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A Novel Iterative Numerical Algorithm for the Solutions of Systems of Fuzzy Initial Value Problems

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Abstract: Behaviors of many dynamic systems with uncertainty can be modelled effectively by systems of fuzzy differential equations. In this paper, we develop new numerical iterative method for solving systems of fuzzy initial value problems based on the reproducing kernel theory under the assumption of Hukuhara differentiability. The exact and approximate solutions are given with series form in terms of their parametric form, where two smooth reproducing kernel functions are used throughout the evolution of the algorithm to obtain the required nodal values. Furthermore, error estimation is proved in order to capture the behavior of fuzzy solutions. Applicability, potentiality, and efficiency of the proposed algorithm for the fuzzy solutions of different fuzzy systems are investigated using computer tables and graphical representation.

Keywords: Fuzzy differential systems; Reproducing kernel theory; Hukuhara derivative

1 Introduction

Theory of systems of differential equations plays a vital role to model physical, engineering, and economical problems, such as in solid and fluid mechanics, dynamic supply and demand, mathematical biology, plasma physics, control theory, and other areas of science [1,2,3,4,5,6,7,8,9,10]. But in actual case, the parameters, variables, or initial conditions involved in the differential systems may be uncertain, or a vague estimation of those are found in general by some observation, experiment, experience, data collection, or maintenance induced error. So, to overcome the uncertainty and vagueness, one may use fuzzy environment in parameters, variables, and initial conditions in place of crisp ones. So, with these uncertainties the general differential systems turn into fuzzy differential systems.

Numerical techniques are widely used by scientists and engineers to solve their problems. A major advantage for numerical techniques is that a numerical answer can be obtained even when a problem has no analytical solution. Anyhow, in most real-life applications, it is too complicated to obtain the exact solutions to systems of fuzzy initial value problems (FIVPs) in terms of elementary functions in a simple manner, so an efficient, reliable numerical algorithm for the solutions of such systems is required; it is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in mathematics and engineering problem solving has increased dramatically in recent years.

In this paper, we introduce a novel iterative technique based on the use of reproducing kernel Hilbert space (RKHS) method for numerically approximating solutions of systems of FIVPs in the space $\bigoplus_{\nu=1}^{2\eta} W_2^2[a,b]$ under the Hukuhara differentiability. The new method has the following characteristics; first, it is of global nature in terms of the solutions obtained as well as its ability to solve other mathematical and engineering problems; second, it is accurate, need less effort to achieve the results, and is developed especially for the nonlinear case; third, in the proposed method, it is possible to pick any point in the interval of integration and as well the approximate solutions and their first Hukuhara derivatives will be applicable; fourth, the method does not require discretization of the variables, and it is not effected by

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computation round off errors and one is not faced with necessity of large computer memory and time; fifth, the proposed approach does not resort to more advanced mathematical tools; that is, the algorithm is simple to understand, implement, and should be thus easily accepted in the mathematical and engineering application's fields. More precisely, we provide numerical approximate solutions on the interval $[a,b]$ for systems of FIVPs of the form

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t), ..., x_\eta(t)), \\ x_2'(t) &= f_2(t, x_1(t), x_2(t), ..., x_\eta(t)), \\ &\vdots \\ x_\eta'(t) &= f_\eta(t, x_1(t), x_2(t), ..., x_\eta(t)), \end{aligned} \quad (1)$$

subject to the fuzzy initial conditions

$$x_1(a) = \alpha_1, x_2(a) = \alpha_2, \cdots, x_\eta(a) = \alpha_\eta, \quad (2)$$

where $f_\upsilon : [a,b] \times \mathbb{R}_{\mathscr{F}}^{\eta} \to \mathbb{R}_{\mathscr{F}}$ are continuous η-tuples fuzzy-valued functions, $x_\upsilon : [a,b] \to \mathbb{R}_{\mathscr{F}}$, $\alpha_\upsilon \in \mathbb{R}_{\mathscr{F}}$, $a, b \in \mathbb{R}$, and $\upsilon = 1, 2, \cdots, \eta$. Throughout this paper $\mathbb{R}$ the set of real numbers and $\mathbb{R}_{\mathscr{F}}$ denote the set of fuzzy real numbers on $\mathbb{R}$.

Reproducing kernel theory has important applications in numerical analysis, differential equations, integral equations, integro-differential equations, probability and statistics, and so fourth [11,12,13]. Recently, a lot of research work has been devoted to the applications of RKHS method for wide classes of stochastic and deterministic problems involving operator equations, differential equations, integral equations, and integro-differential equations. The RKHS method was successfully used by many authors to investigate several scientific applications side by side with their theories. The reader is kindly requested to go through [14,15,16,17,18, 19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35, 36,37] in order to know more details about RKHS method, including its history, its modification for use, its scientific applications, its kernel functions, and its characteristics.

The numerical solvability for systems of FIVPs have been pursued by several authors. To mention a few, in [38] the authors have discussed the geometric approach to solve linear systems of FIVPs. Furthermore, the variational iteration method is carried out in [39] for linear fuzzy differential system. The homotopy analysis method (HAM) has been applied to solve the linear fuzzy system as described in [40]. Recently, the fuzzy neural network approach for solving linear system of FIVPs is proposed in [41]. On the other aspect as well, the numerical solvability of other version of FIVPs can be found in [42,43,44,45] and references therein. As a result, none of previous studies propose a methodical way to solve systems of FIVPs in general. Moreover, previous studies require more effort to achieve the results, they are not accurate and usually they are suited for linear form.

This paper is comprised of 6 sections including the introduction. In the next section, overview of fuzzy calculus theory is collected. In Section 2, 2η dimensional inner product spaces are constructed in order to apply the method. In Section 3, series representation of exact and approximate solutions and theoretical basis of the method are introduced. In Section 4, an iterative algorithm for numerically approximating the solutions is described and the n-truncation approximate solutions are proved to converge to the exact solutions. Software libraries and numerical experiment are presented in Section 5. This article ends in Section 6 with some concluding remarks.

2 Overview of fuzzy calculus theory

The contents of this section is basic in some sense, for the reader's convenience, we present some necessary definitions from fuzzy calculus theory and preliminary results. After that, a numerical algorithm for the solutions of systems of FIVPs based on their r-cut representation form is introduced.

Let S be a nonempty set. A fuzzy set u in S is characterized by its membership function $u : S \to [0,1]$. Thus, $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in S$. A fuzzy set u on $\mathbb{R}$ is called convex if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, $u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$; is called upper semicontinuous if the set $\{s \in \mathbb{R} \mid u(s) \geq r\}$ is closed for each $r \in [0,1]$; and is called normal if there is $s \in \mathbb{R}$ such that $u(s) = 1$. The support of a fuzzy set u is defined as $\{s \in \mathbb{R} : u(s) > 0\}$.

Definition 1 [46] A fuzzy number u is a fuzzy subset of $\mathbb{R}$ with normal, convex, and upper semicontinuous membership function of bounded support.

The concept of a fuzzy real number arises from the fact that many quantifiable phenomena do not lend themselves to being characterized in terms of absolutely precise numbers. In fact, a fuzzy number is one which is described in terms of a number word and a linguistic modifier, such as approximately, nearly, or around.

For each $r \in (0,1]$, set $[u]^r = \{s \in \mathbb{R} : u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R} : u(s) > 0\}}$, where $\overline{\{\cdot\}}$ denote the closure of $\{\cdot\}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is a compact convex subset of $\mathbb{R}$ for each $r \in [0,1]$ and $[u]^1 \neq \phi$ [47]. Thus, if u is a fuzzy number, then $[u]^r = [u_1(r), u_2(r)]$, where

$$\begin{aligned} u_1(r) &= \min\{s : s \in [u]^r\} \\ u_2(r) &= \max\{s : s \in [u]^r\}, \end{aligned}$$

for each $r \in [0,1]$. The symbol $[u]^r$ is called the r-cut representation or parametric form of a fuzzy number u.

The question arises here is, if we have an interval-valued function $[z_1(r), z_2(r)]$ defined on $[0,1]$, then is there a fuzzy number u such that

$[u]^r = [z_1(r), z_2(r)]$. The next theorem is characterizes fuzzy numbers through their r-cut representations.

Theorem 1 [47] Suppose that $u_1, u_2 : [0,1] \to \mathbb{R}$ satisfy the following conditions; first, u_1 is a bounded increasing function and u_2 is a bounded decreasing function with $u_1(1) \leq u_2(1)$; second, for each $k \in (0,1]$, u_1 and u_2 are left-hand continuous functions at $r = k$; third, u_1 and u_2 are right-hand continuous functions at $r = 0$. Then

$$u : \mathbb{R} \to [0,1],$$

defined by

$$u(s) = \sup\{r : u_1(r) \leq s \leq u_2(r)\},$$

is a fuzzy number with parameterization $[u_1(r), u_2(r)]$. Furthermore, if $u : \mathbb{R} \to [0,1]$ is a fuzzy number with parameterization $[u_1(r), u_2(r)]$, then the functions u_1 and u_2 satisfy the aforementioned conditions.

In general, we can represent an arbitrary fuzzy number u by an order pair of functions (u_1, u_2) which satisfy the requirements of Theorem 1. Frequently, we will write simply u_{1r} and u_{2r} instead of $u_1(r)$ and $u_2(r)$, respectively.

Definition 2 [48,49] The complete metric structure on $\mathbb{R}_{\mathscr{F}}$ is given by the Hausdorff distance mapping

$$D : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}^+ \cup \{0\},$$

such that

$$D(u,v) = \sup_{0 \leq r \leq 1} \max\{|u_{1r} - v_{1r}|, |u_{2r} - v_{2r}|\},$$

for arbitrary fuzzy numbers u and v.

Let $u, v \in \mathbb{R}_{\mathscr{F}}$. If there exists an element $w \in \mathbb{R}_{\mathscr{F}}$ such that $u = v + w$, then w is called the Hukuhara difference of u and v, denoted by $u \ominus v$. Here, the sign $\ominus$ stands always for Hukuhara difference and let us mention that $u \ominus v \neq u + (-1)v$. Usually, we denote $u + (-1)v$ by $u - v$, while $u \ominus v$ stands for the Hukuhara difference.

