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A Result for the Formal Local Cohomology Module and Vanishing Results

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Abstract: In this paper, is discuted and studied about the formal local cohomology module. Also is put its connection with the local cohomology module and moreover is put some vanishing results for such module of formal local cohomology.

Keywords: inverse limit; local cohomology; formal local cohomology; serre subcategory.

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity. The local cohomology theory of Grothendieck has proved to be an important tool in commutative algebra. The theory of local cohomology if has developed so much six decades after its introduction by Grothendieck. There exists a relation between local cohomology, given by [1], and formal local cohomology, given by [5]. Not so much is known about these modules.

The motivation of this work is precisely the formal local cohomology of P. Schenzel. Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. For a finitely generated R-module M, let $\mathrm{H}^i_{\mathfrak{a}}(M)$, for $i \in \mathbb{N}$, denote the local cohomology module of M with respect to $\mathfrak{a}$ (cf. [1] for the basic definitions). There are the following integers related to these local cohomology modules, such as

$$\operatorname{grade}(\mathfrak{a},M) = \inf\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\} \text{ and,}$$

$$\operatorname{cd}(\mathfrak{a},M) = \sup\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\},$$

called the grade (respectively the cohomological dimension) of M with respect to $\mathfrak{a}$. In general we have the bounds:

$$\operatorname{height}_M(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a},M) \leq \dim(M).$$

In the case of $\mathfrak{m}$ the maximal ideal it follows that

$$\operatorname{grade}(\mathfrak{m},M) = \operatorname{depth}(M) \text{ and,}$$

$$\operatorname{cd}(\mathfrak{m},M) = \dim(M).$$

From the local cohomology we obtain the formal local cohomology, and the purpose here is to obtain results for the formal local cohomology module. Here we consider the asymptotic behavior of the family of local cohomology modules given by $\left\{\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)\right\}_{n\in\mathbb{N}}$ for an integer $i \in \mathbb{Z}$. By the natural homomorphisms these families form a projective system. Their projective limit given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$$

is called the ith formal local cohomology of M with respect to $\mathfrak{a}$. Not so much is know about these modules.

The main subject of the paper is a systematic study of the formal local cohomology modules.

In Section 2 we present notions of local cohomology modules and some prerequisites, which will be used throughout of the paper.

In Section 3 we provide a theorem interesting about the formal local cohomology module. We observe that the formal local cohomology module is used for to characterize some values. For example, we have the result: "Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. Then

$$\dim(M/\mathfrak{a}M) = \sup\left\{i \in \mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM) \neq 0\right\},$$

for a finitely generated R-module M."

Moreover, the description of

$$\inf\left\{i \in \mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM) \neq 0\right\},$$

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is used for to define the formal grade, i.e., we have that

$$\operatorname{fgrade}(\mathfrak{a},M)=\inf\left\{i\in\mathbb{Z}\mid \varprojlim_{n\in\mathbb{N}}\mathrm{H}^{i}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right)\neq 0\right\},$$

for an ideal $\mathfrak{a}$ of R and a finitely generated R-module M.

For to finishe, in the Section 4 we presented some results for vanishing of formal local cohomology modules.

2 Some definitions and preliminaries

Let I be an ideal of R, and let M be an R-module. In [1], the ith local cohomology module $\mathrm{H}^{i}_{I}(M)$ of M with respect to I is defined by

$$\mathrm{H}^{i}_{I}(M)=\varinjlim_{t\in\mathbb{N}}\operatorname{Ext}^{i}_{R}\left(R/I^{t},M\right),$$

for all $0\leq i\in\mathbb{Z}$. By [2, Remark 3.5.3 (a)], we have $\mathrm{H}^{0}_{I}(M)\cong\Gamma_{I}(M)$, where we have that

$$\Gamma_{I}(M):=\left\{m\in M\mid I^{t}m=0\text{ for some }t\in\mathbb{N}\right\},$$

is an R-submodule of the R-modulo M.

We can also see this definition of the following form.

