v_1)=u_1+\frac{v_1}{u_1+p\beta+\gamma-1}+\delta,$$

where $\delta$ is given by (6). From (13) and (14), we thus find that

$$\Re\left\{\mathfrak{H}\left(q(z);zq'(z)\right)\right\}>0 \qquad (z\in\mathbb{U}).$$

In order to verify the validity of the following condition:

$$\Re\{\mathfrak{H}(is;t)\}\leqq 0 \qquad \left(s\in\mathbb{R};\ t\leqq-\frac{1+s^2}{2}\right), \tag{15}$$

we proceed as follows:

$$\begin{aligned}\Re\{\mathfrak{H}(is;t)\}&=\Re\left(is+\frac{t}{is+a}+\delta\right)\\&=\delta+\frac{t\,\Re(a)}{|is+a|^2}\\&\leqq-\frac{E_\delta(s)}{2|a+is|^2},\end{aligned}$$

where

$$\begin{aligned}E_\delta(s)&=[\Re(a)-2\delta]s^2-4\delta[\Im(a)]s\\&\quad+\left(\Re(a)-2\delta|a|^2\right).\end{aligned} \tag{16}$$

For $\delta$ given by (2.2), the coefficient of $s^2$ in the quadratic expression for $E_\delta(s)$ given by (16) is positive or equal to zero and $E_\delta(s)\geqq 0$. Thus, clearly, we see that

$$\Re\{\mathfrak{H}(is;t)\}\leqq 0 \qquad \left(\forall s\in\mathbb{R};\ t\leqq-\frac{1+s^2}{2}\right).$$

Thus, by using Lemma 2, we conclude that

$$\Re\{q(z)\}>0 \qquad (z\in\mathbb{U}),$$

that is, that $V(z)$ defined by (10) is convex (univalent) in $\mathbb{U}$.

We next prove that the subordination condition (7) implies that

$$U(z)\prec V(z),$$

for the functions $U(z)$ and $V(z)$ defined by (9) and (10), respectively. Without loss of generality, we assume that the function $V(z)$ is analytic and univalent on $\overline{\mathbb{U}}$ and

$$V'(\zeta)\neq 0 \qquad (|\zeta|=1).$$

If not, then we replace $U(z)$ and $V(z)$ by $U(\rho z)$ and $V(\rho z)$, respectively, where $0<\rho<1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, so we can use them in the proof of our result. The asserted result would follow by letting $\rho\to 1$. Consider the function $L(z,t)$ given by

$$L(z,t)=\left(1-\frac{1}{p\beta+\gamma}\right)V(z)+\frac{1+t}{p\beta+\gamma}zV'(z) \tag{17}$$

$$(0\leqq t<\infty;\ z\in\mathbb{U}).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0}=\left(1+\frac{t}{p\beta+\gamma}\right)V'(0)\neq 0$$

$$(0\leqq t<\infty;\ z\in\mathbb{U}).$$

This shows that the function $L(z,t)$ given by

$$L(z,t)=a_1(t)z+a_2(t)z^2+\cdots$$

satisfies the following conditions:

$$\lim_{t\to\infty}|a_1(t)|=\infty$$

and

$$a_1(t)\neq 0 \qquad (0\leqq t<\infty).$$

Furthermore, we have

$$\Re\left(\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right)=\Re\left\{p\beta+\gamma-1+(1+t)\left(1+\frac{zV''(z)}{V'(z)}\right)\right\}>0$$

$$(0 \leqq t < \infty;\ z \in \mathbb{U}).$$

Since the function $V(z)$ is convex and $\Re(p\beta+\gamma-1) > 0$, by using the well-known sharp growth and distortion inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. It follows from the definition of a subordination chain that

$$\begin{aligned} v(z) &= \left(1-\frac{1}{p\beta+\gamma}\right)V(z)+\frac{1}{p\beta+\gamma}\,zV'(z) \\ &= L(z,0) \end{aligned}$$

and

$$L(z,0) \prec L(z,t) \qquad (0 \leqq t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = v(\mathbb{U}) \tag{18}$$

