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Fractal Aesthetics in Architecture

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Abstract: This paper deals with fractal aesthetics and proposes a new fractal analysis method for the perceptual study of architecture. The authors believe in the universality of formulas and aim to complement the architectural description in terms of proportion. Although a well established fractal analysis method to describe the complexity of facades across different scales already exists, box-counting is imprecise because of too many influences coming along with the method itself. The authors consider the self-similarity as an important part of aesthetic quality in architecture. This is due to the fact that it describes a concept of consistency that holds everything together from the whole to the smallest detail which refers to the classical meaning of the word symmetry. Hence, a new fractal analysis method is introduced which so far has been applied to quantitative linguistics. Basically, elements of different order, called construct/constituent pairs, are counted and related in a formula. In architecture the pairing consists of likewise elements belonging to different orders, from the overview, the fundamental elements to the smaller details. As a conjecture, some preferable fractal dimensions (from the aesthetical point of view) are proposed for architectural structures.

Keywords: Architectural analysis, design analysis, fractal geometry, complexity, harmonic proportion, construct/constituent pairing, box-counting

1 Introduction

1.1 Symmetry

The classical meaning of symmetry dates back to Greek times and was described by Vitruvius Pollio (1914) as *"... a proper agreement between the members of the work itself, and relation between the different parts and the whole general scheme, in accordance with a certain part selected as standard"*. This understanding of the word symmetry was common to Gothic Master Builders and architects of the Renaissance (Ghyka 1977). It describes the harmonic arrangement of elements which is achieved by a certain relation between each (important) part and between (important) parts and the whole. The linking element is a common measure, the proportion. Hence symmetry is measurement and can be calculated from an even smaller part like the thickness of a column or a modulus (Vitruvius 1914, Ghyka 1977).

1.2 Self-similarity as a factor of quality in architecture

The classical notion of symmetry strongly reminds us of Benot Mandelbrot's (1983) term 'self-similarity'. In fact, unlike mathematicians, physicists still understand by symmetry the invariantness under transformations and, in particular, under scaling.

Architectural quality includes many different factors. One of them concerns the linkage of architectural components across different scales. This aspect is linked to a specific property of fractals, known as 'self-similarity' or, more generally, 'self-affinity'. With theoretical (mathematical) fractals, even an infinitely small part can represent an exact or at least a somehow similar copy of the whole. This property is common in nature (Mandelbrot 1981), but only for a limited range of scales. That means from a certain scale onwards no more copies of the whole can be identified. The same is true in architecture. Moreover, in nature and architecture, each part usually includes some variation. Nevertheless, there are many situations when these parts are similar to each other and to the whole. Thus, because of its definition as a

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structure with infinitely small and self-similar parts, a fractal is only a product of theory. As a consequence, the authors recommend calling objects that exhibit fractal characteristics for a limited range of scales, fractal-like objects (Lorenz 2013b). What these objects have in common is their high complexity, whereas the underlying laws are very simple.

In architecture, self-similarity equals the concept that holds the building together, across many different scales; from the whole to a very small detail (e.g. a window-frame or an ornament). This is achieved e.g. by the application of the same proportion ratio ($a : b$; e.g. ratio between width and height) for each (important) element, from the overview and main structures (e.g. bays and main edges) to openings and even to the interior. Finally, if a whole range of characteristic ratios of a building are linked together by a modulus, the term proportion describes the classical meaning of symmetry.

2 Motivation

This paper aims to develop further the application of fractal analysis methods in order to measure both the complexity of facades and the organising depth of similar architectural elements (Jencks 1995). The authors' opinion is that this will open up the possibility to describe aesthetics in architecture by a number or a set of numbers. The concept is linked to Pythagoras of Samos' axiom "Everything is arranged according to number and mathematical shape" (Ghyka 1977), which was taken up by Plato and their followers. In this context, proportion becomes the fundament of order which is finally perceived as harmony.

Currently, a well established fractal analysis method introduced by Mandelbrot (1983) exists in form of 'box-counting', an approach which measures roughness (density of lines). In architecture, the method was first time used in the 1990s (Batty and Longley 1994, Bovill 1996; for more recent references see Eglash 2005, Gullet 2012, Harris 2012, Ostwald 2009, Ostwald and Vaughan 2008, Ostwald et al. 2008, Sala 2012, Wen and Kao 2005). Since then researchers have continued to apply box-counting to facades of various architects (Bovill 1996, Ostwald and Vaughan 2008, Lorenz 2012). However, only a few researchers have dealt with architecture-specific influence factors of the method itself (Ostwald et al. 2008, Lorenz 2009). Nonetheless, the big advantages of the technique are:

– it is a simple algorithm, which is easily implemented, and
– it offers a possibility to measure 'self-similar' as well as 'non-self-similar' (or even non-fractal) objects which is the reason why the method is called universal.

The latter aspect is important, because architecture will never produce pure fractals but only fractal-like structures. However, due to various inadequacies (detailed out later in this paper) box-counting only serves as a comparison for another fractal analysis method. Continuing the linguistic fractal analysis of texts by the second author (Andres 2009, 2010, 2014), this paper discusses its application to architecture. The idea behind the method is to define construct/constituent pairs whose lengths can be easily counted, and subsequently brought into a formula. In linguistic terms, the first set of pairs consists of semantic constructs and their constituents, which are sentences/clauses. In the next step, words are, in turn, the constituents of sentences/clauses, etc. The paper will discuss architectural analogies of text segmentations into different units of length. That means the main focus lies on how a facade subdivision into elements looks like to get a similar construct/constituent to linguistic analysis. An advantage of using this method is the minimization of influences and its universality. One can therefore speak with this respect about the *language of architecture*.