Definition 1([2, Definition 3.5.2]) The local cohomology functors, denoted by $\mathrm{H}^{i}_{I}(\bullet)$, are the right derived functors of $\Gamma_{I}(\bullet)$. In other words, if $\mathbf{I}^{\bullet}$ is an injective resolution of the R-module M, then $\mathrm{H}^{i}_{I}(M)\cong\mathrm{H}^{i}(\Gamma_{I}(\mathbf{I}^{\bullet}_{M}))$ for all $i\geq 0$, where $\mathbf{I}^{\bullet}_{M}$ denotes the deleted injective resolution of M.

We present now the definition of the formal local cohomology module, the object main of the study of our work.

Suppose that $(R,\mathfrak{m})$ is a Noetherian local ring. For $\mathfrak{a}$ other ideal of R, consider the family of local cohomology modules given as it follows by $\left\{\mathrm{H}^{i}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right)\right\}_{n\in\mathbb{N}}$, for all $i\geq 0$, with M a finitely generated R-module. According to [5], for every $n\in\mathbb{N}$, there exists a natural homomorphism

$$\phi_{n+1,n}:\mathrm{H}^{i}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n+1}M\right)\to\mathrm{H}^{i}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right).$$

These families form an inverse system. Their inverse limit that is given by

$$\varprojlim_{n\in\mathbb{N}}\mathrm{H}^{i}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right),$$

is called, according to [5], the ith formal local cohomology module of M with respect to $\mathfrak{a}$, and will be denoted by $\mathfrak{F}^{i}_{\mathfrak{a},\mathfrak{m}}(M)$.

Let R be a Noetherian ring. Recall that a *Serre subcategory* $\mathfrak{S}$ of the category of R-modules is a subclass of R-modules (i.e., is a subset of R-modules) such that for any short exact sequence of R-modules, we say

$$0\to M^{'}\to M\to M^{''}\to 0,$$

we have that the R-module M is in $\mathfrak{S}$ if and only if $M^{'}$ and $M^{''}$ are in $\mathfrak{S}$.

Example 1 The following classes of R-modules are Serre subcategories of the category of R-modules.

(a)The class of zero R-modules.
(b)The class of finite length R-modules.
(c)The class of finitely generated R-modules.
(d)The class of Artinian R-modules.

Let R be a Noetherian ring. Let $\lambda:\mathfrak{S}\to\tau$ be a function from a Serre subcategory of the category of R-modules $\mathfrak{S}$ to a partially ordered Abelian monoid $(\tau,\star,\prec)$. We say that λ is a *subadditive function* if $\lambda(0)=0$ and for any short exact sequence, we say

$$0\to M^{'}\to M\to M^{''}\to 0,$$

in which all the terms (the R-modules) belong to $\mathfrak{S}$, we have $\lambda\left(M^{'}\right)\prec\lambda(M)$, $\lambda\left(M^{''}\right)\prec\lambda(M)$ and $\lambda(M)\prec\lambda\left(M^{'}\right)\star\lambda\left(M^{''}\right)$.

Example 2 The following function is subadditive: the function $\lambda(M)=\mathfrak{l}_{R}(M)$, length of M, from the class of finite length R-modules to the partially ordered Abelian monoid $(\mathbb{Z},+,\leq)$. Other examples it follows below:

(a)The function $\lambda(M)=(0:_{R}M)=\{x\in R\mid xM=0\}$, annihilator of the R-module M, from the category of R-modules to the partially ordered Abelian monoid $(\operatorname{Ideals}(R),\cdot,\supseteq)$.

The following remark will be used in the next section.

Remark 1 Let R be a Noetherian ring, and let M be an arbitrary R-module. Then the following statements are true.

$(a)$$\Gamma_{I}(M)\cong\operatorname{Hom}_{R}(R,\Gamma_{I}(M))$.
(b)If $\operatorname{Supp}_{R}(M)\subseteq V(I)$, then $\mathrm{H}^{i}_{I}(M)\cong\operatorname{Ext}^{i}_{R}(R,M)$ for all $i\geq 0$.

3 Application

In this section, we presented a result interesting about the formal local cohomology module. For the concept of linearly compact modules, see [4, Definition 3.1].