$$(0 \leqq t < \infty;\ \zeta \in \partial\mathbb{U}).$$

If $U(z)$ is not subordinate to $V(z)$, by using Lemma 4, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\begin{aligned} U(z_0) = V(\zeta_0) \ \text{ and } \ z_0U'(z_0) \\ = (1+t)\,\zeta_0 V'(\zeta_0) \end{aligned} \tag{19}$$

$$(0 \leqq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0,t) &= \left(1-\frac{1}{p\beta+\gamma}\right)V(\zeta_0)+\frac{1+t}{p\beta+\gamma}\,\zeta_0V'(\zeta_0) \\ &= \left(1-\frac{1}{p\beta+\gamma}\right)U(z_0)+\frac{1}{p\beta+\gamma}\,z_0U'(z_0) \\ &= z\left(\frac{f(z)}{z^p}\right)^{\alpha}\phi(z) \in v(\mathbb{U}). \end{aligned}$$

This contradicts (18). Consequently, we deduce that

$$U(z) \prec V(z).$$

Considering the case when

$$U(z) = V(z) \qquad (z \in \mathbb{U}),$$

we see that the function $V(z)$ is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

**Theorem 2.** *Let $f,g \in \mathscr{A}_{p,\alpha,\delta}$ and*

$$\Re\left(1+\frac{zv''(z)}{v'(z)}\right) > -\delta \qquad \left(v(z) = z\left(\frac{g(z)}{z^p}\right)^{\alpha}\phi(z)\right),$$

*where $\delta$ is given by (6). If the function*

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha}\phi(z)$$

*is univalent in $\mathbb{U}$ and*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\Phi(z) \in \mathscr{H}[1,1]\cap\mathscr{Q}.$$

*Then the following subordination condition*:

$$v(z) \prec u(z) = z\left(\frac{f(z)}{z^p}\right)^{\alpha}\phi(z) \tag{20}$$

*implies that*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z^p}\right)^{\beta}\Phi(z) \prec z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\Phi(z) \tag{21}$$

*and the function*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z^p}\right)^{\beta}\Phi(z)$$

*is the best subordinant.*

*Proof.* Suppose that the functions $U(z)$, $V(z)$ and $q(z)$ are defined by (9), (10) and (11), respectively. We will use a method similar to that used in the proof of Theorem 1. Indeed, as in Theorem 1, we have

$$\begin{aligned} v(z) &= \left(1-\frac{1}{p\beta+\gamma}\right)V(z) \\ &\quad+\frac{1}{p\beta+\gamma}\,zV'(z) = \varphi\left(V(z),zV'(z)\right) \end{aligned}$$

and we obtain

$$\Re\{q(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Next, in order to obtain the desired result, we show that $V(z) \prec U(z)$. For this purpose, we suppose that the function $L(z,t)$ given by

$$L(z,t) = \left(1-\frac{1}{p\beta+\gamma}\right)V(z)+\frac{t}{p\beta+\gamma}\,zV'(z)$$

$$(0 \leqq t < \infty;\ z \in \mathbb{U}).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \left(1-\frac{1}{p\beta+\gamma}\right)V'(0) \neq 0$$

$$(0 \leqq t < \infty;\ z \in \mathbb{U}).$$

This shows that the function $L(z,t)$ given by

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$$

satisfies the following conditions:

$$\lim_{t\to\infty} |a_1(t)| = \infty$$

and

$$a_1(t) \neq 0 \qquad (0 \leqq t < \infty).$$

Furthermore, we have

$$\Re\left(\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right) = \Re\left\{p\beta + \gamma - 1 + t\left(1 + \frac{zV''(z)}{V'(z)}\right)\right\} > 0$$

$$(0 \leqq t < \infty;\ z \in \mathbb{U}).$$

Since the function $V(z)$ is convex and

$$\Re(p\beta + \gamma - 1) > 0,$$

by using the well-known sharp growth and distortion inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. Therefore, by using Lemma 5, we conclude that the superordination condition (20) must imply the superordination given by (21). Moreover, since the differential equation has a univalent solution $V$, it is the best subordinant. This completes the proof of Theorem 2.