2.1 Universality of formulas

Although the formulas are universal, they are not sufficient in architecture. One exception is the usage of proportion ratios, representing an attempt to establish the order. In connection with aesthetics in architecture, the authors are convinced of the importance of proportion systems and especially about the specific significance of the so-called metallic means such as the golden, silver and bronze ratio (see equation 1). Basically, a metallic mean expresses a continued fraction of the form (Andres 2015, Andres and Fišer 2005, de Spinadel 1998):

$$\lambda = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{\cdots}}}. \tag{1}$$

with $k = 1$ for the golden ratio ($\lambda = 1.6180$), $k = 2$ for the silver ratio ($\lambda = 2.4142$) and $k = 3$ for the bronze ratio ($\lambda = 3.3028$). Because of its significance the authors will relate the proportion ratio with the results of their measurements using the construct/constituent pairing. Finally and as a result, a formula will be established that characterises the considered building. This may lead, in turn, to specific formulas of architectural styles. In short, the authors focus on formalism for architecture, as was done in quantitative linguistics. Therein lies the key difference between the two measurement methods: While box-counting dimension describes the structure, the formalism of construct/constituent pairing expresses a statement about the style.

3 Background

3.1 Self-similarity dimension

Each part of mathematical self-similar fractals like the Koch curve represents the exactly same structure as the whole. Thus, a precise relation exists between the number of single parts and the considered scale. It is the instruction rule that provides information about this relation: In order to generate the Koch curve a line is divided into three parts, whereby the middle part is replaced by an equilateral triangle without its base (see Figure 2). Hence, for the first iteration, the number (N) of single parts is 4 and its scale (s) is $1/3^{\text{rd}}$. Finally, the relation between number and scale defines the self-similar dimension (D_s):

$$D_s = \frac{\log N}{\log \frac{1}{s}}, \tag{2}$$

provided the single parts are totally disconnected or, at least, just touching.

3.2 Box-counting dimension

Since the self-similarity dimension cannot be calculated for structures which are not self-similar, certain alternative fractal analysis methods have become established, including box-counting. With box-counting, so to speak, pixelated representations of different resolutions replace the analyzed object. For doing so a mesh is placed over the object (see Figure 3 right). The mesh size equals the reciprocal number of lattice boxes across one row or the absolute size of a single box. The number of boxes that completely covers the object is counted. In order to minimize influences coming along with the method, the position of the mesh in relation to the measured object changes several times; Foroutan-pour et al. (1999) recommend 100 grid offsets. Subsequently, for a certain mesh-size only the smallest number of covering boxes is taken into account. This is in accordance with the fact that the box-counting dimension demands the smallest number of covered boxes for a certain grid-size (Peitgen et al. 1992). In the next step the grid-size is reduced and measurement starts again. The box-counting dimension between two mesh-sizes (mesh-size 1 and mesh-size 2) is given as follows:

$$D_{B_{\overline{12}}} = \frac{\log \frac{N_{s_2}}{N_{s_1}}}{\log \frac{s_1}{s_2}} \tag{3}$$

The repeated reduction of the mesh-size results in several data points that are the product of the number of covering boxes (N) and the particular mesh-size (s). With normalizing the results in a double logarithmic graph ($\log N$ versus $\log s$), three different characteristics of the data curve are possible: First, a clear trend becomes apparent. In this case all points (at least within a limited range of scales) approximately follow a straight line. That indicates a clear connection between the number of covering boxes and the mesh-size. It displays a specific range of coherence where the whole and all elements demonstrate similar roughness. The lower limit of the range in architecture depends on the limited scale of perception and can be derived from the smallest size of used individual components. Visually in the graph, from a certain grid-size onwards the data curve approximates to a 45 degree incline. This identifies the turnaround where only single (one-dimensional) curves of the object are measured. Since the slope equals the box-counting dimension of the specific range this is supported by the calculation of the 45 degrees slope with $D_s = 1$ (see Figure 4). The second characteristic, which is similar to the first, displays two clearly separate trends before the data curve approximates to a 45 degree incline (e.g. south-east elevation of Le Corbusiers Villa Savoye, Lorenz 2013b). Finally, the last case concerns fluctuating data points, which means no connection exists between number of covering boxes and grid-size.

As a consequence of the above, the curve progression is of a great importance. The coefficient of determination provides quantifiable information. A value close to one indicates a high correlation of the data points and, consequently, a similar roughness across the considered range of mesh-sizes (as described in the first two cases). Vice versa a value close to zero indicates no correlation (as described in the last case of fluctuating data points). According to Bovill (1996), the box-counting dimension – which is equivalent to the fractal Hausdorff dimension (Mandelbrot 1983)- is a characteristic value of how much texture an object has (e.g. a facade). It is also equivalent to the complexity of the whole structure for a specific range of mesh-sizes – the higher the value, the more twisted the curve is or the more texture an object has (Bovill 1996). Consequently, the range of mesh-sizes, the box-counting dimension (the slope of the regression line) and the coefficient of determination provides significant values of the composition and a possible comparison for architecture in terms of roughness across scales. If, in particular, the single parts of self-similar structures are totally disconnected or, at least, just touching (like those of the Koch curve), the box-counting dimension coincides with the self-similarity dimension (Falconer 2003).