Definition 3 [50] Let $x : [a,b] \to \mathbb{R}_{\mathscr{F}}$ and $t_0 \in [a,b]$. We say that x is Hukuhara differentiable at t_0, if there exists an element $x'(t_0) \in \mathbb{R}_{\mathscr{F}}$ such that for each $h > 0$ sufficiently close to 0, the Hukuhara differences $x(t_0 + h) \ominus x(t_0)$, $x(t_0) \ominus x(t_0 - h)$ exist and

$$\begin{aligned} x'(t_0) &= \lim_{h \to 0^+} \tfrac{x(t_0+h) \ominus x(t_0)}{h} \\ &= \lim_{h \to 0^+} \tfrac{x(t_0) \ominus x(t_0-h)}{h}. \end{aligned}$$

Here, the limit is taken in the metric space $(\mathbb{R}_{\mathscr{F}}, D)$ and at the endpoints of $[a,b]$, we consider only one-sided derivatives. Next theorem shows us a way to translate a differential system from fuzzy setting into ordinary setting.

Theorem 2 [51,52] Let $x : [a,b] \to \mathbb{R}_{\mathscr{F}}$ be Hukuhara differentiable function and $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$. Then the endpoints functions x_{1r} and x_{2r} are differentiable on $[a,b]$ and

$$\left[x'(t)\right]^r = \frac{d}{dt}[x(t)]^r = \left[x'_{1r}(t), x'_{2r}(t)\right],$$

for each $r \in [0,1]$.

In some applications, the behavior of an object is determined by physics laws and is crisp. However, if the initial values are obtained from measurements, for example, this value can be uncertain and often there are more suitable to model them using fuzzy numbers. Next, we consider and study systems involving fuzzy equations and/or fuzzy initial conditions. In other word, if the initial values are fuzzy numbers, the solutions are fuzzy functions, and consequently the derivatives must be considered as fuzzy derivatives. Let us consider the following system of first-order equations described the crisp ordinary differential equations (ODEs) on the interval $[a,b]$:

$$\begin{aligned} x'_1(t) &= f_1(t, x_1(t), x_2(t), \ldots, x_\eta(t)), \\ x'_2(t) &= f_2(t, x_1(t), x_2(t), \ldots, x_\eta(t)), \\ &\vdots \\ x'_\eta(t) &= f_\eta(t, x_1(t), x_2(t), \ldots, x_\eta(t)), \end{aligned} \tag{3}$$

subject to the crisp initial conditions

$$x_1(a) = \alpha_1, x_2(a) = \alpha_2, \ldots, x_\eta(a) = \alpha_\eta, \tag{4}$$

where $f_\upsilon : [a,b] \times \mathbb{R}^\eta \to \mathbb{R}$ are continuous η-tuples real-valued functions, $x_\upsilon : [a,b] \to \mathbb{R}$, $\alpha_\upsilon, a, b \in \mathbb{R}$, and $\upsilon = 1, 2, \ldots, \eta$.

Assume that the initial conditions α_υ in Eq. (4) are uncertain and modeled by fuzzy numbers. Also, assume that the function f_υ in system of ODE (3) contain uncertain parameters modeled by fuzzy numbers. Then, we obtain system of FIVP (1) and (2). Anyhow, in order to solve this new system, we rewrite the fuzzy functions $x_\upsilon(t)$ as $[x_\upsilon(t)]^r = \left[x_{(2\upsilon-1)r}(t), x_{(2\upsilon)r}(t)\right]$ and $[x_\upsilon(a)]^r = \left[\alpha_{(2\upsilon-1)r}, \alpha_{(2\upsilon)r}\right]$. Indeed, according to Nguyen theorem [53,54] it follows that:

$$\begin{aligned} &[f_\upsilon(t, x_1(t), x_2(t), \ldots, x_\eta(t))]^r \\ &\quad = f_\upsilon(t, [x_1(t)]^r, [x_2(t)]^r, \ldots, [x_\eta(t)]^r) \\ &\quad = \left\{f_\upsilon(t, y_1, y_2, \ldots, y_\eta) : y_\upsilon \in [x_\upsilon(t)]^r\right\} \\ &\quad = \left[f_{(2\upsilon-1)r}(t, x_r(t)), f_{(2\upsilon)r}(t, x_r(t))\right], \end{aligned}$$

where $\upsilon = 1, 2, \ldots, \eta$.

Definition 4 Let $x_\upsilon : [a,b] \to \mathbb{R}_{\mathscr{F}}$ such that x'_υ exists. If x_υ and x'_υ satisfy system of FIVP (1) and (2), we say that x_υ are system fuzzy solutions, where $\upsilon = 1, 2, \ldots, \eta$.

Before using RKHS method as an efficient solver for fuzzy differential systems, we shall now introduce and implement a procedure to transform system of FIVP (1) and (2) into parametric form in order to find system fuzzy solutions.

Algorithm 1 To find fuzzy solutions of system of FIVP (1) and (2), there are four main steps:

Input: The interval $[a,b]$, the unit interval $[0,1]$, and the endpoints functions $f_{(2\upsilon-1)r}(t,x_r(t)), f_{(2\upsilon)r}(t,x_r(t))$ of $[f_\upsilon(t,x_1(t),x_2(t),...,x_\eta(t))]^r$.

Output: Exact fuzzy solutions $x_\upsilon(t)$ for each $t\in[a,b]$.

Step 1: For $v=1,...,\eta$, do the following:

Set $[x_\upsilon(t)]^r=\left[x_{(2\upsilon-1)r}(t),x_{(2\upsilon)r}(t)\right]$;
Set $[x'_\upsilon(t)]^r=\left[x'_{(2\upsilon-1)r}(t),x'_{(2\upsilon)r}(t)\right]$;
Set $[x_\upsilon(0)]^r=\left[\alpha_{(2\upsilon-1)r},\alpha_{(2\upsilon)r}\right]$;
Set

$$[f_\upsilon(t,x_r(t))]^r=\left[f_{(2\upsilon-1)r}(t,x_r(t)),f_{(2\upsilon)r}(t,x_r(t))\right];$$

Step 2: Solve the following system of ODEs for $x_r(t)$:

$$\begin{aligned}
&x'_{1r}(t)=f_{1r}(t,x_r(t)),\\
&x'_{2r}(t)=f_{2r}(t,x_r(t)),\\
&x'_{3r}(t)=f_{3r}(t,x_r(t)),\\
&x'_{4r}(t)=f_{4r}(t,x_r(t)),\\
&\vdots\\
&x'_{(2\eta-1)r}(t)=f_{(2\eta-1)r}(t,x_r(t)),\\
&x'_{(2\eta)r}(t)=f_{(2\eta)r}(t,x_r(t)),
\end{aligned}\tag{5}$$

subject to

$$\begin{aligned}
&x_{1r}(t_0)=\alpha_{1r},\ x_{2r}(t_0)=\alpha_{2r},\\
&x_{3r}(t_0)=\alpha_{3r},\ x_{4r}(t_0)=\alpha_{4r},\\
&\vdots\\
&x_{(2\eta-1)r}(t_0)=\alpha_{(2\eta-1)r},\ x_{(2\eta)r}(t_0)=\alpha_{(2\eta)r}.
\end{aligned}\tag{6}$$

Step 3: For $\upsilon=1,...,\eta$ and each $t\in[a,b]$ and $r\in[0,1]$, do the following:

Ensure that the solutions $\left[x_{(2\upsilon-1)r}(t),x_{(2\upsilon)r}(t)\right]$ are valid level sets;

Ensure that the derivatives $\left[x'_{(2\upsilon-1)r}(t),x'_{(2\upsilon)r}(t)\right]$ are valid level sets;

Construct the fuzzy solutions $x_\upsilon(t)$ such that $[x_\upsilon(t)]^r=\left[x_{(2\upsilon-1)r}(t),x_{(2\upsilon)r}(t)\right]$.

Step 4: Stop.

3 Multidimensional inner product spaces

In functional analysis, RKHS is a Hilbert space of functions in which pointwise evaluation is a continuous linear functional. Equivalently, they are spaces that can be defined by reproducing kernels. In this section, we firstly formulate several reproducing kernel functions in order to generate and construct an orthogonal normal basis on the spaces $W_2^2[a,b]$ and $W_2^1[a,b]$. After that, new spaces $\bigoplus_{v=1}^{2\eta}W_2^2[a,b]$ and $\bigoplus_{v=1}^{2\eta}W_2^1[a,b]$ are building in order to formulate and utilize the solutions of system of FIVP (1) and (2) using RKHS method.

An abstract set is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for the number of elements.

Definition 5 [14] Let E be a nonempty abstract set. A function $K:E\times E\to\mathbb{C}$ is a reproducing kernel of the Hilbert space H if

1. $\forall t\in E$; $K(\cdot,t)\in H$,
2. $\forall t\in E$ and $\varphi\in H$; $\langle\varphi(\cdot),K(\cdot,t)\rangle=\varphi(t)$.

Remark 1 The condition (2) in Definition 5 is called "the reproducing property" which means that the value of a function φ at a point t is reproducing by the inner product of $\varphi(\cdot)$ with $K(\cdot,t)$. A Hilbert space which possesses a reproducing kernel is called a RKHS.

An important subset of the RKHSs are the RKHSs associated to a continuous kernel. These spaces have wide applications, including complex analysis, harmonic analysis, quantum mechanics, statistics and machine learning. Next, in order to apply the RKHS method, we shall define and construct a reproducing kernel space $W_2^2[a,b]$ in which every function satisfies the initial conditions $z(a)=0$.

Definition 6 [15] The inner product space $W_2^2[a,b]$ is defined as $W_2^2[a,b]=\{z(t):z,z'$ are absolutely continuous real-valued functions on $[a,b]$, $z''\in L^2[a,b]$, and $z(a)=0\}$. The inner product and the norm in $W_2^2[a,b]$ are given by

$$\begin{aligned}\langle z_1(t),z_2(t)\rangle_{W_2^2}&=z_1(a)z_2(a)\\&+z'_1(a)z'_2(a)+\int_a^b z''_1(t)z''_2(t)\,dt,\end{aligned}\tag{7}$$

and $||z_1||_{W_2^2}=\sqrt{\langle z_1(t),z_1(t)\rangle_{W_2^2}}$, respectively, where $z_1,z_2\in W_2^2[a,b]$.