By combining Theorem 1 and Theorem 2, the following *sandwich-type* results are derived.

**Theorem 3.** *Let* $f, g_j \in \mathscr{A}_{p,\alpha,\delta}$ $(j = 1,2)$ *and*

$$\Re\left(1 + \frac{zv_j''(z)}{v_j'(z)}\right) > -\delta \tag{22}$$

$$\left(v_j(z) = z\left(\frac{g_j(z)}{z^p}\right)^{\alpha} \phi(z)\right),$$

*where* $\delta$ *is given by* (6). *If the function*

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha} \phi(z)$$

*is univalent in* $\mathbb{U}$ *and*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta} \Phi(z) \in \mathscr{H}[1,1] \cap \mathscr{Q},$$

*then the following subordination condition*:

$$v_1(z) \prec u(z) = z\left(\frac{f(z)}{z^p}\right)^{\alpha} \phi(z) \prec v_2(z) \tag{23}$$

*implies that*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta} \Phi(z) \prec z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta} \Phi(z) \prec z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta} \Phi(z), \tag{24}$$

*Moreover, the functions*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta} \Phi(z)$$

*and*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta} \Phi(z)$$

*are the best subordinant and the best dominant, respectively.*

The assumption of Theorem 3 that the functions

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha} \phi(z) \quad \text{and} \quad z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta} \Phi(z)$$

need to be univalent in $\mathbb{U}$ may be replaced as in the following corollary.

**Corollary.** *Let* $f, g_j \in \mathscr{A}_{p,\alpha,\delta}$ $(j = 1,2)$. *Suppose also that the condition* (22) *is satisfied and*

$$\Re\left(1 + \frac{z\Theta''(z)}{\Theta'(z)}\right) > -\delta \tag{25}$$

$$\left(\Theta(z) = z\left(\frac{f(z)}{z^p}\right)^{\alpha} \phi(z);\ z \in \mathbb{U}\right),$$

*where* $\delta$ *is given by* (6). *Then the following subordination condition*:

$$v_1(z) \prec u(z) = z\left(\frac{f(z)}{z^p}\right)^{\alpha} \phi(z) \prec v_2(z)$$

*implies that*

$$z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta}\Phi(z) \prec z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\Phi(z) \prec z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta}\Phi(z). \quad (26)$$

*Proof.* In order to prove the above Corollary, we have to show that the condition (25) implies the univalence of $\Theta(z)$ and

$$U(z)=z\left(\frac{I^{p,\Phi,\phi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\Phi(z).$$

Since $0 \leqq \delta < \frac{1}{2}$, it follows that the function $\Theta(z)$ is close-to-convex in $\mathbb{U}$ (see [9]) and hence $\Theta(z)$ is univalent in $\mathbb{U}$. Also, by using the same techniques as in the proof of Theorem 1, we can easily show that the function $U(z)$ is convex (univalent) in $\mathbb{U}$. The details involved are being omitted here. Therefore, by applying Theorem 3, we obtain the desired result asserted by the above Corollary.

**Remark 2.**

(i) For $p=1$ in our results, we obtain the results obtained by Cho *et al.* [5];

(ii) For $p=1$ and $\Phi(z)=1$ in our results, we obtain the results obtained by Cho and Bulboacă [4];

(iii) For $p=1$ and $\Phi(z)=\phi(z)=1$ in our results, we obtain some of the results obtained by Cho and Kwon [6].

## 3 Applications to the Gauss Hypergeometric Function

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by (see [17] and [21])

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$\left(z\in\mathbb{U};\ a,b\in\mathbb{C};\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \mathbb{Z}_0^-=\{0,-1,-2,\cdots\}\right),$$

and

$$(\lambda)_n=\begin{cases}1 & (n=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & (n\in\mathbb{N};\ \lambda\in\mathbb{C}).\end{cases}$$

For the function ${}_2F_1(a,b;c;z)$, the following Eulerian integral representation is known (see [21]):