3.3 Influences on the method

Box-counting depends on several influences. The first influence concerns the choice of elements. In architecture, it is the selection of edges that defines the 2-dimensional representation of a facade (the contour, projections and recesses, changes of material or the differentiation between various architectural elements). The selection is either a question of the scale of perception (distance of the observer) or of the level of design (overall main idea versus detail including cornices and ornaments). In addition, the specific selection changes when coming closer or when considering design elements on smaller scales (see Figure 5). Door handles or window frames will not be included in an overview but at smaller scales of detail. Moreover, when analysing the furthest location of an observer (i.e. the closed silhouette), box-counting is no more an adequate method. The authors recommend the structured walk method for closed curves instead (measured dimension). The second influence concerns line thickness of the representation. This aspect can be avoided by using the vector graphics instead of pixel graphics (Lorenz 2013b). The third influence relates to the range of grid-sizes. The largest grid-size is recommended as one fourth of the smallest side of the object under consideration. If the object is of less complexity, the value increases to one third (Foroutan-pour et al. 1999). The smallest grid-size in turn depends on the smallest detail and is reflected in the double logarithmic graph by the point where the data curve approximates to a 45 degree incline. Fourth, both the position and the orientation of the mesh influence the result. These influences can be countered by repeatedly changing the starting position for one and the same mesh-size. Finally, the reduction factor of the grid has to be considered as well. Usually the grid-size is reduced by one half, which implies larger steps between larger mesh-sizes. With smaller reduction ratios, in turn, the influence by the position of the mesh increases. Therefore, different starting positions for each mesh-size are recommended (Foroutan-pour et al. 1999, Lorenz 2013b).

3.4 Usability of box-counting

Although box-counting can be applied to fractals as well as non-fractal objects and is therefore universal, the method is not optimal for measuring aesthetics in architecture. The method does not distinguish between overlaps and collisions. In particular, overlapping elements and subsequently hidden architectural elements are not taken into account (e.g. doors or other architectural elements that are hidden by a balcony parapet wall). However, for analysis of the design intention overlapping is important. In this case all relevant parts have to be considered, including all hidden elements (edges). One possible solution is a 3-dimensional version of box-counting that uses cubes in a 3-dimensional lattice instead of boxes. In this case the difficulties mainly concern the lack of required 3-dimensional digital plans, especially in a detailed form on smaller scale. If the focus is on perception, in turn, the analysis of elevations (plans) conforms only partly to perception: an observer typically looks at an elevation from street level, a fact that demonstrates the need of a perspective instead of perpendicular representations (Ostwald and Tucker 2007).

Nevertheless, box-counting applies for several usages. Above all, it analyses the development of elements (density of lines) from the whole to the smallest detail. The data curve indicates whether or not the design provides similar distribution on every scale. However, box-counting does not identify the kind of connection between these elements of different scales; it just shows that a connection exists. One possibility of holding all elements of different size together is the application of a certain proportion ratio. As shown elsewhere (Lorenz 2013a), box-counting also serves as a method for comparison between predecessor and successor. Regardless of whether or not the interpretation is just a copy of the original design, similar characteristics reflect that the new building implements the underlying idea. Such characteristics include the same detail richness across scales and by that the same distribution of edges (similar slope of data curve and similar D_B). Furthermore, box-counting is also a good method to evaluate architectural compositions in their context. That is the fitness of the characteristics in relation to the environment. In this way, not only facades, but also mountain ridges or city maps can be analyzed and compared (Bovill 1996, Lorenz 2003).

4 Fractal language of architecture

4.1 Construct/constituent pairing

Fractal language of architecture must not be confused with language of architecture in the traditional sense (Jencks 2002). It is similar to the language developed for quantitative linguistics. The fractal analysis method applied to quantitative linguistics uses the idea of deconstruction. It is based on three binarisms (Andres 2009, 2010, 2014, Andres and Rypka 2012), listed as a construct/constituent pairing. The pairing includes a language unit on a higher level versus a language unit of a lower level:

- semantic constructs (their length is calculated in the integer number of sentences resp. clauses) / sentences resp. clauses (their length is calculated in the average number of words),
- sentences resp. clauses (in the integer number of words) / words (in the average number of syllables),
- words (in the integer number of syllables) / syllables (in the average number of phonemes).

The method counts the element lengths of pairing ($x \ldots$ the integer length of constructs; $y \ldots$ the average length of constituents) and puts them into perspective:

$$y = A * x^{-b}, \tag{4}$$

where A, b are real parameters, characterising the given structure under our consideration, to be specified.

Since this (truncated) formula was verified to hold statistically in quantitative linguistics, it is nowadays called the *Menzerath–Altmann law* (shortly, MAL). The heuristic version of MAL says that the longer a language construct, the shorter its components (constituents) are. Mathematically, it means that the parameters A, b should be positive, $A > 0$, $b > 0$.