Definition 7 [14] The Hilbert space $W_2^2[a,b]$ is called a reproducing kernel if for each fixed $t\in[a,b]$ and any $z(s)\in W_2^2[a,b]$, there exist $G(t,s)\in W_2^2[a,b]$ (simply $G_t(s)$) and $s\in[a,b]$ such that $\langle z(s),G_t(s)\rangle_{W_2^2}=z(t)$.

It is very important to obtain the representation form of the reproducing kernel function $G_t(s)$, because it is the basis of our algorithm. In the following theorem, we will give the representation form of the reproducing kernel function $G_t(s)$ in the space $W_2^2[a,b]$. After that, we construct the space $W_2^1[a,b]$ in order to define the linear bounded operators as shown later in the next section.

Theorem 3 [15] The Hilbert space $W_2^2[a,b]$ is a complete reproducing kernel and its reproducing kernel function $G_t(s)$ is given by

$$G_t(s) = \begin{cases} \Lambda(s,t), s \le t, \\ \Lambda(t,s), s > t. \end{cases}$$

where

$$\Lambda(s,t) = \frac{1}{6}(s-a) \\ (2a^2 - s^2 + 3t(2+s) - a(6+3t+s)).$$

Definition 8 [16] The inner product space $W_2^1[a,b]$ is defined as $W_2^1[a,b] = \{z(t) : z$ is absolutely continuous real-valued function on $[a,b]$ and $z' \in L^2[a,b]\}$. The inner product and the norm in $W_2^1[a,b]$ are defined as $\langle z_1(t), z_2(t)\rangle_{W_2^1} = \int_a^b (z_1'(t) z_2'(t) + z_1(t) z_2(t))\, dt$ and $\|z_1\|_{W_2^1} = \sqrt{\langle z_1(t), z_1(t)\rangle_{W_2^1}}$, respectively, where $z_1, z_2 \in W_2^1[a,b]$.

Theorem 4 [16] The Hilbert space $W_2^1[a,b]$ is a complete reproducing kernel and its reproducing kernel function $H_t(s)$ is given by

$$H_t(s) = \begin{cases} \Delta(s,t), s \le t, \\ \Delta(t,s), s > t. \end{cases}$$

where

$$\Delta(s,t) = \frac{1}{2}\text{csch}(b-a) \times \\ (\cosh(t+s-b-a) + \cosh(t-s-b+a))$$

The spaces $W_2^2[a,b]$ and $W_2^1[a,b]$ are complete Hilbert with some special properties. So, all the properties of the Hilbert space will be hold. Further, theses spaces possesses some special and better properties which could make some problems be solved easier. For instance, many problems studied in $L^2[a,b]$ space, which is a complete Hilbert space, requires large amount of integral computations and such computations may be very difficult in some cases. Thus, the numerical integrals have to be calculated in the cost of losing some accuracy. However, the properties of $W_2^2[a,b]$ and $W_2^1[a,b]$ require no more integral computation for some functions, instead of computing some values of a function at some nodes. In fact, this simplification of integral computation not only improves the computational speed, but also improves the computational accuracy. Henceforth and not to conflict unless stated otherwise, we denote

$$W[a,b] = \oplus_{\nu=1}^{2\eta} W_2^2[a,b]$$
$$H[a,b] = \oplus_{j=1}^{2\eta} W_2^1[a,b].$$

Definition 9 The inner product space $W[a,b]$ can be constructed as

$$W[a,b] = \{(z_1(t), z_2(t), ..., z_{2\eta}(t))^T\},$$

where $z_j \in W_2^2[a,b]$ and $j = 1,...,2\eta$. The inner product and the norm in $W[a,b]$ are building as

$$\langle z(t), w(t)\rangle_W = \sum_{j=1}^{2\eta} \langle z_j(t), w_j(t)\rangle_{W_2^2}$$

and $\|z\|_W = \sqrt{\sum_{j=1}^{2\eta} \|z_j\|_{W_2^2}^2}$, respectively, where $z, w \in W[a,b]$.

Clearly, $W[a,b]$ is a Hilbert space. On the other aspect as well, the inner product space $H[a,b]$ can be defined in a similar manner with similar inner product and norm, and it is also a Hilbert space.

4 Series representation of solutions

In this section, formulation of differential linear operator and implementation method are presented in the spaces $W[a,b]$ and $H[a,b]$. Meanwhile, we construct an orthogonal function system of the space $W[a,b]$ based on Gram-Schmidt orthogonalization process in order to obtain the exact and approximate solutions of system of FIVP (1) and (2). Through remainder sections, the lowercase letter r whenever used means for each $r \in [0,1]$.

Now, to apply the RKHS method, we will define the differential linear operator $L_{jr} : W_2^2[a,b] \to W_2^1[a,b]$ such that $L_{jr} x_{jr}(t) = x'_{jr}(t)$, $j = 1,2,...,2\eta$. Put $f_r = (f_{1r}, f_{2r}, ..., f_{(2\eta)r})^T$, $x_r = (x_{1r}, x_{2r}, ..., x_{(2\eta)r})^T$, $\alpha_r = (\alpha_{1r}, \alpha_{2r}, ..., \alpha_{(2\eta)r})^T$, and $L_r = \text{diag}(L_{1r}, L_{2r}, ..., L_{(2\eta)r})$, where

$$L_r : W[a,b] \to H[a,b]$$

Based on this, the system of ODEs (5) and (6) can be converted into the equivalent form as follows:

$$L_r x_r(t) = f_r(t, x_r(t)) \\ = f_r(t, x_{1r}(t), x_{2r}(t), ..., x_{(2\eta)r}(t)), \tag{8}$$

subject to

$$x_r(a) = \alpha_r, \tag{9}$$

in which $x_r \in W[a,b]$ and $f_r \in H[a,b]$.

Lemma 1 The operators $L_{jr} : W_2^2[a,b] \to W_2^1[a,b]$, $j = 1,2,...,2\eta$ are bounded and linear.

Proof The linearity part is obvious, for boundedness part, we need to prove that $\|L_{jr} x_{jr}\|_{W_2^1}^2 \le M_{jr} \|x_{jr}\|_{W_2^2}^2$, where $M_{jr} > 0$. From the definition of the inner product and the norm of $W_2^1[a,b]$, we have

$$\|L_{jr} x_{jr}\|_{W_2^1}^2 = \int_a^b \left\{ \left[(L_{jr} x_{jr})'(t)\right]^2 + \left[(L_{jr} x_{jr})(t)\right]^2 \right\} dt.$$

By reproducing property of the kernel function $G_t(s)$, we have

$$x_{jr}(t)=\langle x_{jr}(s),G_t(s)\rangle_{W_2^2}$$
$$(L_{jr}x_{jr})(t)=\langle x_{jr}(s),(L_{jr}G_t)(s)\rangle_{W_2^2}$$
$$(L_{jr}x_{jr})'(t)=\langle x_{jr}(s),(L_{jr}G_t)'(s)\rangle_{W_2^2}.$$

Again, by Schwarz inequality, we get

$$\begin{aligned}\left|(L_{jr}x_{jr})(t)\right| &= \left|\langle x_{jr}(t),(L_{jr}G_t)(t)\rangle_{W_2^2}\right| \\ &\le \left\|(L_{jr}G_t)(t)\right\|_{W_2^2}\left\|x_{jr}(t)\right\|_{W_2^2} \\ &= M_{jr}^1\left\|x_{jr}(t)\right\|_{W_2^2},\end{aligned}$$

$$\begin{aligned}\left|(L_{jr}x_{jr})'(t)\right| &= \left|\langle x_{jr}(t),(L_{jr}G_t)'(t)\rangle_{W_2^2}\right| \\ &\le \left\|(L_{jr}G_t)'(t)\right\|_{W_2^2}\left\|x_{jr}(t)\right\|_{W_2^2} \\ &= M_{jr}^2\left\|x_{jr}(t)\right\|_{W_2^2},\end{aligned}$$

where $M_{jr}^1,M_{jr}^2>0$. Thus,

$$\begin{aligned}\left\|L_{jr}x_{jr}\right\|_{W_2^1}^2 &= \int_a^b\left\{\left[(L_{jr}x_{jr})'(t)\right]^2+\left[(L_{jr}x_{jr})(t)\right]^2\right\}dt \\ &\le \left(M_{jr}^1+M_{jr}^2\right)(b-a)\left\|x_{jr}(t)\right\|_{W_2^2}^2\end{aligned}$$

or

$$\left\|L_{jr}x_{jr}\right\|_{W_2^1}\le M_{jr}\left\|x_{jr}(t)\right\|_{W_2^2},$$

where $M_{jr}=\sqrt{\left(M_{jr}^1+M_{jr}^2\right)(b-a)}$.

Theorem 5 The operator $L_r:W[a,b]\to H[a,b]$ is bounded and linear.

Proof Clearly, L_r is a linear operator. A boundedness is shown as follows: for each $x_r\in W[a,b]$, one can write

$$\begin{aligned}\|L_rx_r\|_H &= \sqrt{\sum_{j=1}^{2\eta}\left\|L_{jr}x_{jr}\right\|_{W_2^1}^2} \\ &\le \sqrt{\sum_{j=1}^{2\eta}\left\|L_{jr}\right\|^2\left\|x_{jr}\right\|_{W_2^2}^2} \\ &\le \sqrt{\left(\sum_{j=1}^{2\eta}\left\|L_{jr}\right\|^2\right)\left(\sum_{j=1}^{2\eta}\left\|x_{jr}\right\|_{W_2^2}^2\right)} \\ &= \sqrt{\sum_{j=1}^{2\eta}\left\|L_{jr}\right\|^2}\,\|x_r\|_W\end{aligned}$$

The boundedness of L_{jr} implies that L_r is bounded. So, the proof of the theorem is complete.