Theorem 1*Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let M be a linearly compact R-module and finitely generated; moreover, let i be a non-negative integer such that*

$$\varprojlim_{n\in\mathbb{N}}\operatorname{Ext}^{i-r}_{R}\left(R,\mathrm{H}^{r}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right)\right)\text{ is in }\mathfrak{S}\text{ for all }r,0\leq r\leq i.$$

Then $\mathfrak{F}^{i}_{\mathfrak{a},\mathfrak{m}}(M)\in\mathfrak{S}$, and

$$\lambda\left(\mathfrak{F}^{i}_{\mathfrak{a},\mathfrak{m}}(M)\right)\prec\star^{i}_{r=0}\,\lambda\left(\varprojlim_{n\in\mathbb{N}}\operatorname{Ext}^{i-r}_{R}\left(R,\mathrm{H}^{r}_{\mathfrak{m}}\left(M/\mathfrak{a}^{n}M\right)\right)\right).$$

*Proof.*We prove by using induction on i. The case $i=0$: from the Remark 1 (a), we have that

$$\mathrm{Ext}_R^0\left(R,\mathrm{H}_{\mathfrak{m}}^0\left(M/\mathfrak{a}^nM\right)\right)=\mathrm{Hom}_R\left(R,\Gamma_{\mathfrak{m}}\left(M/\mathfrak{a}^nM\right)\right)\cong\Gamma_{\mathfrak{m}}\left(M/\mathfrak{a}^nM\right).$$

As, by hypothesis, $\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^0\left(R,\mathrm{H}_{\mathfrak{m}}^0\left(M/\mathfrak{a}^nM\right)\right)\in\mathfrak{S}$ it follows that

$$\varprojlim_{n\in\mathbb{N}}\Gamma_{\mathfrak{m}}\left(M/\mathfrak{a}^nM\right)=\varprojlim_{n\in\mathbb{N}}\mathrm{H}_{\mathfrak{m}}^0\left(M/\mathfrak{a}^nM\right):=\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^0\left(M\right)\in\mathfrak{S}.$$

And, moreover, it follows that as $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^0\left(M\right)=\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^0\left(R,\mathrm{H}_{\mathfrak{m}}^0\left(M/\mathfrak{a}^nM\right)\right)$ and the relation of partial order is reflexive, we have

$$\lambda\left(\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^0\left(M\right)\right)\prec\lambda\left(\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^0\left(R,\mathrm{H}_{\mathfrak{m}}^0\left(M/\mathfrak{a}^nM\right)\right)\right).$$

Now, suppose that $i>0$ and that the result is true for $i-1\geq 0$. Consider that $\tilde{M}:=M/\mathfrak{a}^nM$. Let $\overline{X}=\tilde{M}/\Gamma_{\mathfrak{m}}\left(\tilde{M}\right)$ and let $L=\mathrm{E}\left(\overline{X}\right)/\overline{X}$, where $\mathrm{E}\left(\overline{X}\right)$ is an injective hull of $\overline{X}$. Note that $\Gamma_{\mathfrak{m}}\left(\overline{X}\right)=0=\Gamma_{\mathfrak{m}}\left(\mathrm{E}\left(\overline{X}\right)\right)$. Thus, consider the short exact sequence

$$0\to\overline{X}\to\mathrm{E}\left(\overline{X}\right)\to L\to 0\quad(*).$$

Applying the derived functors of $\Gamma_{\mathfrak{m}}\left(\bullet\right)$, according to the Definition 1, in the sequence $(*)$ we obtain, for all $t>0$, the isomorphisms

$$\mathrm{H}_{\mathfrak{m}}^{t-1}\left(L\right)\cong\mathrm{H}_{\mathfrak{m}}^{t}\left(\overline{X}\right)\left(\cong\mathrm{H}_{\mathfrak{m}}^{t}\left(\tilde{M}\right)\right).$$