With the correspondence

$$D_s \sim \frac{1}{b} \quad \text{and} \quad \frac{1}{b} = \frac{\ln x}{\ln \frac{A}{y}} = \frac{\log x}{\log \frac{A}{y}} \tag{5}$$

this allows us the calculation of the self-similarity dimension D_s out of the construct/constituent pairing, namely (cf. (2))

$$D_s \sim \frac{\log N}{\log \frac{1}{s}} \quad \text{with } N \sim x \text{ and } \frac{1}{s} \sim \frac{A}{y}, \tag{6}$$

because the formulas (4) and (5) are equivalent.

The method of deconstruction avoids overlapping of single parts, and therefore ensures that all design elements are included. Moreover, it employs self-similarity. Although, in nature (and architecture) self-similarity stops at a certain point, the method assures to be continued. Therefore, although only a limited level of binarisms is valid in nature, our assumption allows us to handle with self-similar structures in a mathematical way. Since a strict self-similarity can be rather restrictive in our investigation in general, one can relax such a hypothesis by a more literal presumption of a cyclic self-similarity, when self-similarity is repeated in blocks of levels. For more details, see Andres 2014, and Andres and Rypka 2012.

4.2 Elements of architecture

According to the construct/constituent pairing of linguistic analysis the concept of different unit-lengths in architecture – the disconnection in major and minor parts – is similar to the analogy of coming closer with box-counting. First, the silhouette with its main parts of design (in size and strong emergence) comes in the observers focus. This includes several elements and their division of the main composition: strong significant changes of the material, projections and recesses on a large scale. Roofs and walls are typical elements of this category (see Figure 6). Areas are separated from each other distinctly (primary design features). Minor parts of this level are arches, windows and doors. The next smaller level contains secondary design features including chimneys, horizontal banding features, handrails and gutters. Categorizing is not only a question of size, but also of the clarity of the difference.

In architecture a large number of elements are connected together to a smaller number of larger units that are, again, linked with each other and the whole and, finally, form the whole building (Meiss 1990). This characteristic is the prerequisite of an architectural parallel to the construct/constituent pairing of linguistic analysis. Parts may then concern volumes, spaces or elements. Following from above, a first classification includes:

– major parts of primary design features (solid parts such as roof, walls, columns, beams, significant changes of material) and minor parts of primary design features (openings such as arches, windows and doors);
– minor parts of primary design features and secondary design features (chimneys, horizontal banding features, handrails, gutters, frames, cornices and rustication);
– secondary design features and tertiary design features (doorknobs, components of a cornice).

As an example the following description gives a possible hierarchy of elements recognised in an antic Greek temple (Ionic order). According to figure 7 (also compare Gibbs 1732, Rattner et al. 1998, Chitham 1987) the front elevation of the temple consists of (from top to down) gable, entablature, column and basement. The entablature, for example, in turn, consists of cornice, frieze and architrave, while a single column can be separated into capital, shaft (with flute and fillet) and base. The cornice, again, consists of sima, corona, (egg-and-dart, bead-and-reel) dentil band and bed mold, while the capitals' subparts are volute, abacus, echinus and cincture. But even the dentil band is not the smallest part as it can still be divided into smaller elements (the single dentil). The point is that all parts of greek temples such as columns or their flutes (20-24) are not only in an hierarchical order but are also clearly distinguishable and therefore countable.

4.3 Usability

As described earlier, construct/constituent pairing considers self-similarity, which is (resp. whose consequences are) a possible approach to describe quality in architecture (see Introduction). Herein lies, compared with box-counting, the main advantage of this method. However, a couple of aspects have still to be considered. The first one concerns the right definition of construct/constituent pairing. Furthermore, in order to retain comparability, facades should be similar. This mainly concerns the same ratio between width and height rather than same absolute size. In addition for compared buildings an index of significance (or of change) should be defined. Another aspect concerns the plan to apply the method to simple photographs. This includes identifying automatically overlapping elements. Finally, with construct/constituent pairing symmetry is not taken into account. Consequently, a proportion analysis should be used as supplement (Kulcke et al. 2015). On the other

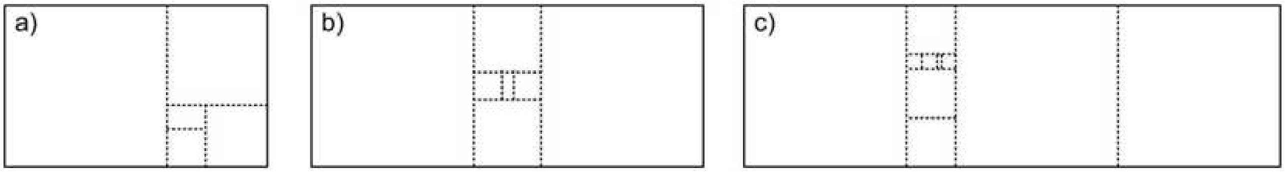

Fig. 1: Metallic means: a) golden ratio, b) silver ratio, c) bronze ratio.