Next, we construct an orthogonal function system of $W[a,b]$ as follows: put $\varphi_{ij}(t)=H_{t_i}(t)e_j$ and $\psi_{ij}(t)=L_r^*\varphi_{ij}(t)$, where $e_j=\left(0,...0,1_{j\text{th}},0,...,0\right)^T$, $L_r^*=\text{diag}\left(L_{1r}^*,L_{2r}^*,...,L_{(2\eta)r}^*\right)$ is the adjoint operator of L_r, $H_t(s)$ is the reproducing kernel function of $W_2^1[a,b]$, and $\{t_i\}_{i=1}^\infty$ is dense on $[a,b]$. The orthonormal function system $\left\{\overline{\psi}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2\eta)}$ of $W[a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\left\{\psi_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2\eta)}$ as follows: set

$$\overline{\psi}_{ij}(t)=\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(t), \tag{10}$$

where $i=1,2,3,...$, $j=1,2,...,2\eta$ and β_{lk}^{ij} are orthogonalization coefficients.

The subscript s by the operator L_r, denoted by L_{rs}, indicates that the operator L_r applies to the function of s. Indeed, it is easy to see that, $\psi_{ij}(t)=L_r^*\varphi_{ij}(t)=\left\langle L_r^*\varphi_{ij}(s),G_t(s)\right\rangle_W=\left\langle \varphi_{ij}(s),L_{rs}G_t(s)\right\rangle_H=L_{rs}G_t(s)|_{s=t_i}\in W[a,b]$. Thus, $\psi_{ij}(t)$ can be expressed in the form $\psi_{ij}(t)=L_{rs}G_t(s)|_{s=t_i}$.

Theorem 6 For Eqs. (8) and (9), if $\{t_i\}_{i=1}^\infty$ is dense on $[a,b]$, then $\left\{\psi_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2\eta)}$ is the complete function system of the space $W[a,b]$.

Proof $\forall x_r(t)\in W[a,b]$, let $\left\langle x_r(t),\psi_{ij}(t)\right\rangle_W=0$, which gives

$$\begin{aligned}\left\langle x_r(t),\psi_{ij}(t)\right\rangle_W &= \left\langle x_r(t),L_r^*\varphi_{ij}(t)\right\rangle_W \\ &= \left\langle L_rx_r(t),\varphi_{ij}(t)\right\rangle_H \\ &= L_rx_r(t_i)=0.\end{aligned}$$

Whilst

$$\begin{aligned}x_r(t) &= \sum_{j=1}^{2\eta}x_{jr}(t)e_j \\ &= \sum_{j=1}^{2\eta}\langle x_r(\cdot),G_t(\cdot)e_j\rangle_W e_j.\end{aligned}$$

Hence, $L_rx_r(t)=\sum_{j=1}^{2\eta}\left\langle L_rx_r(t),\varphi_{ij}(t)\right\rangle_W e_j=0$. But since $\{t_i\}_{i=1}^\infty$ is dense on $[a,b]$, we must have $L_rx_r(t)=0$. It follows that $x_r(t)=0$ from the existence of L_r^{-1}. So, the proof of the theorem is complete.

The internal structure of the following theorem is to utilize the representation form of the exact and approximate solutions of system of FIVP (1) and (2) in the space $W[a,b]$.

Theorem 7 If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of Eqs. (8) and (9) is unique, then the exact solution of Eqs. (8) and (9) satisfies the expansion form

$$x_r(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{ik}^{ij} f_{kr}(t_l, x_r(t_l))\,\overline{\psi}_{ij}(t). \tag{11}$$

Proof Applying Theorem 6, it is easy to see that $\left\{\overline{\psi}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2\eta)}$ is the complete orthonormal basis of $W[a,b]$. Thus, using Eq. (10), we have

$$\begin{aligned}
x_r(t) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\left\langle x_r(t), \overline{\psi}_{ij}(t)\right\rangle_W \overline{\psi}_{ij}(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\left\langle x_r(t), \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(t)\right\rangle_W \overline{\psi}_{ij}(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\left\langle x_r(t), L_r^{*}\varphi_{lk}(t)\right\rangle_W \overline{\psi}_{ij}(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\left\langle L_r x_r(t), \varphi_{lk}(t)\right\rangle_H \overline{\psi}_{ij}(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\left\langle f_{kr}(t, x_r(t)), \varphi_{lk}(t)\right\rangle_H \overline{\psi}_{ij}(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij} f_{kr}(t_l, x_r(t_l))\,\overline{\psi}_{ij}(t).
\end{aligned}$$

Therefore, the form of Eq. (11) is the exact solution of Eqs. (8) and (9). The proof is complete.

Remark 2 We mention here that, the approximate solution $x_r^n(t)$ of $x_r(t)$ for Eqs. (8) and (9) can be obtained directly by taking finitely many terms in the series representation form of $x_r(t)$ for Eq. (11) and is given as

$$x_r^n(t) = \sum_{i=1}^{n}\sum_{j=1}^{2\eta}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij} f_{kr}(t_l, x_r(t_l))\,\overline{\psi}_{ij}(t). \tag{12}$$

5 Implementation of iterative algorithm

In this section we develop an iterative algorithm to find the solutions of system of FIVP (1) and (2) in the space $W[a,b]$ for linear and nonlinear case. Also, the solutions of same system, obtained by using proposed method with existing fuzzy numbers are proved to converge to the exact solutions with decreasing absolute difference between the exact values and the values obtained using RKHS method.

The basis of our RKHS solutions method for solving Eqs. (8) and (9) is summarized below for the exact and approximate solutions. Firstly, we shall make use of the following facts about linear and nonlinear case depending on the internal structure of the function f_r.

Case 1 If Eq. (8) is linear, then the exact and approximate solutions can be obtained directly from Eqs. (11) and (12), respectively.

Case 2 If Eq. (8) is nonlinear, then the exact and approximate solutions can be obtained by using the following iterative process. According to Eq. (11), the representation form of the solution of Eqs. (8) and (9) will be

$$x_r(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta} L_{ijr}\overline{\psi}_{ij}(t),$$

where $L_{ijr} = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{ik}^{ij} f_{kr}(t_l, x_r(t_l))$. Put $t_1 = a$, it follows that $x_r(t_1)$ is known from the initial conditions of Eq. (9); so $f_r(t_1, x_r(t_1))$ is known. For numerical computations, we put initial function $x_r^0(t_1) = x_r(t_1)$ and define the n-term approximations to $x_r(t)$ by

$$x_r^n(t) = \sum_{i=1}^{n}\sum_{j=1}^{2\eta} B_{ij}\overline{\psi}_{ij}(t), \tag{13}$$

where the coefficients B_{ij} and the successive approximations $x_r^i(t)$, $i = 1, 2, \ldots, n$ are given as follows:

$$\begin{aligned}
B_{1j} &= \sum_{l=1}^{1}\sum_{k=1}^{j}\beta_{1k}^{1j} f_{kr}\left(t_1, x_r^0(t_1)\right);\\
x_r^1(t) &= \sum_{j=1}^{2\eta} B_{1j}\overline{\psi}_{1j}(t),\\
B_{2j} &= \sum_{l=1}^{2}\sum_{k=1}^{j}\beta_{lk}^{2j} f_{kr}\left(t_l, x_r^{l-1}(t_l)\right);\\
x_r^2(t) &= \sum_{i=1}^{2}\sum_{j=1}^{2\eta} B_{ij}\overline{\psi}_{ij}(t),\\
&\vdots\\
B_{nj} &= \sum_{l=1}^{n}\sum_{k=1}^{j}\beta_{lk}^{nj} f_{kr}\left(t_l, x_r^{l-1}(t_l)\right);\\
x_r^n(t) &= \sum_{l=1}^{n}\sum_{j=1}^{2\eta} B_{ij}\overline{\psi}_{ij}(t).
\end{aligned} \tag{14}$$

In the iterative process of Eq. (14), we can guarantee that the approximation $x_r^n(t)$ satisfies the initial condition of Eq. (9). Now, we will proof that $x_r^n(t)$ in the iterative formula (14) is converge to the exact solution $x_r(t)$ of Eq. (8). In fact, this result is a fundamental rule in the RKHS theory and its applications.

Lemma 2 If $z(t) \in W_2^2[a,b]$, then

$$\begin{aligned}
|z(t)| &\le \left(1 + b - a + \sqrt{(b-a)^3}\right)\|z\|_{W_2^2}\\
|z'(t)| &\le \left(1 + \sqrt{b-a}\right)\|z\|_{W_2^2}.
\end{aligned}$$

Proof For the first part, noting that $z'(t) - z'(a) = \int_a^t z''(p)\,dp$, where $z'(t)$ is absolute continuous on $[a,b]$. If this is integrated again from a to t, the result is $z(t)$ itself as;

$$z(t) - z(a) - z'(a)(t-a) = \int_a^t\left(\int_a^y z''(p)\,dp\right)dy$$

So,

$$|z(t)| \leq |z(a)| + |z'(a)|(b-a) + (b-a)\int_a^b |z''(p)|\,dp$$

By using Holder's inequality and Eq. (7), we can note the following relation: $|z(a)| \leq \|z\|_{W_2^2}$, $|z'(a)| \leq \|z\|_{W_2^2}$, and $\int_a^b |z''(p)|\,dp \leq \sqrt{(b-a)}\,\|z\|_{W_2^2}$. Thus,

$$|z(t)| \leq \left(1+b-a+\sqrt{(b-a)^3}\right)\|z\|_{W_2^2}.$$

For the second part, since $z'(t) = z'(a) + \int_a^t z''(p)\,dp$, this means that $|z'(t)| \leq |z'(a)| + \int_a^b |z''(p)|\,dp$. In other word, one can find $|z'(t)| \leq \left(1+\sqrt{(b-a)}\right)\|z\|_{W_2^2}$.

Theorem 8 If $\|x_r^n(t) - x_r(t)\|_W \to 0$, $t_n \to s$ as $n \to \infty$, $\|x_r^n\|_W$ is bounded, and $f_r(t, x_r(t))$ is continuous, then $f_r(t_n, x_r^{n-1}(t_n)) \to f_r(s, x_r(s))$ as $n \to \infty$.