From the previous isomorphisms, for all r, $0\leq r\leq i-1$, we have $\mathrm{Ext}_R^{(i-1)-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(L\right)\right)\cong\mathrm{Ext}_R^{i-(r+1)}\left(R,\mathrm{H}_{\mathfrak{m}}^{r+1}\left(\tilde{M}\right)\right)$, the which implies

$$\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{(i-1)-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(L\right)\right)\cong\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i-(r+1)}\left(R,\mathrm{H}_{\mathfrak{m}}^{r+1}\left(\tilde{M}\right)\right),$$

which is in $\mathfrak{S}$ by assumptions. Thus, from the induction hypothesis on L, $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i-1}\left(L\right)\in\mathfrak{S}$ and

$$\lambda\left(\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i-1}\left(L\right)\right)\prec\star_{r=0}^{i-1}\lambda\left(\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{(i-1)-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(L/\mathfrak{a}^nL\right)\right)\right).$$

Therefore, $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(\overline{X}\right)\in\mathfrak{S}$ and

$$\lambda\left(\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(\overline{X}\right)\right)\prec\star_{r=1}^{i}\lambda\left(\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(\tilde{M}\right)\right)\right).$$

Now, by the short exact sequence $0\to\Gamma_{\mathfrak{m}}\left(\tilde{M}\right)\to M/\mathfrak{a}^nM\to\overline{X}\to 0$ and by Remark 1 (b), we get the long exact sequence

$$\ldots\to\mathrm{Ext}_R^{i}\left(R,\Gamma_{\mathfrak{m}}\left(\tilde{M}\right)\right)\to\mathrm{H}_{\mathfrak{m}}^{i}\left(\tilde{M}\right)\to\mathrm{H}_{\mathfrak{m}}^{i}\left(\overline{X}\right)\to\ldots.$$

By [4, Properties 3.14 and 3.5], we have that $\tilde{M}$ is a linearly compact R-module, and then, by [3, Lemma 2.4], we get the long exact sequence

$$\ldots\to\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i}\left(R,\Gamma_{\mathfrak{m}}\left(\tilde{M}\right)\right)\to\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(M\right)\to\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(\tilde{M}\right),$$

the which shows that $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(M\right)\in\mathfrak{S}$ and

$$\lambda\left(\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(M\right)\right)\prec\star_{r=0}^{i}\lambda\left(\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(\tilde{M}\right)\right)\right),$$

as required.

Remark 2 Note that we have, in the Theorem 1, the following

$$\mathrm{Ext}_R^{i-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(M/\mathfrak{a}^nM\right)\right)=0$$

for all $0\leq r<i$, since R is a projective R-module. Hence we have:

$$\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(M/\mathfrak{a}^nM\right)\right)=\varprojlim_{n\in\mathbb{N}}\mathrm{Hom}_R\left(R,\mathrm{H}_{\mathfrak{m}}^{i}\left(M/\mathfrak{a}^nM\right)\right)=\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^{i}\left(M\right),$$

for $r=i$ and

$$\varprojlim_{n\in\mathbb{N}}\mathrm{Ext}_R^{i-r}\left(R,\mathrm{H}_{\mathfrak{m}}^{r}\left(M/\mathfrak{a}^nM\right)\right)=0,$$

for $0\leq r<i$. Now, the result it follows.

4 Vanishing results

Here want discuss about the vanishing of formal local cohomology modules. As a motivation for a result of vanishing of such modules, we presented the following vanishing result given by P. Schenzel: “Let $(R,\mathfrak{m})$ be a Noetherian local ring. Let M denote a finitely generated R-module. Let $j\in\mathbb{Z}$ and let $\mathfrak{a}$ be an ideal of R. Suppose that

$$\varprojlim_{n\in\mathbb{N}}\mathrm{H}_{\mathfrak{m}}^{j}\left(M/\mathfrak{a}^nM\right)=\mathfrak{a}\left(\varprojlim_{n\in\mathbb{N}}\mathrm{H}_{\mathfrak{m}}^{j}\left(M/\mathfrak{a}^nM\right)\right).$$

Then, we have that:

$$\varprojlim_{n\in\mathbb{N}}\mathrm{H}_{\mathfrak{m}}^{j}\left(M/\mathfrak{a}^nM\right)=0.\text{”}$$

Now, note that in [1], we have that:

$$\mathrm{H}_I^0\left(M\right)=\Gamma_I\left(M\right):=\left\{m\in M\mid I^nm=0\text{ for some }n\in\mathbb{N}\right\}.$$

An R-module M is said to be I-torsion, if $M=\Gamma_I\left(M\right)$.