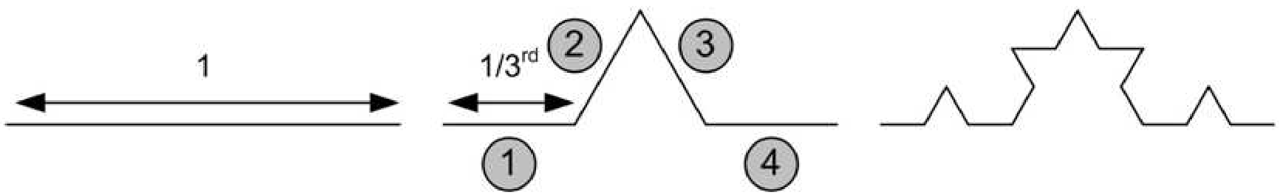

Fig. 2: With $N = 4$ and $s = 1/3^{\mathrm{rd}}$ the self-similar dimension of the Koch curve equals 1.2619.

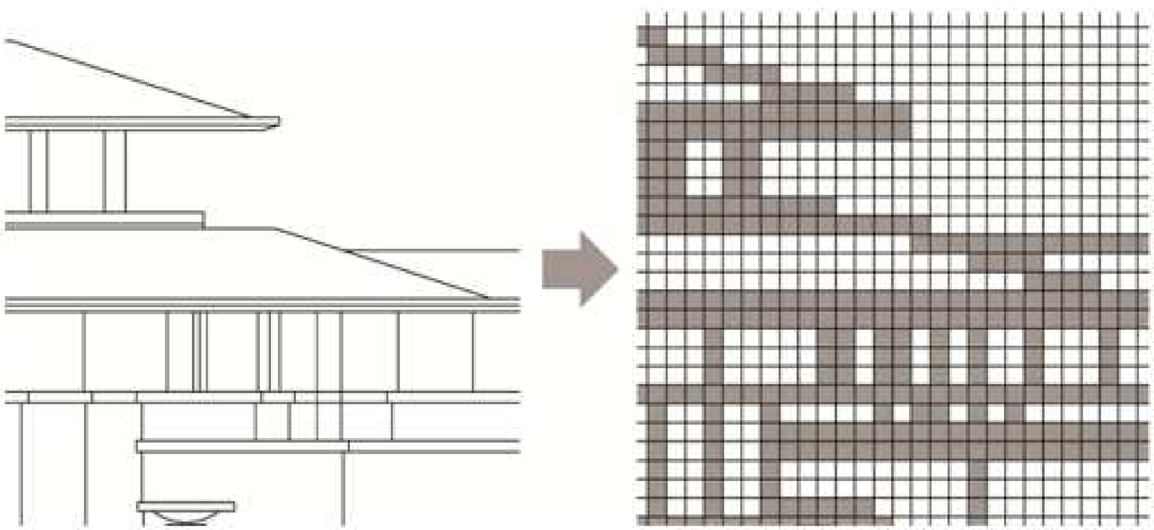

Fig. 3: Analyzed object (left) and pixelated representation (right).

hand, if we are able to detect specific parameters A, b in formulas (4) resp. (5), then formula (4) with those fixed parameters might be applied for a reconstruction or for an imitation of a given architectural style. Anyway, it is a question whether or not the analogy of the Menzerath–Altmann law is also valid for architectural compositions. This question is quite legal and promissible to be affirmatively answered, because formally the same law was observed to hold under various names (Pareto's law, Zipf–Mandelbrot's law) in nature and society.

5 Possible applications (fractal aesthetics)

Hence, the final idea of construct/constituent pairing is to develop a formula out of the results. That may lead to a specific characterization of a certain style, e.g. of an outstanding architect such as Frank Lloyd Wright. Furthermore, differences between styles may be deduced (e.g. between Baroque and Renaissance, or between Bauhaus and Adolf Loos).

If the single parts of fractal structures are not overlapped, then even the box-counting dimension D_B can be equivalently used instead of the self-similarity dimension D_s in our modelling, when taking $b \sim 1/D_s = 1/D_B$ in formula (4).

Last, but not least, an important aspect of the fractal aesthetics in architecture (whence the title of our contribution) must be still mentioned. There are not many empirical studies about fractal aesthetics, i.e. whether some fractal dimensions are preferable to others, and

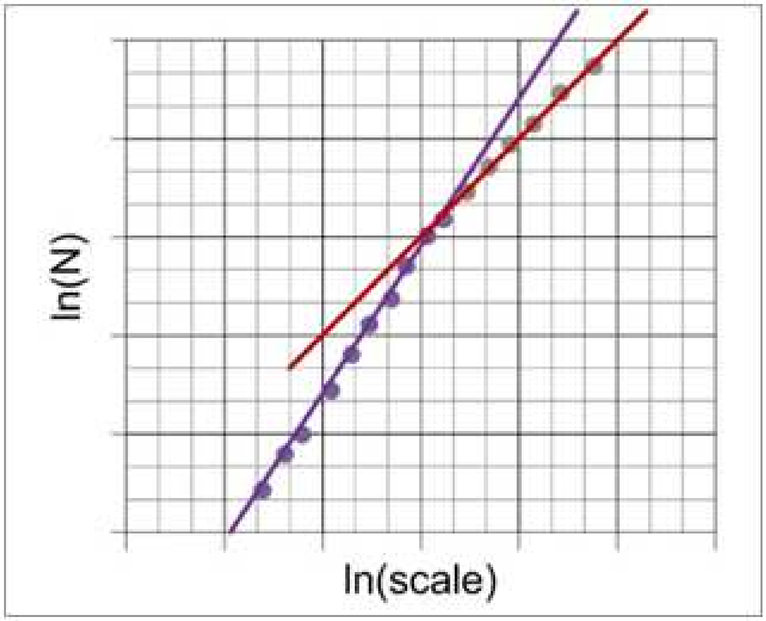

Fig. 4: Double logarithmic graph of number of counted boxes versus grid-size; two regression lines (red line: approximates to a 45 degree incline).