Proof Firstly, we will prove that $x_r^{n-1}(t_n) \to x_r(s)$. Since, we can note that

$$\begin{aligned}
&\left|x_r^{n-1}(t_n) - x_r(s)\right| \\
&\quad = \left|x_r^{n-1}(t_n) - x_r^{n-1}(s) + x_r^{n-1}(s) - x_r(s)\right| \\
&\quad \leq \left|x_r^{n-1}(t_n) - x_r^{n-1}(s)\right| + \left|x_r^{n-1}(s) - x_r(s)\right| \\
&\quad \leq \left|\left(x_r^{n-1}\right)'(\xi)\right| |t_n - s| + \left|x_r^{n-1}(s) - x_r(s)\right|,
\end{aligned}$$

where ξ lies between t_n and s. From Lemma 2, it follows that

$$\left|x_r^{n-1}(s) - x_r(s)\right| \leq \left(1+b-a+\sqrt{(b-a)^3}\right) \times \left\|x_r^{n-1}(s) - x_r(s)\right\|_W,$$

which is gives $\left|x_r^{n-1}(s) - x_r(s)\right| \to 0$ as $n \to \infty$, while on the other hand, we have

$$\left|\left(x_r^{n-1}\right)'(\xi)\right| \leq \left(1+\sqrt{(b-a)}\right)\left\|x_r^{n-1}(\xi)\right\|_W.$$

In terms of the boundedness of $\left\|x_r^{n-1}(t)\right\|_W$, one obtains that $\left|x_r^{n-1}(t_n) - x_r(s)\right| \to 0$ as $n \to \infty$. Thus, by means of the continuation of $f_r(t, x_r(t))$, it is implies that $f_r(t_n, x_r^{n-1}(t_n)) \to f_r(s, x_r(s))$ as $n \to \infty$. So, the proof of the theorem is complete.

Theorem 9 Suppose that $\|x_r^n\|_W$ is bounded in Eq. (13), and Eqs. (8) and (9) has a unique solution. If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then the n-term approximate solution $x_r^n(t)$ in the iterative formula of Eq. (13) converges to the exact solution $x_r(t)$ of Eqs. (8) and (9), and $x_r(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2\eta} B_{ij}\overline{\psi}_{ij}(t)$.

Proof Similar to the proof of Theorem 4 in []

6 Software libraries and numerical experiment

In order to solve system of FIVP (1) and (2) approximately on a computer, the system is approximated by a discrete one. Continuous functions are approximated by finite arrays of values. Algorithms are then sought which approximately solve the mathematical problem efficiently, accurately and reliably. While scientific computing focuses on the design and the implementation of such algorithms, numerical analysis may be viewed as the theory behind them. To show behavior, properties, efficiency, and applicability of the present RKHS method, two linear and one nonlinear fuzzy differential systems will be solved numerically in this section.

An algorithm is a finite sequence of rules for performing computations on a computer such that at each instant the rules determine exactly what the computer has to do next. Next algorithm is utilizes to implement a procedure to solve FIVP (1) and (2) in numeric form in terms of their grid nodes based on the use of RKHS method.

Algorithm 2 To approximate the solution $x_r^n(t)$ of $x_r(t)$ for Eqs. (8) and (9), we do the following steps:

Input The interval $[a,b]$, the unit interval $[0,1]$ the integers n, the integers m, the kernel functions $G_t(s)$ and $H_t(s)$, the differential operator L_r, and the function f_r.

Output Approximate solution $x_r^n(t)$ of $x_r(t)$.

Step 1 Fixed t in $[a,b]$ and set $s \in [a,b]$;

If $s \leq t$, set $G_t(s) = \Lambda(s,t)$;
Else set $G_t(s) = \Lambda(t,s)$;
For $i = 1,2,\ldots,n$, $h = 1,2,\ldots,m$, and $j = 1,2,\ldots,2\eta$, do the following:
Set $t_i = \frac{i-1}{n-1}$;
Set $r_h = \frac{h-1}{m-1}$;
Set $\psi_{i,j}(t) = L_{r_h s}\left[G_t(s)\right]_{s=t_i}$;
Output: the orthogonal function system $\psi_{i,j}(t)$.

Step 2 For $l = 2,3\ldots,n-1$ and $k = 1,2\ldots,l-1$, do the following:

Set $\overline{\psi}_{ij}(t) = \sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{lk}^{ij}\psi_{lk}(t)$;
Output: the orthonormal function system $\overline{\psi}_{ij}(t)$.

Step 3 Set $x_{r_h}^0(t_1) = x_{r_h}(t_1) = 0$;

Set $B_{ij} = \sum_{l=1}^{i}\sum_{k=1}^{2} \beta_{lk}^{ij} f_{kr_h}\left(t_l, x_{r_h}^{l-1}(t_l)\right)$;
Set $x_{r_h}^i(t) = \sum_{i=1}^{i}\sum_{j=1}^{2} B_{ij}\overline{\psi}_{ij}(t)$;
Output: the approximate solution $x_r^n(t)$ of $x_{r_h}(t)$.

Step 4 Stop.

Remark 3 Throughout this paper, we will try to give the results of the three examples; however, in some cases we will switch between the results obtained for the examples in order not to increase the length of the paper without the loss of generality for the remaining examples and results. In the process of computation, all the symbolic and numerical computations are performed by using MAPLE 13 software package.

Next, we show by example that the system of crisp initial value problems can be modeled in a natural way as system of FIVPs. To illustrate this, consider the dynamic supply and demand system. The system of ODE corresponding to this problem is $p'(t) = \theta - k_1(s - s_0)$ and $s'(t) = k_2(p - p_0)$, where p is the price, s is the supply, p_0 is the equilibrium price, s_0 is equilibrium supply, θ is the rate of inflation, and k_1, k_2 are positive constant corresponding to the dynamic nature of the system. Here, we are considering an item such that increasing its price p results in an increase in supply s but that increasing its supply s will ultimately decrease its price p. Furthermore, we will assume there are two factors that influence price; inflation and supply. The factor $s - s_0$ means that; firstly, if $s > s_0$, the supply is too large and price is to decrease; secondly, if $s < s_0$, supply is too low and price tends to increase, while on the other hand, the factor $p - p_0$ means that; firstly, if $p > p_0$, price is high and supply increasing; secondly, if $p < p_0$, price is low and supply decreases. Uncertainty in determining the initial values, inaccuracy in element modeling, and other parameters cause uncertainty in the aforementioned system. Considering them instead as system of FIVPs yields more realistic results.

Example 1 [41] Consider the following dynamic supply and demand differential system of fuzzy equations on $[0,1]$:

$$\begin{aligned} p'(t) &= \theta - k_1(s - s_0), \\ s'(t) &= k_2(p - p_0), \end{aligned} \tag{15}$$

subject to the fuzzy initial conditions

$$x_1(0) = \alpha_1, x_2(0) = \alpha_2, \tag{16}$$

where

$$\begin{aligned} [\alpha_1]^r &= [20 + 5r, 30 - 5r] \\ [\alpha_2]^r &= [550 + 50r, 650 - 50r]. \end{aligned}$$

For numerical results and comparisons, the following values, for parameters, are considered [41]: $\theta = 0.05$, $s_0 = 1200$, $p_0 = 25$, and $k_1 = k_2 = 0.5$. The exact fuzzy solutions of system of FIVP (15) and (16) in parametric form are

$$\begin{aligned} &[p(t)]^r \\ &\quad = \left[\left(\tfrac{45}{2} - \tfrac{45}{2}r\right)e^{-\frac{t}{2}} - \left(\tfrac{55}{2} - \tfrac{55}{2}r\right)e^{\frac{t}{2}},\right. \\ &\quad\quad \left.\left(\tfrac{45}{2}r - \tfrac{45}{2}\right)e^{-\frac{t}{2}} + \left(\tfrac{55}{2} - \tfrac{55}{2}r\right)e^{\frac{t}{2}}\right] \\ &\quad\quad + \tfrac{6001}{10}\sin\left(\tfrac{t}{2}\right) + 25, \\ &[s(t)]^r \\ &\quad = \left[\left(\tfrac{45}{2}r - \tfrac{45}{2}\right)e^{-\frac{t}{2}} - \left(\tfrac{55}{2} - \tfrac{55}{2}r\right)e^{\frac{t}{2}},\right. \\ &\quad\quad \left.\left(\tfrac{45}{2} - \tfrac{45}{2}r\right)e^{-\frac{t}{2}} + \left(\tfrac{55}{2} - \tfrac{55}{2}r\right)e^{\frac{t}{2}}\right] \\ &\quad\quad - \tfrac{6001}{10}\cos\left(\tfrac{t}{2}\right) + \tfrac{12001}{10}. \end{aligned}$$

Using RKHS method, taking $t_i = \frac{i-1}{n-1}$, $i = 1,2,\ldots,n$, $n = 251$ and $r_j = \frac{j-1}{m-1}$, $j = 1,2,\ldots,m$, $m = 5$ with the reproducing kernel functions $G_t(s)$ and $H_t(s)$ on $[0,1]$ in which Algorithms 1 and 2 are used throughout the computations; some graphical results and tabulate data are presented and discussed quantitatively to illustrate the fuzzy approximate solutions and the approximate Hukuhara derivatives.

As we mentioned earlier, it is possible to pick any point in the interval of integration $[0,1]$ and as well the fuzzy approximate solutions and their first Hukuhara derivatives will be applicable. Next, numerical results of approximating the sets $[p(t)]^r$ and $[p'(t)]^r$ of system of FIVP (15) and (16) at $t = 1/\sqrt{2}$ and various r are given in Tables 1 and 2, respectively, while in Tables 3 and 4 the approximate solutions for $[s(t)]^r$ and $[s'(t)]^r$ have been tabulated.