Remark 3 We take $x\in I$ and let M be an I-torsion R-module. Then $M\xrightarrow{x\cdot}M$ is injective if and only if $M=0$.

Let M be an R-module. A finite sequence $x_1,\dots,x_r$ of elements of R is said to be an *M-sequence* if $x_i \in \mathrm{NZD}_R\left(M/\sum_{j=1}^{i-1} x_j M\right)$ for $i \in \{1,\dots,r\}$, where NZD means non-zero divisors. Use the convention that $\sum_{j=1}^{0} x_j M = 0$. So, this means that $x_1 \in \mathrm{NZD}_R(M)$, $x_2 \in \mathrm{NZD}_R(M/x_1 M)$, $x_3 \in \mathrm{NZD}_R(M/(x_1 M + x_2 M))$, ... Note that the empty sequence of elements of I is an M-sequence. If $x_1,\dots,x_r$ is an M-sequence such that $x_i \in I$ for $i \in \{1,\dots,r\}$, then $x_1,\dots,x_r$ is called an M-sequence in I. If $x_1,\dots,x_r$ is an M-sequence, r is called its *length*.

Now, we take $x_1,\dots,x_r \in R$, $s \in \{1,\dots,r-1\}$ and let M be an R-module. Then, we have the following properties of M-sequences:

(1) $x_1,\dots,x_r$ is an M-sequence if and only if $x_1,\dots,x_s$ is an M-sequence and $x_{s+1},\dots,x_r$ is an $M/\sum_{l=1}^{s} x_l M$-sequence;
(2) As a special case of (1) we obtain that, $x_1,\dots,x_r$ is an M-sequence if and only if $x_1 \in \mathrm{NZD}_R(M)$ and $x_2,\dots,x_r$ is an $M/(x_1)M$-sequence.

Lemma 1 *Let M be an R-module. Then:*

(*a*) *If $\Gamma_I(M) \neq 0$, then $I \subseteq \mathrm{ZD}_R(M)$, where ZD means zero divisors.*
(*b*) *Let R be a Noetherian ring and let M be a finitely generated R-module. If $I \subseteq \mathrm{ZD}_R(M)$, then $\Gamma_I(M) \neq 0$.*

Proof. (a): It follows directly of the definition of $\Gamma_I(M)$.
(b): By the hypothesis, we have that M is a Noetherian R-module and so the set $\mathrm{Ass}_R(M)$ is finite, so that we can write

$$\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}.$$

Since R is a Noetherian ring, it follows that

$$\mathrm{ZD}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$$

and thus, we obtain that $I \subseteq \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$.

So, by prime avoidance, there exists some $i \in \{1,\dots,r\}$ such that $I \subseteq \mathfrak{p}_i$. As $\mathfrak{p}_i \in \mathrm{Ass}_R(M)$, there exists some $v \in M \setminus 0$ with

$$\mathfrak{p}_i = \mathrm{ann}(v) = \{x \in R \mid xv = 0\}.$$

It follows that $Iv \subseteq \mathfrak{p}_i v = 0$ and thus, $v \in \Gamma_I(M) \setminus \{0\}$.

For to finishe, we have the following proposition.

Proposition 1 *Let M be a finitely generated R-module, and let $(R,\mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}$ be an ideal of R. Let $x_1,\dots,x_r$ be an $M/\mathfrak{a}^n M$-sequence in $\mathfrak{m}$ for all $n \in \mathbb{N}$. Then $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M) = 0$, for all $i < r$.*

Proof. The proof is by induction on r. We take $r = 1$. Then, as seen above, $x_1 \in \mathfrak{m} \cap \mathrm{NZD}_R(M/\mathfrak{a}^n M)$ and hence, by Lemma 1 (a), $\Gamma_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$. Thus, $\mathrm{H}^0_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $n \in \mathbb{N}$ and so we have that

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^0_{\mathfrak{m}}(M/\mathfrak{a}^n M) := \mathfrak{F}^0_{\mathfrak{a},\mathfrak{m}}(M) = 0.$$

Now, we take $r > 1$. Then $x_1,\dots,x_{r-1}$ is an $M/\mathfrak{a}^n M$-sequence in $\mathfrak{m}$. Hence, by induction, $\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $i < r-1$ and for all $n \in \mathbb{N}$. Therefore, we have that

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$$

for all $i < r-1$ and thus, $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M) = 0$ for all $i < r-1$.