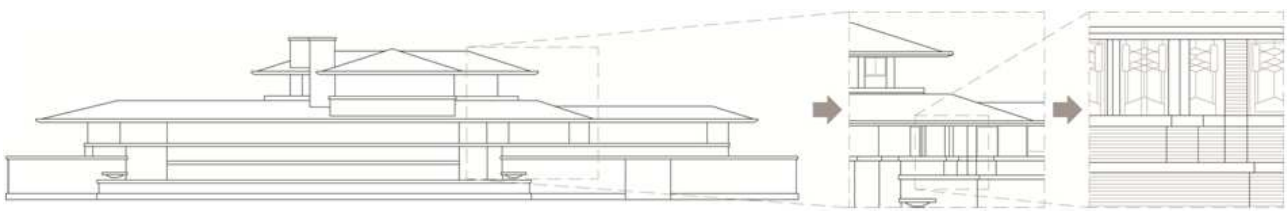

Fig. 5: Changing in perception: In general, when coming closer to a building on every level of scale, there are elements that fit to the scale of the observer at the specific distance.

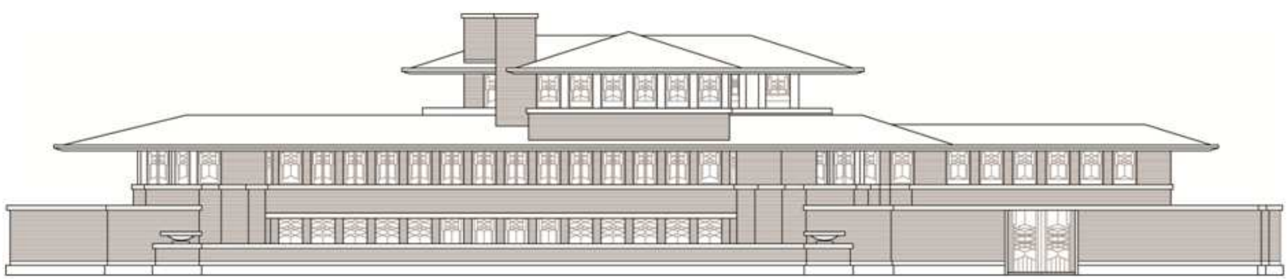

Fig. 6: Primary design features (major parts: roof and wall, minor parts: openings and smaller wall surfaces) and secondary design features (major parts: openings and smaller wall surfaces, minor parts: window frames, stained glass and facing bricks).

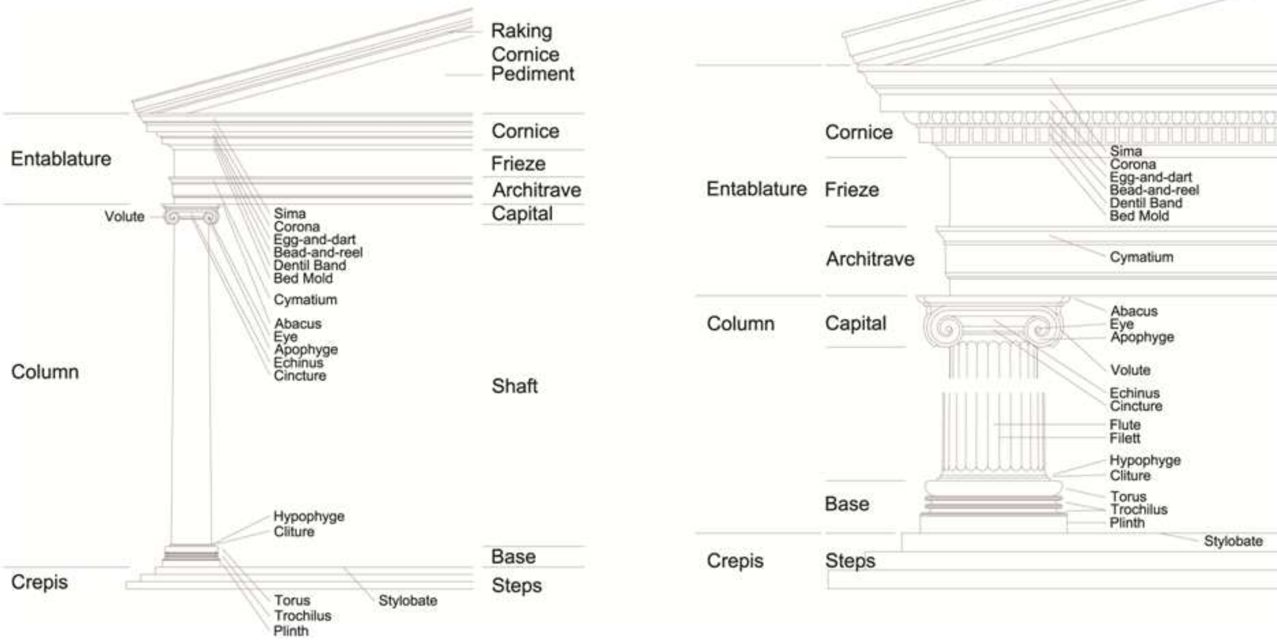

Fig. 7: Greek temple (Ionic order).