Example 2 [41] Consider the following linear differential system of fuzzy equations on $[0,1]$:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(t), \\ x_2'(t) &= -x_1(t) + x_2(t), \end{aligned} \tag{17}$$

subject to the fuzzy initial conditions

$$x_1(0) = \alpha_1, x_2(0) = \alpha_2, \tag{18}$$

where

$$\alpha_1(s) = \begin{cases} s - 1, & 1 \le s \le 2, \\ 3 - s, & 2 \le s \le 3, \end{cases}$$

and

$$\alpha_2(s) = \begin{cases} s, & 0 \le s \le 1, \\ 2 - s, & 1 \le s \le 2, \end{cases}$$

The exact fuzzy solutions of system of FIVP (17) and (18) in fuzzy setting are

$$\begin{aligned} x_1(t) &= \alpha_3(s)e^{2t} + e^t\sin(x) + 2e^t\cos(t) \\ x_2(t) &= \alpha_3(s)e^{2t} + e^t\cos(t) - 2e^t\sin(t), \end{aligned}$$

where

$$\alpha_3(s) = \begin{cases} s + 1, & -1 \le s \le 0, \\ 1 - s, & 0 \le s \le 1, \end{cases}$$

Table 1: The fuzzy exact and approximate solutions of $[p(t)]^r$ for system of FIVP (15) and (16) at $t = 1/\sqrt{2}$.

r	$\left[p\left(1/\sqrt{2}\right)\right]^r$	$\left[p^{251}\left(1/\sqrt{2}\right)\right]^r$
0	$[209.4107478718200, 256.1388113835788]$	$[209.4107457778047, 256.1388096296631]$
0.25	$[215.2517558107898, 250.2978034446090]$	$[215.2517537592875, 250.2978016481808]$
0.5	$[221.0927637497568, 244.4567955056391]$	$[221.0927617407695, 244.4567936666986]$
0.75	$[226.9337716887295, 238.6157875666692]$	$[226.9337697222519, 238.6157856852160]$
1	$[232.7747796276994, 232.7747796276994]$	$[232.7747777037349, 232.7747777037349]$

Table 2: The Hukuhara derivative of fuzzy exact and approximate solutions of $[p'(t)]^r$ for system of FIVP (15) and (16) at $t = 1/\sqrt{2}$.

r	$\left[p'\left(1/\sqrt{2}\right)\right]^r$	$\left[(p')^{251}\left(1/\sqrt{2}\right)\right]^r$
0	$[254.0101507708822, 308.9726650864563]$	$[254.0101469199274, 308.9726619205239]$
0.25	$[260.8804650603299, 302.1023507970087]$	$[260.8804612950028, 302.1023475454491]$
0.5	$[267.7507793497783, 295.2320365075638]$	$[267.7507756700765, 295.2320331703741]$
0.75	$[274.6210936392236, 288.3617222181178]$	$[274.6210900451513, 288.3617187952993]$
1	$[281.4914079286687, 281.4914079286687]$	$[281.4914044202282, 281.4914044202282]$

Table 3: The fuzzy exact and approximate solutions of $[s(t)]^r$ for system of FIVP (15) and (16) at $t = 1/\sqrt{2}$.

r	$\left[s\left(1/\sqrt{2}\right)\right]^r$	$\left[s^{251}\left(1/\sqrt{2}\right)\right]^r$
0	$[582.1546698270840, 692.0796984582332]$	$[582.1546694374060, 692.0796981604634]$
0.25	$[595.8952984059777, 678.3390698793396]$	$[595.8952980277882, 678.3390695700812]$
0.5	$[609.6359269848714, 664.5984413004459]$	$[609.6359266181703, 664.5984409796985]$
0.75	$[623.3765555637650, 650.8578127215522]$	$[623.3765552085529, 650.8578123893166]$
1	$[637.1171841426586, 637.1171841426586]$	$[637.1171837989350, 637.1171837989350]$

Here, $\alpha_1(s)$, $\alpha_2(s)$, and $\alpha_3(s)$ are vanished outside the intervals $[1,3]$, $[0,2]$, and $[-1,1]$, respectively. In fact this system is a generalization of the system of ODE $x_1'(t) = x_1(t) + x_2(t)$ and $x_2'(t) = -x_1(t) + x_2(t)$ subject to initial conditions $x_1(0) \approx 2$ and $x_2(0) \approx 1$. Anyhow, if one put $r = s-1$, then $s = r+1$, again if $r = 3-s$, then $s = 3-r$; hence, $[\alpha_1]^r = [r+1, 3-r]$; similarly, $[\alpha_2]^r = [r, 2-r]$ and $[\alpha_3]^r = [r-1, 1-r]$. In order to apply the RKHS method, we first apply Algorithm 1 as follows; put $[x_1(t)]^r = [x_{1r}(t), x_{2r}(t)]$ and $[x_2(t)]^r = [x_{3r}(t), x_{4r}(t)]$. Then we have the following system of ODE:

$$\begin{aligned} x_{1r}'(t) &= x_{1r}(t) + x_{3r}(t), \\ x_{2r}'(t) &= x_{2r}(t) + x_{4r}(t), \\ x_{3r}'(t) &= -x_{2r}(t) + x_{3r}(t), \\ x_{4r}'(t) &= -x_{1r}(t) + x_{4r}(t), \end{aligned} \tag{19}$$

subject to the initial conditions

$$\begin{aligned} &x_{1r}(0) = r+1,\ x_{2r}(0) = 3-r, \\ &x_{3r}(0) = r,\ x_{4r}(0) = 2-r. \end{aligned} \tag{20}$$

Using RKHS method, taking $t_i = \frac{i-1}{n-1}$, $i = 1,2,\ldots,n$, $n = 251$ and $r_j = \frac{j-1}{m-1}$, $j = 1,2,\ldots,m$, $m = 5$ with the reproducing kernel functions $G_t(s)$ and $H_t(s)$ on $[0,1]$ in which Algorithms 1 and 2 are used throughout the computations; some graphical results, comparison feedback, and tabulate data are presented and discussed quantitatively to illustrate the fuzzy approximate solutions.

Result from numerical analysis is an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. Next, the absolute difference between the exact values and the values obtained using RKHS method (absolute error) of numerically approximating $x_r(t)$ by $x_r^{251}(t)$ for system of ODE (19) and (20) have been calculated for various t and r as shown in Tables 5, 6, 7, and 8. From the tables, it can be seen that with the few tens of iterations, the RKHS approximate solutions with high accuracy are achievable.

Numerical comparisons for system of FIVP (17) and (18) are studied next. The numerical methods that are used for comparison with RKHS method include the variational iteration method [39], the HAM [40], and the fuzzy neural network method [41]. Anyhow, Table 9 shows a comparison between the absolute errors of our method together with other aforementioned methods in approximating $x_{1r}(t)$ and $x_{2r}(t)$ of $[x_1(t)]^r$ at $t = 0.2$ and various r, while Table 10 shows a comparison in approximating $x_{3r}(t)$ and $x_{4r}(t)$ of $[x_2(t)]^r$ at $t = 0.2$ and various r. It is clear from the tables that the absolute errors of the RKHS method are the lowest one among all other numerical and analytical ones.

Table 4: The Hukuhara derivative of fuzzy exact and approximate solutions of $[s'(t)]^r$ for system of FIVP (15) and (16) at $t = 1/\sqrt{2}$.

r	$\left[s'\left(1/\sqrt{2}\right)\right]^r$	$\left[(s')^{251}\left(1/\sqrt{2}\right)\right]^r$
0	$[92.20537393591667, 115.5694056917909]$	$[92.20537249563444, 115.5694045426889]$
0.25	$[95.12587790539287, 112.6489017223145]$	$[95.12587650151613, 112.6489005368070]$
0.5	$[98.04638187488287, 109.7283977528246]$	$[98.04638050739791, 109.7283965309244]$
0.75	$[100.9668858443705, 106.8078937833370]$	$[100.9668845132806, 106.8078925250427]$
1	$[103.8873898138538, 103.8873898138538]$	$[103.8873885191619, 103.8873885191619]$

Table 5: The absolute error of approximating $x_{1r}(t)$ for system of ODE (19) and (20).

t	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.1	6.99146×10^{-8}	5.08162×10^{-8}	3.17177×10^{-8}	1.26192×10^{-8}	6.47922×10^{-9}
0.2	1.29834×10^{-7}	9.32631×10^{-8}	5.66921×10^{-8}	2.01211×10^{-8}	1.64500×10^{-8}
0.3	1.78264×10^{-7}	1.26074×10^{-7}	7.38839×10^{-8}	2.16936×10^{-8}	3.04967×10^{-8}
0.4	2.13219×10^{-7}	1.47598×10^{-7}	8.19769×10^{-8}	1.63557×10^{-8}	4.92655×10^{-8}
0.5	2.32128×10^{-7}	1.55730×10^{-7}	7.93321×10^{-8}	2.93410×10^{-9}	7.34639×10^{-8}
0.6	2.31722×10^{-7}	1.47827×10^{-7}	6.39312×10^{-8}	1.99643×10^{-8}	1.0386×10^{-7}
0.7	2.07895×10^{-7}	1.20601×10^{-7}	3.33079×10^{-8}	5.39855×10^{-8}	1.41279×10^{-7}
0.8	1.55529×10^{-7}	6.99966×10^{-8}	1.55357×10^{-8}	1.01068×10^{-7}	1.86600×10^{-7}
0.9	6.82846×10^{-8}	8.97415×10^{-9}	8.62329×10^{-8}	1.63492×10^{-7}	2.40750×10^{-7}
1	6.16605×10^{-8}	1.22419×10^{-7}	1.83178×10^{-7}	2.43936×10^{-7}	3.04695×10^{-7}

Table 6: The absolute error of approximating $x_{2r}(t)$ for system of ODE (19) and (20).

t	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.1	8.28731×10^{-8}	6.37746×10^{-8}	4.46761×10^{-8}	2.55777×10^{-8}	6.47922×10^{-9}
0.2	1.62734×10^{-7}	1.26163×10^{-7}	8.95920×10^{-8}	5.30210×10^{-8}	1.64500×10^{-8}
0.3	2.39258×10^{-7}	1.87068×10^{-7}	1.34877×10^{-7}	8.26870×10^{-8}	3.04967×10^{-8}
0.4	3.11750×10^{-7}	2.46129×10^{-7}	1.80508×10^{-7}	1.14887×10^{-7}	4.92655×10^{-8}
0.5	3.79056×10^{-7}	3.02658×10^{-7}	2.26260×10^{-7}	1.49862×10^{-7}	7.34639×10^{-8}
0.6	4.39442×10^{-7}	3.55546×10^{-7}	2.71651×10^{-7}	1.87755×10^{-7}	1.03860×10^{-7}
0.7	4.90453×10^{-7}	4.03159×10^{-7}	3.15866×10^{-7}	2.28572×10^{-7}	1.41279×10^{-7}
0.8	5.28730×10^{-7}	4.43197×10^{-7}	3.57665×10^{-7}	2.72133×10^{-7}	1.86600×10^{-7}
0.9	5.49786×10^{-7}	4.72527×10^{-7}	3.95268×10^{-7}	3.18009×10^{-7}	2.40750×10^{-7}
1	5.47729×10^{-7}	4.86971×10^{-7}	4.26212×10^{-7}	3.65454×10^{-7}	3.04695×10^{-7}

Nonlinear phenomena's are of fundamental importance in various fields of science and engineering, and other disciplines, since most phenomena in our world are essentially nonlinear and are described by nonlinear equations. Anyhow, in most real-life situations, the differential systems that models the uncertainty systems are too complicated to solve analytically, and there is a practical need to approximate the solutions. In the next example, the fuzzy Hukuhara differentiable exact solutions cannot be found analytically in terms of closed form solutions.