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z) \quad (27)$$

$$\left(\Re(c)>\Re(b)>0\right).$$

If we set

$$g(z)=\frac{z^p}{(1-z)^{\eta}} \quad (\eta>0) \qquad \text{and} \qquad \phi(z)=\Phi(z)=1,$$

then (4) yields

$$I^{p}_{\alpha,\beta,\gamma,\delta}(g)(z)=z^p\left[{}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z)\right]^{\frac{1}{\beta}}.$$

**Theorem 4.** *Let $f\in\mathscr{A}_{p,\alpha,\delta}$ with $\phi(z)=1$. Suppose that*

$$\eta\alpha<2\delta+1 \qquad (\eta>0),$$

*where $\delta$ is given by (7). Then the following subordination condition*:

$$\left(\frac{f(z)}{z^p}\right)^{\alpha}\prec(1-z)^{-\eta\alpha}$$

*implies that*

$$\left(\frac{I^{p}_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^{\beta}\prec {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z)$$

*and the function*

$${}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z)$$

*is the best dominant.*

**Theorem 5.** *Let $f\in\mathscr{A}_{p,\alpha,\delta}$ with $\phi(z)=1$. Suppose that*

$$\eta\alpha<2\delta+1 \qquad (\eta>0),$$

*where $\delta$ is given by (7). If the function*

$$\left(\frac{f(z)}{z^p}\right)^{\alpha}$$

*is univalent in $\mathbb{U}$ and*

$$\left(\frac{I^{p}_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^{\beta}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

*then the following subordination condition*:

$$(1-z)^{-\eta\alpha}\prec\left(\frac{f(z)}{z^p}\right)^{\alpha}$$

*implies that*

$$ {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) \prec \left(\frac{I^p_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^\beta $$

*and the function*

$$ {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) $$

*is the best subordinant.*

If we set

$$ g(z)=\frac{z^p}{(1-z)^\eta} \qquad (\eta>0), \qquad \phi(z)=\frac{1}{1-z} $$

and

$$ \Phi(z)=1, $$

then (4) yields

$$ I^p_{\alpha,\beta,\gamma,\delta}(g)(z) = z^p\left[{}_2F_1(p\alpha+\delta,\eta\alpha+1;p\alpha+\delta+1;z)\right]^{\frac{1}{\beta}}. $$

**Theorem 6.** *Let $f\in\mathscr{A}_{p,\alpha,\delta}$ with*

$$ \phi(z)=\frac{1}{1-z}. $$

*Suppose that*

$$ \eta\alpha<2\delta+1 \qquad (\eta>0), $$

*where $\delta$ is given by (7). Then the following subordination condition:*

$$ \left(\frac{f(z)}{z^p}\right)^\alpha \frac{1}{1-z} \prec (1-z)^{-\eta\alpha-1} $$

*implies that*

$$ \left(\frac{I^p_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^\beta \prec {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) $$

*and the function*

$$ {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) $$

*is the best dominant.*

**Theorem 7.** *Let $f\in\mathscr{A}_{p,\alpha,\delta}$ with*

$$ \phi(z)=\frac{1}{1-z}. $$

*Suppose that*

$$ \eta\alpha<2\delta+1 \qquad (\eta>0), $$

*where $\delta$ is given by (7). If the function*

$$ \left(\frac{f(z)}{z^p}\right)^\alpha $$

*is univalent in $\mathbb{U}$ and*

$$ \left(\frac{I^p_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^\beta \in \mathscr{H}[1,1]\cap\mathscr{Q}, $$

*then the following subordination condition:*

$$ (1-z)^{-\eta\alpha-1} \prec \left(\frac{f(z)}{z^p}\right)^\alpha \frac{1}{1-z} $$

*implies that*

$$ {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) \prec \left(\frac{I^p_{\alpha,\beta,\gamma,\delta}(f)}{z^p}\right)^\beta $$

*and the function*

$$ {}_2F_1(p\alpha+\delta,\eta\alpha;p\alpha+\delta+1;z) $$

*is the best subordinant.*

## 4 Concluding Remarks and Observations

In our present investigation, we have derived several subordination, superordination and sandwich type results which are related to a certain family of integral operators defined on the space of $p$-valent functions in the open unit disk $\mathbb{U}$. We have also successfully applied these subordination and superordination theorems to the familiar Gauss hypergeometric function. Our results provide generalizations of a number of previously known subordination, superordination and sandwich-type theorems. We have indicated some of these connections of our results with those that were proven in earlier works.