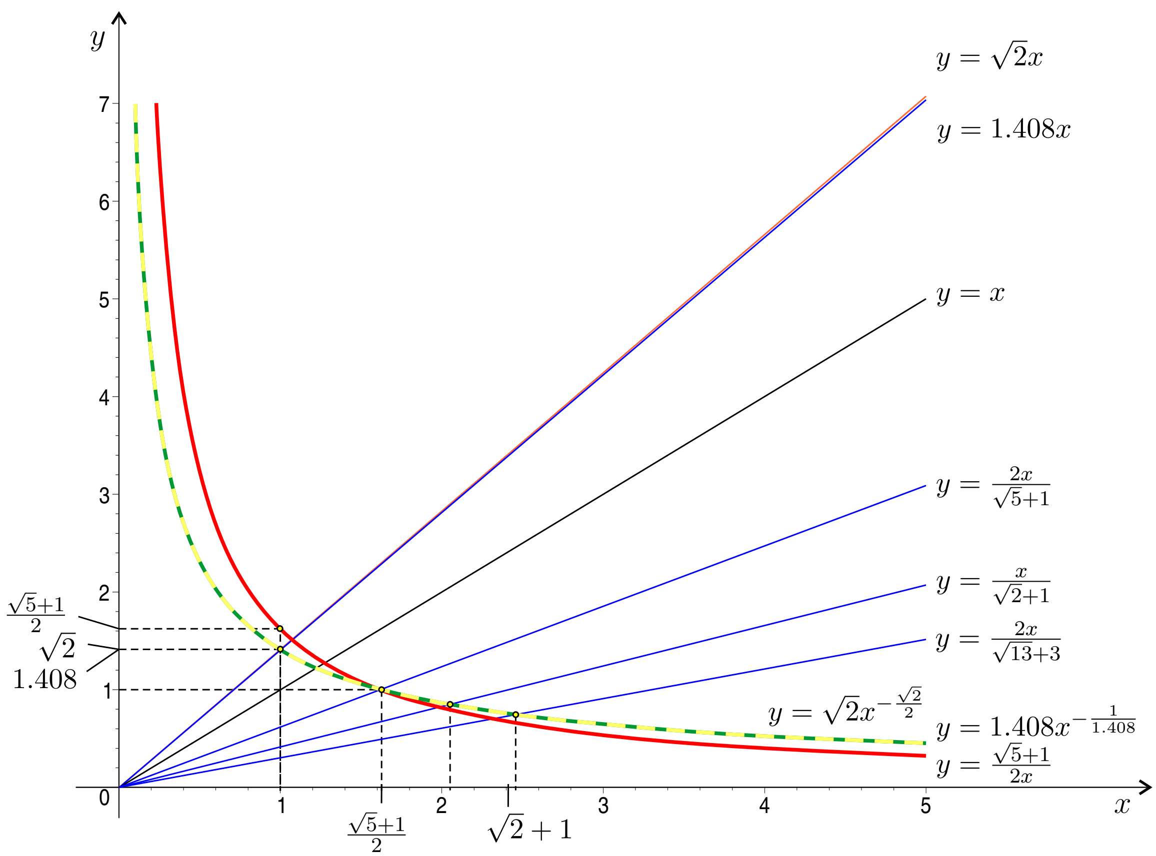

Fig. 8: Graphs of “aesthetic” functions.

eventually why. Moreover, the deduced criteria are rather rough, and so not very convincing. For instance, in 2D (like on the photos), the values of "nicest" fractal dimensions D were detected as $D = 1.51 \pm 0.43$ (Sprott 1994), $D = 1.52 \pm 0.23$ (Draves et al. 2008), $D = 1.3$ (Taylor 2006), etc. Since the tolerance is too big, the inconsistency suggests that there is so far no universally preferred fractal dimension.

Our first attempt is therefore to make a possible link between the self-similarity dimension and metallic means which are traditionally regarded as the most aesthetical (but static) proportions. Thus, the dynamic (functional) proportionality between constructs and their constituents in architecture should be somehow taken into account from such a point of view (cf. Figure 8). Taking, for instance,[1] $y = 1$ and $A = 10$, by which $\log\frac{A}{y} = 1$, one can normalize formula (5) in the sense that then $D_s = \log x$. In this way, we get for x equal to metallic means λ (cf. (1), and observe that the values of x are no longer integers) that:

$$D_s = \log\frac{\sqrt{5}+1}{2} \doteq \log 1.61803 \doteq 0.208987 \quad \text{(golden mean)},$$

$$D_s = \log(\sqrt{2}+1) \doteq \log 2.414213 \doteq 0.382776 \quad \text{(silver mean)},$$

$$D_s = \log\frac{\sqrt{13}+3}{2} \doteq \log 3.302775 \doteq 0.518879 \quad \text{(bronze mean)}.$$

Let us note that these calculations are conditioned by the chosen parameter value $A = 10$. In practice, A can be put more naturally equal to y, for $x = 1$, because $A = y$ for $x = 1$ in formula (4). In particular, formula (4) might then take the form

$$y = \lambda * x^{\frac{1}{D_s}}, \tag{7}$$

where λ is a metallic number and D_s is the "aesthetic" self-similarity dimension.