Example 3 Consider the following nonlinear differential system of fuzzy equations on $[0,1]$:

$$\begin{aligned} x_1'(t) &= e^{x_2(t)} + \alpha, \\ x_2'(t) &= x_1^3(t), \end{aligned} \tag{21}$$

subject to the fuzzy initial conditions

$$x_1(0) = 0, x_2(0) = \beta, \tag{22}$$

where

$$\alpha(s) = \max_{s\in\mathbb{R}}\left(0, 1-(4s)^{\frac{2}{3}}\right)$$
$$\beta(s) = \max_{s\in\mathbb{R}}\left(0, 1-(5s)^{2}\right).$$

For the conduct of proceedings in the solution and depending on Algorithm 1, it is clear that

$$\left[x_1^3(t)\right]^r = \left[x_{1r}^3(t), x_{2r}^3(t)\right]$$
$$\left[e^{x_2(t)}\right]^r = \left[e^{x_{3r}(t)}, e^{x_{4r}(t)}\right].$$

This is due to the fact that s^3 and e^s are strictly increasing continuous functions on $(-\infty,\infty)$. On the other hand, if one set $r = 1-(4s)^{\frac{2}{3}}$, then $s = -\frac{1}{4}(1-r)^{\frac{3}{2}}$ or $s = \frac{1}{4}(1-r)^{\frac{3}{2}}$; hence,

$$[\alpha]^r = \left[-\tfrac{1}{4}\sqrt{(1-r)^3}, \tfrac{1}{4}\sqrt{(1-r)^3}\right]$$
$$[\beta]^r = \left[-\tfrac{1}{5}\sqrt{1-r}, \tfrac{1}{5}\sqrt{1-r}\right].$$

Table 7: The absolute error of approximating $x_{3r}(t)$ for system of ODE (19) and (20).

t	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.1	1.18488×10^{-7}	9.93898×10^{-8}	8.02914×10^{-8}	6.11929×10^{-8}	4.20945×10^{-8}
0.2	2.29597×10^{-7}	1.93026×10^{-7}	1.56455×10^{-7}	1.19884×10^{-7}	8.33126×10^{-8}
0.3	3.33077×10^{-7}	2.80887×10^{-7}	2.28696×10^{-7}	1.76506×10^{-7}	1.24316×10^{-7}
0.4	4.28326×10^{-7}	3.62705×10^{-7}	2.97084×10^{-7}	2.31463×10^{-7}	1.65841×10^{-7}
0.5	5.14312×10^{-7}	4.37914×10^{-7}	3.61516×10^{-7}	2.85118×10^{-7}	2.08720×10^{-7}
0.6	5.89478×10^{-7}	5.05583×10^{-7}	4.21687×10^{-7}	3.37792×10^{-7}	2.53896×10^{-7}
0.7	6.51618×10^{-7}	5.64325×10^{-7}	4.77031×10^{-7}	3.89738×10^{-7}	3.02444×10^{-7}
0.8	6.97723×10^{-7}	6.12190×10^{-7}	5.26658×10^{-7}	4.41126×10^{-7}	3.55593×10^{-7}
0.9	7.23783×10^{-7}	6.46525×10^{-7}	5.69266×10^{-7}	4.92007×10^{-7}	4.14748×10^{-7}
1	7.24547×10^{-7}	6.63789×10^{-7}	6.03030×10^{-7}	5.42271×10^{-7}	4.81513×10^{-7}

Table 8: The absolute error of approximating $x_{4r}(t)$ for system of ODE (19) and (20).

t	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.1	3.42994×10^{-8}	1.52009×10^{-8}	3.89753×10^{-9}	2.29960×10^{-8}	4.20945×10^{-8}
0.2	6.29715×10^{-8}	2.64005×10^{-8}	1.01706×10^{-8}	4.67416×10^{-8}	8.33126×10^{-8}
0.3	8.44453×10^{-8}	3.22551×10^{-8}	1.99352×10^{-8}	7.21255×10^{-8}	1.24316×10^{-7}
0.4	9.66433×10^{-8}	3.10221×10^{-8}	3.45991×10^{-8}	1.00220×10^{-7}	1.65841×10^{-7}
0.5	9.68714×10^{-8}	2.04735×10^{-8}	5.59245×10^{-8}	1.32322×10^{-7}	2.08720×10^{-7}
0.6	8.16858×10^{-8}	2.20970×10^{-9}	8.61052×10^{-8}	1.70001×10^{-7}	2.53896×10^{-7}
0.7	4.67297×10^{-8}	4.05638×10^{-8}	1.27857×10^{-7}	2.15151×10^{-7}	3.02444×10^{-7}
0.8	1.34640×10^{-8}	9.89963×10^{-8}	1.84529×10^{-7}	2.70061×10^{-7}	3.55593×10^{-7}
0.9	1.05713×10^{-7}	1.82972×10^{-7}	2.60231×10^{-7}	3.37489×10^{-7}	4.14748×10^{-7}
1	2.38478×10^{-7}	2.99237×10^{-7}	3.59996×10^{-7}	4.20754×10^{-7}	4.81513×10^{-7}

Table 9: Numerical comparison of approximate solution $[x_1(t)]^r$ for system of FIVP (17) and (18) at $t=0.2$.

	method of [27]		method of [26]		method of [25]		RKHS method	
r	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$
0	2.1×10^{-5}	1.4×10^{-5}	6.7×10^{-6}	5.3×10^{-6}	1.2×10^{-4}	5.9×10^{-5}	1.3×10^{-7}	1.6×10^{-7}
0.2	6.0×10^{-6}	1.4×10^{-5}	5.5×10^{-6}	4.1×10^{-6}	1.1×10^{-4}	4.0×10^{-5}	1.0×10^{-7}	1.3×10^{-7}
0.4	1.3×10^{-5}	1.6×10^{-5}	4.3×10^{-6}	2.9×10^{-6}	8.8×10^{-5}	2.2×10^{-5}	7.1×10^{-8}	1.0×10^{-7}
0.6	1.6×10^{-5}	1.9×10^{-5}	3.1×10^{-6}	1.7×10^{-6}	6.9×10^{-5}	3.9×10^{-6}	4.2×10^{-8}	7.5×10^{-8}
0.8	8.0×10^{-6}	1.3×10^{-5}	1.9×10^{-6}	5.2×10^{-7}	5.1×10^{-5}	1.4×10^{-5}	1.3×10^{-8}	4.6×10^{-8}
1	1.3×10^{-5}	1.3×10^{-5}	6.9×10^{-6}	6.9×10^{-7}	3.3×10^{-5}	3.3×10^{-5}	8.3×10^{-8}	8.3×10^{-8}

For finding fuzzy approximate solutions of system of FIVP (21) and (22), which are corresponding to their parametric form, we have the following system of ODE:

$$\begin{aligned} x'_{1r}(t) &= e^{x_{3r}(t)} - \tfrac{1}{4}\sqrt{(1-r)^3}, \\ x'_{2r}(t) &= e^{x_{4r}(t)} + \tfrac{1}{4}\sqrt{(1-r)^3}, \\ x'_{3r}(t) &= x_{1r}^3(t), \\ x'_{4r}(t) &= x_{2r}^3(t), \end{aligned} \quad (23)$$

subject to the initial conditions

$$\begin{aligned} & x_{1r}(0)=0,\, x_{2r}(0)=0, \\ & x_{3r}(0) = -\tfrac{1}{5}\sqrt{1-r},\, x_{4r}(0) = \tfrac{1}{5}\sqrt{1-r}. \end{aligned} \quad (24)$$

Our next goal is to present the HAM approximate solutions for system of ODE (23) and (24) in order to measure the extent of agreement with unknowns closed form solutions which are inapplicable in general for such nonlinear systems, in order to employ again the obtained expansions to measure the accuracy of the RKHS method in finding and predicting the fuzzy approximate solutions. To do so, we report the series formulas for the HAM solutions in which the obtained results are generated from the 10-truncated series solutions for each $x_{jr}(t)$, $j=1,2,3,4$. Henceforth, for simplicity and not to conflict, we will let $x_{jr}^{\mathrm{HAM}}(t)$, $j=1,2,3,4$ to denote the HAM series solutions of $x_{jr}(t)$, as follows:

$$\begin{aligned} x_{1r}^{\mathrm{HAM}}(t) &= \left(e^{\beta_{1r}}+\alpha_{1r}\right)t + \tfrac{1}{20}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^3 e^{\beta_{1r}} t^5 \\ &\quad + \left(\tfrac{1}{288}e^{\beta_{1r}}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^6 \right. \\ &\quad \left. + \tfrac{1}{480}e^{2\beta_{1r}}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^5\right)t^9 \end{aligned}$$