It remains to be shown that $\mathfrak{F}^{r-1}_{\mathfrak{a},\mathfrak{m}}(M) = 0$.

We have, as seen above, that $x_1 \in \mathrm{NZD}_R(M/\mathfrak{a}^n M)$ and that $x_2,\dots,x_r$ is an $\frac{M}{\mathfrak{a}^n M}/(x_1)\frac{M}{\mathfrak{a}^n M}$-sequence in $\mathfrak{m}$. In particular, the cohomology sequence (see [1]) gives us an exact sequence of R-modules:

$$\mathrm{H}^{r-2}_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}/(x_1)\frac{M}{\mathfrak{a}^n M}\right) \to \mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M) \xrightarrow{x_1\cdot} \mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M).$$

As $x_2,\dots,x_r$ is an $\frac{M}{\mathfrak{a}^n M}/(x_1)\frac{M}{\mathfrak{a}^n M}$-sequence in $\mathfrak{m}$ we get by induction that $\mathrm{H}^{r-2}_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}/(x_1)\frac{M}{\mathfrak{a}^n M}\right) = 0$. Thus, the multiplication homomorphism

$$x_1\cdot : \mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M) \to \mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

is injective. Now, by [1], $\mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is $\mathfrak{m}$-torsion, and as $x_1 \in \mathfrak{m}$, we get, by Remark 3, that $\mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $n \in \mathbb{N}$.

Therefore, it follows that

$$\mathfrak{F}^{r-1}_{\mathfrak{a},\mathfrak{m}}(M) := \varprojlim_{n\in\mathbb{N}} \mathrm{H}^{r-1}_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0,$$

as required, finalizing so the proof.

5 Conclusion

The main idea of this work was to do a more specific study about the formal local cohomology module.

Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. Let M be a finitely generated R-module. There exists a systematic study of the formal local cohomology modules, which, as we have already seen in the text, are given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M), \text{ for } i \in \mathbb{Z}, \text{ with } i \geq 0.$$

We analyze their R-module structure, vanishing and non-vanishing results in terms of intrinsic data of M, and its functorial behavior.

These cohomology modules occur in relation to the formal completion of the punctured spectrum $\mathrm{Spec}(R) \setminus \mathrm{V}(\mathfrak{m})$.

In our discussion in the article, we cover the definition of Serre subcategory of the category of R-modules and also we define a subadditive function from of Serre subcategory of the category of R-modules to a partially ordered Abelian monoid, and with these definitions, together with some additional hypotheses, we obtain that the formal local cohomology module $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M) \in \mathfrak{S}$, a Serre subcategory, and

$$\lambda\left(\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M)\right) \prec \star^i_{r=0}\, \lambda\left(\varprojlim_{n\in\mathbb{N}} \mathrm{Ext}^{i-r}_R\left(R, \mathrm{H}^r_{\mathfrak{m}}\left(M/\mathfrak{a}^n M\right)\right)\right).$$

(cf. Theorem 1.)

We can not fail to mention in our conclusion that we use many things from the commutative algebra theory to make the results of the theory of local cohomology and formal local cohomology. For example, we put for M a finitely generated R-module, R a commutative Noetherian ring, that $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ is a set finite, where $\mathrm{Ass}_R(M)$ denote the set of associated prime ideals to the R-module M.

We can then observe that the formal local cohomology theory can be further explored in order that we can obtain more interesting results, which relate more definitions and concepts. For example, for future studies on the subject we can establish conditions for that the formal local cohomology module satisfies the conditions for to be an co-Cohen-Macaulay R-module.

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