Finally, let us try to indicate, by virtue of formula (4), a possible relationship of the golden mean $\lambda \doteq 1.618033$ with the above values deduced by Sprott, Taylor, Draves et al. Hence, in order to get $D_s \doteq 1.3$ or $D_s \doteq 1.4$ or $D_s \doteq 1.5$, the associated parameter A value should satisfy $A \doteq 1.447960$ or $A \doteq 1.410186$ or $A \doteq 1.378239$, respectively, when taking $y = 1$ and $\log x = \log\lambda = 0.208987$. In other words,

$$\log A \doteq \frac{0.208987}{D_s}, \quad \text{resp. } A \doteq 10^{\frac{0.208987}{D_s}},$$

where D_s is taken as hopefully the most "aesthetic" value.

Let us observe that A does not differ much from D_s, for $A \doteq 1.410186$ and $D_s \doteq 1.4$. Moreover, 1.41 is the mean value of 1.3 and 1.52 (see the calculations by Taylor 2006 and Draves et al. 2008), while 1.405 is the mean value of 1.3 and 1.51(see again the calculations by Taylor 2006 and this time by Sprott 1994). Is it by chance? If not, then the solution $A = D_s \doteq 1.407579814$ of the equation

$$z\log z = \log\frac{\sqrt{5}+1}{2}$$

could play an important role in the fractal aesthetics and, in particular, in architectural compositions exhibiting the golden proportion between the associated constructs and their constituents whose normalized length is $y = 1$.

Since, for $A = D_s = \sqrt{2} \doteq 1.414$, we are not far from the solution of the equation $z\log z = \log\frac{\sqrt{5}+1}{2}$, i.e. from the value $1.407579814\ldots$, the form of MAL

$$y = \sqrt{2}x^{-\frac{\sqrt{2}}{2}}$$

(whose graph cannot be practically distinguished from the one of $y = 1.408x^{-\frac{1}{1.408}}$ in Figure 8) could also determine aesthetic architectural structures. Observe that, for $x = 1$, we get $y = \sqrt{2}$ which is the number well known as a gate of harmony. Furthermore, the coordinates of the points

$$[1, \sqrt{2}],\ \left[\frac{\sqrt{5}+1}{2}, 1\right],\ \left[2, \frac{2}{\sqrt{2}+1}\right],\ \left[\sqrt{2}+1, \frac{2\sqrt{2}+2}{\sqrt{13}+3}\right]$$

of–more or less–the same MAL exhibit the gate of harmony, golden, silver and bronze proportions, respectively. The first three are useful for our goal, because one coordinate is an integer. The associated fractal (self-similar) dimensions D_s of the second (golden) and third (silver) proportions are $D_s \doteq 1.388$ and $D_s \doteq 1.296$ which slightly differ from an earlier "aesthetic" value $D_s \doteq 1.408$, but it is still in accordance with the estimates due to Sprott (1994), Taylor (2006) and Draves et al. (2008).

By similar arguments, the values $A = D_s \doteq 1.686444370$ and $A = D_s \doteq 1.884883269$, as solutions of the equation $z\log z = \log\lambda$, where $\lambda = \sqrt{2}+1$ and $\lambda = \frac{\sqrt{13}+3}{2}$ are respectively the silver and bronze numbers, could play some role in fractal aesthetics, too. Nevertheless, since our arguments are only speculative, they must be supported by experimental investigations and checked statistically.

6 Conclusion/Outlook

Box-counting as a description of architecture is limited in its functionality due to influences of parameters. Furthermore, the result only displays whether or not a similar distribution of architectural elements of different

[1] Observe that $y = 1$ is the only positive solution of the equation $xy = x/y = \lambda$, where $x/y = \lambda$ stands for a direct proportion, while $xy = \lambda$ for an indirect one (cf. the heuristic version of MAL). Thus, this seems to be the only reasonable way how to match suitably a direct static metallic proportion with an indirect dynamic (functional) proportions of MAL. Otherwise, more precisely, $y = (A/\lambda^b)^{1/(b+1)}$ which is rather cumbersome for calculations.

size exists; it neglects the kind and quality of the relation between elements. Therefore, other methods have to be investigated: Construct/constituent pairing uses clearly distinguishable components of single elements (potentially distinguished by height, width and depth). This time elements of similar kind are counted. The key benefit (but, at the same time, also a non-universality restriction) is that self-similarity is taken into account. The method consists of

– identifying the construct/constituent pairs,
– calculation of parameters A, b in formula (4),
– a possible application of formula (4) to reconstruction or imitation of a given architectural style.

The applicability of this method will be checked by ourselves elsewhere on specific architectures; in particular with facades, which have been analyzed using the box-counting method.

We have also formulated a conjecture to be tested that the solutions of the equation $z \log z = \log \lambda$, where λ denotes the metallic means, can play an important role for preferable values of fractal dimensions in architectural aesthetics.

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