## References


[1] T. Bulboacă, Integral operators that preserve the subordination, *Bull. Korean Math. Soc.* **32** (1997), 627–636.

[2] T. Bulboacă, On a class of integral operators that preserve the subordination, *Pure Math. Appl.* **13** (2002), 87–96.

[3] T. Bulboacă, A class of superordination-preserving integral operators, *Indag. Math.* (*New Ser.*) **13** (2002), 301–311.

[4] N. E. Cho and T. Bulboacă, A class of integral operators preserving subordination and superordination, *Complex Var. Elliptic Equ.* **58** (2013), 909–921.

[5] N. E. Cho, T. Bulboacă and H. M. Srivastava, A general family of integral operators and associated subordination and superordination properties of some special analytic function classes, *Appl. Math. Comput.* **219** (2012), 2278–2288.

[6] N. E. Cho and O. S. Kwon, A class of integral operators preserving subordination and superordination, *Bull. Malays. Math. Sci. Soc.* **33** (2010), 429–437.
[7] N. E. Cho and H. M. Srivastava, A class of nonlinear integral operators preserving subordination and superordination, *Integral Transforms Spec. Funct.* **18** (2007), 95–107.
[8] D. J. Hallenbeck and T. H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Series of Monographs and Studies in Mathematics, Pitman Advanced Publishing Program, Boston and London, 1984.
[9] W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.* **2** (1952), 169–185.
[10] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* **28** (1981), 157–172.
[11] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, *J. Differential Equations* **56** (1985), 297-309.
[12] S. S. Miller and P. T. Mocanu, Integral operators on certain classes of analytic functions, In: *Univalent Functions*, *Fractional Calculus*, *and Their Applications* (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989, pp. 153–166.
[13] S. S. Miller and P. T. Mocanu, Classes of univalent integral operators, *J. Math. Anal. Appl.* **157** (1991), 147–165.
[14] S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. **225**, Marcel Dekker, New York and Basel, 2000.
[15] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Variables Theory Appl.* **48** (2003), 815–826.
[16] S. S. Miller and P. T. Mocanu and M. O. Reade, Starlike integral operators, *Pacific J. Math.* **79** (1978), 157–168.
[17] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (1987), 1057–1077.
[18] S. Owa and H. M. Srivastava, Some subordination theorems involving a certain family of integral operators, *Integral Transforms Spec. Funct.* **15** (2004), 445–454.
[19] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
[20] H. M. Srivastava, D. Răducanu and P. Zaprawa, A certain subclass of analytic functions defined by means of differential subordination, *Filomat* **30** (2016), 3743–3757.
[21] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *With an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.

**H. M. Srivastava** For the author's biographical and other professional details (including the lists of his most recent publications such as Journal Articles, Books, Monographs and Edited Volumes, Book Chapters, Encyclopedia Chapters, Papers in Conference Proceedings, Forewords to Books and Journals, *et cetera*), the interested reader should look into the following Web Site:
http://www.math.uvic.ca/faculty/harimsri.

**M. K. Aouf** was born on July 16, 1951 in Mansoura in Egypt. He graduated in 1973 and got his Ph.D. degree in 1981 in Pure Mathematics (Complex Analysis) from the Faculty of Science at Mansoura University in Egypt. Since 1991 he is a Professor of Mathematics at Mansoura University in Egypt. He received the degree of Doctor of Science (D.Sc.) in Pure Mathematics (Complex Analysis) from Mansoura University in Egypt in the year 2012.

**A. O. Mostafa** is an Assistant Professor in the Department of Mathematics (Faculty of Science) at Mansoura University in Egypt. She got her Ph.D. degree in Mathematics in the year 1990 from Mansoura University in Egypt. Her research interest is Complex Analysis.

**H. M. Zayed** is currently a Lecturer in the Department of Mathematics (Faculty of Science) at Menofia University in Shebin Elkom in Egypt. Her research interests are Complex Analysis and Functional Analysis.