Table 10: Numerical comparison of approximate solution $[x_2(t)]^r$ for system of FIVP (17) and (18) at $t = 0.2$.

	method of [27]		method of [26]		method of [25]		RKHS method	
r	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$	$x_{1r}(t)$	$x_{2r}(t)$
0	2.4×10^{-5}	1.4×10^{-5}	4.5×10^{-6}	7.5×10^{-6}	7.9×10^{-5}	1.0×10^{-4}	2.3×10^{-7}	6.3×10^{-8}
0.2	1.3×10^{-5}	4.0×10^{-6}	3.3×10^{-6}	6.3×10^{-6}	6.1×10^{-5}	8.5×10^{-5}	2.0×10^{-7}	3.4×10^{-8}
0.4	2.0×10^{-5}	1.3×10^{-5}	2.1×10^{-6}	5.1×10^{-6}	4.3×10^{-5}	6.7×10^{-5}	1.7×10^{-7}	4.5×10^{-9}
0.6	1.3×10^{-5}	0.8×10^{-5}	9.3×10^{-7}	3.9×10^{-6}	2.4×10^{-5}	4.9×10^{-5}	1.4×10^{-7}	2.5×10^{-8}
0.8	1.4×10^{-5}	9.0×10^{-6}	2.8×10^{-7}	2.7×10^{-6}	6.1×10^{-6}	3.0×10^{-5}	1.1×10^{-7}	5.4×10^{-8}
1	1.0×10^{-5}	1.0×10^{-5}	1.5×10^{-6}	1.5×10^{-6}	1.2×10^{-5}	1.2×10^{-5}	1.6×10^{-8}	1.6×10^{-8}

Table 11: The values of absolute residual error functions for system of ODE (23) and (24) at $t = 0.5$.

r	$\mathrm{Res}_{1r}^{\mathrm{HAM}}(t)$	$\mathrm{Res}_{2r}^{\mathrm{HAM}}(t)$	$\mathrm{Res}_{3r}^{\mathrm{HAM}}(t)$	$\mathrm{Res}_{4r}^{\mathrm{HAM}}(t)$
0	$1.1656990040\times10^{-8}$	$2.3244814053\times10^{-3}$	$1.3235259190\times10^{-7}$	$8.2380214832\times10^{-5}$
0.25	$5.2844108311\times10^{-8}$	$1.3639342244\times10^{-3}$	$5.0779808894\times10^{-7}$	$4.4089302308\times10^{-5}$
0.5	$1.7734610724\times10^{-7}$	$7.7122024269\times10^{-4}$	$1.4936037025\times10^{-6}$	$2.4555858754\times10^{-5}$
0.75	$4.9090555176\times10^{-7}$	$3.8672144886\times10^{-4}$	$3.7071763640\times10^{-6}$	$1.5286806524\times10^{-5}$
1	$1.7943953232\times10^{-6}$	$1.7943953232\times10^{-6}$	$1.1855064168\times10^{-5}$	$1.1855064168\times10^{-5}$

Table 12: Numerical comparison of approximate solution of $[x_1(t)]^r$ for system of FIVP (21) and (22) at $t = 0.5$.

r	HAM solution	RKHS solution
0	$[0.2846010588829784, 0.7417813098616158]$	$[0.2846010604602641, 0.7417813504005433]$
0.25	$[0.3397042479611809, 0.6803323123735007]$	$[0.3397042507217690, 0.6803323429733797]$
0.5	$[0.3905128029340397, 0.6235854675651437]$	$[0.3905128072739875, 0.6235854905123175]$
0.75	$[0.4377406928550377, 0.5707492345416490]$	$[0.4377406992449165, 0.5707492515434681]$
1	$[0.5015733506944444, 0.5015733506944444]$	$[0.5015733613715239, 0.5015733613715239]$

$$
\begin{aligned}
x_{2r}^{\mathrm{HAM}}(t) &= \left(e^{\beta_{2r}}+\alpha_{2r}\right)t+\tfrac{1}{20}\left(e^{\beta_{2r}}+\alpha_{2r}\right)^3 e^{\beta_{2r}}t^5\\
&\quad+\left(\tfrac{1}{288}e^{\beta_{1r}}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^6\right.\\
&\quad\left.+\tfrac{1}{480}e^{2\beta_{1r}}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^5 t^9\right),\\
x_{3r}^{\mathrm{HAM}}(t) &= \beta_{1r}+\tfrac{1}{4}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^3 t^4\\
&\quad+\tfrac{3}{160}e^{\beta_{1r}}\left(e^{\beta_{1r}}+\alpha_{1r}\right)^5 t^8,\\
x_{4r}^{\mathrm{HAM}}(t) &= \beta_{2r}+\tfrac{1}{4}\left(e^{\beta_{2r}}+\alpha_{2r}\right)^3 t^4\\
&\quad+\tfrac{3}{160}e^{\beta_{2r}}\left(e^{\beta_{2r}}+\alpha_{2r}\right)^5 t^8
\end{aligned}
$$

While one cannot know the absolute error without knowing the exact solution, in most cases the residual error, denoted by $\mathrm{Res}(t)$, can be used as a reliable indicators in the iteration progresses. In Table 11, the value of the following residual error functions:

$$
\begin{aligned}
&\mathrm{Res}_{1r}^{\mathrm{HAM}}(t)\\
&\quad=\left|\tfrac{d}{dt}x_{1r}^{\mathrm{HAM}}(t)-\left(e^{x_{3r}^{\mathrm{HAM}}(t)}-\tfrac{1}{4}\sqrt{(1-r)^3}\right)\right|,\\
&\mathrm{Res}_{2r}^{\mathrm{HAM}}(t)\\
&\quad=\left|\tfrac{d}{dt}x_{2r}^{\mathrm{HAM}}(t)-\left(e^{x_{4r}^{\mathrm{HAM}}(t)}+\tfrac{1}{4}\sqrt{(1-r)^3}\right)\right|,\\
&\mathrm{Res}_{3r}^{\mathrm{HAM}}(t)=\left|\tfrac{d}{dt}x_{3r}^{\mathrm{HAM}}(t)-\left(x_{1r}^{\mathrm{HAM}}(t)\right)^3\right|,\\
&\mathrm{Res}_{4r}^{\mathrm{HAM}}(t)=\left|\tfrac{d}{dt}x_{4r}^{\mathrm{HAM}}(t)-\left(x_{2r}^{\mathrm{HAM}}(t)\right)^3\right|,
\end{aligned}
\tag{25}
$$

for the 10-truncated series HAM approximate solutions $x_{jr}^{\mathrm{HAM}}(t)$, $j = 1,2,3,4$ have been calculated at $t = 0.5$ and various r for system of ODE (23) and (24). From the table, it can be seen that the HAM provides us with the accurate approximate solutions with attention to that, more accurate solution can be found at the beginning values of r in comparison with large r.

Now, we will return to our RKHS method in order to display new numerical and comparison results. Anyhow, using RKHS method, taking $t_i = \frac{i-1}{n-1}$, $i = 1,2,\ldots,n$, $n = 251$ and $r_j = \frac{j-1}{m-1}$, $j = 1,2,\ldots,m$, $m = 5$ with the reproducing kernel functions $G_t(s)$ and $H_t(s)$ on $[0,1]$ in which Algorithms 1 and 2 are used throughout the computations; some graphical results, comparison feedback, and tabulate data are presented and discussed quantitatively to illustrate the fuzzy approximate solutions.

Numerical comparisons are carried out to verify the mathematical results and the theoretical statement of the solutions. Next, some tabulated data are presented to show the extent between the HAM solutions and the RKHS method solutions. However, Table 12 shows a comparison of approximate solution for $[x_1(t)]^r$ at $t = 0.5$ and various r for system of FIVP (21) and (22), while Tables 13 shows a comparison of approximate solution for $[x_2(t)]^r$ at $t = 0.5$ and various r. As it is evident from the comparison results, it was found that our method in

Table 13: Numerical comparison of approximate solution of $[x_2(t)]^r$ for system of FIVP (21) and (22) at $t = 0.5$.

r	HAM solution	RKHS solution
0	$[-0.1971220785059322, 0.2503923858174381]$	$[-0.1971220669419136, 0.2503925964361660]$
0.25	$[-0.1683138165457321, 0.2121685332672482]$	$[-0.1683137968506898, 0.2121686948804722]$
0.5	$[-0.1339957069088525, 0.1714814924642561]$	$[-0.1339956769374649, 0.1714816163367717]$
0.75	$[-0.0895493664478423, 0.1230851973330545]$	$[-0.0895493241507733, 0.1230852919240308]$
1	$[0.0156982421875000, 0.0156982421875000]$	$[0.0156983060546904, 0.0156983060546904]$

comparison with the mentioned method is similar with a view to accuracy and utilization.

The aforementioned computational results provide a numerical estimate for the RKHS solutions. Also, it is clear that the accuracy obtained using present method is in advanced by using only few tens of iteration, where higher accuracy can be achieved by increasing the number n in Algorithms 2.

7 Concluding remarks

In various subjects of science and engineering, nonlinear systems of fuzzy differential equations subject to given fuzzy initial conditions, as well as their exact and numerical solutions, are essentially important, therefore systems of FIVPs should be solved. In the present paper, we have studied exact and numerical solutions for system of FIVPs (1) and (2) based on the reproducing kernel theory. Some results on the behavior of fuzzy solutions, convergence theorem, and errors estimation have also been studied. In terms of numerical computations, several improvements have been made; first, the dependency problem has been highlighted in constructing numerical methods for the solutions of systems of FIVPs. Second, an efficient computational algorithm has been proposed in order to guarantee the validity of fuzzy solutions on the given interval, especially for nonlinear cases, where this issue had been largely neglected in the literature on numerically solving systems of FIVPs. The solving procedure reveals that the RKHS method is a straightforward, succinct, and promising tool for solving linear and nonlinear systems of FIVPs of ordinary types.

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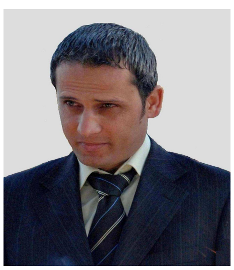

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