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Generalisation of Implication-Based Fuzzy Semiautomaton Over a Finite Group

M. Selva Rathi[1,*] *and J. Michael Anna Spinneli* [2]

[1] Department of Mathematics, Karunya University, Coimbatore, Tamil Nadu, India
[2] Department of Mathematics, L.R.G.Government Arts College for Women, Tirupur, Tamil Nadu, India

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Abstract: We have defined the concept of *implication-based T-fuzzy semiautomaton* (IB-T-FSA) over a finite group. The ideas of an *implication-based T-fuzzy kernel* and *implication-based T-fuzzy subsemiautomaton* of an IB-T-FSA over a finite group are introduced using the notion of *implication-based T-fuzzy subgroup* and *implication-based T-fuzzy normal subgroup*. The necessary and sufficient condition for an *implication-based T-fuzzy normal subgroup* to be an *implication-based T-fuzzy kernel* as well as the necessary and sufficient condition for an *implication-based T-fuzzy subgroup* to be an *implication-based T-fuzzy subsemiautomaton* of an IB-T-FSA are proved in this paper.

Keywords: Implication-based T-fuzzy subgroup, Implication-based T-fuzzy semiautomaton, Implication-based T-fuzzy kernel, Implication-based T-fuzzy subsemiautomaton

1 Introduction

In 1965, the concept of *fuzzy* set was first put forth by Zadeh [1]. Later in 1969 the notion of *fuzzy automata* was developed by Wee [2]. In 1971, Rosenfeld [3] applied the concept of fuzzy sets introduced by Zadeh to groups and constituted the elementary theory of groupoids and groups. From then on many research works had been carried out on their algebraic structures. Several studies had also been made on the *fuzzy normal subgroup* by [4, 5] and [6]. Anthony and Sherwood [7] redefined *fuzzy* subgroups using t-norm.

Hofer [8] and Fong [9] had further documented an intensive work on group semiautomata intensively. Malik et al. [10,11,12] had made an elaborate attempt in researching about fuzzy finite state machine and its subsystems. The notion of group semiautomaton was fuzzified by P. Das [13] and he had established the concept of *fuzzy semiautomaton* over a finite group. A *fuzzy semiautomaton* (FSA) over a finite group $(Q,+)$ is a triple (Q,X,μ) where X is a finite set and μ is a *fuzzy* subset of $Q\times X\times Q$. He defined fuzzy kernel and fuzzy subsemiautomaton of a fuzzy semiautomaton over a finite group. Youn-Hee Kim [14] proposed T-generalised state machines and T-generalised transformation semigroups and investigated their algebraic structures. Sung-Jin Cho [15] established T-fuzzy semiautomata over finite groups. Yuan [16] proposed the definitions of fuzzy subgroup with thresholds of a group and an implication-based fuzzy subgroup of a group. He also evolved the relation between them.

We [17,18] made a use of the implication-based fuzzy subgroup of a group and extended the prior work by consequently formulating the concept of implication-based fuzzy normal subgroup of a finite group and implication-based fuzzy semiautomaton over a finite group which enabled in probing their properties. Further t-norm was applied to the implication-based fuzzy subgroup and implication-based T-fuzzy subgroup of a finite group was developed. In this paper, *implication-based T-fuzzy semiautomaton* (IB-T-FSA) over a finite group has been defined. The ideas of *implication-based T-fuzzy kernel* and *implication-based T-fuzzy subsemiautomaton* of an IB-T-FSA have been formulated using the notions of *implication-based T-fuzzy subgroup* and *implication-based T-fuzzy normal subgroup* of a finite group. We also have proved that the product of an *implication-based T-fuzzy kernel* and an *implication-based T-fuzzy subsemiautomaton* of an IB-T-FSA is an *implication-based T-fuzzy*

* Corresponding author e-mail: selvarathi.maths@gmail.com

subsemiautomaton. Other basic results have also been proved.

2 Preliminaries

Definition 21. [3] Let $(G,\cdot)$ be a group. Let a fuzzy set in G be a function A from G to $[0,1]$. A will be called a *fuzzy subgroup* of G, if for all x,y in G,

$$A(xy) \geq min\big(A(x),A(y)\big)$$
$$A(x^{-1}) \geq A(x)$$

Let X be an universe of discourse and $(G,\cdot)$ be a group. In *fuzzy* logic, truth value of *fuzzy* proposition α is denoted by $[\alpha]$. The *fuzzy* logical and the corresponding set theoretical notations used in this paper are

$(x \in A) = A(x)$;
$(\alpha \wedge \beta) = min\{[\alpha],[\beta]\}$;
$(\alpha \rightarrow \beta) = min\{1, 1-[\alpha]+[\beta]\}$;
$(\forall x\ \alpha(x)) = inf_{x\in X}[\alpha(x)]$;
$(\exists x\ \alpha(x)) = sup_{x\in X}[\alpha(x)]$;
$\models \alpha$ if and only if $[\alpha] = 1$ for all valuations.

The truth valuation rules given above are those in Łukasiewicz system of continuous-valued logic.

Definition 22. [16] If a *fuzzy* subset A of a group G satisfies for any $x,y \in G$

(i)$\models (x \in A) \wedge (y \in A) \rightarrow (xy \in A)$
(ii)$\models (x \in A) \rightarrow (x^{-1} \in A)$

Then A is called a fuzzifying subgroup.

The concept of λ - tautology is $\models_\lambda \alpha$ if and only if $[\alpha] \geq \lambda$ for all valuation by Ying [19].

Definition 23. [16] Let A be a *fuzzy* subset of a finite group G and $\lambda \in (0,1]$ is a fixed number. If for any $x,y \in G$

(i)$\models_\lambda (x \in A) \wedge (y \in A) \rightarrow (xy \in A)$
(ii)$\models_\lambda (x \in A) \rightarrow (x^{-1} \in A)$

Then A is called an *implication-based fuzzy subgroup* of G.

Definition 24. [17] Let A be a *fuzzy* subset of a finite group G and $\lambda \in (0,1]$ be a fixed number. For any $x,y \in G$, if $\models_\lambda (x \in A) \wedge (y \in A) \rightarrow (xy \in A)$. Then A is called an *implication-based fuzzy subgroupoid* of G.

Definition 25. [17] Let A be an *implication-based fuzzy subgroup* of G, $\lambda \in (0,1]$ is a fixed number and $f: G \rightarrow G$ be a function defined on G. Then the *implication-based fuzzy subgroup* B of $f(G)$ is defined by $\models_\lambda \big(\exists x\{(x \in A)\}; x \in f^{-1}(y)\big) \rightarrow (y \in B)$, for all $y \in f(G)$ Similarly if B is an *implication-based fuzzy subgroup* of $f(G)$ then the *implication-based fuzzy subgroup* $A = f \circ B$ in G is defined as $\models_\lambda (f(x) \in B) \rightarrow (x \in A)$ for all $x \in G$ and is called the pre-image of B under f.

Definition 26. [17] An *implication-based fuzzy subgroup* A of G is called an *implication-based fuzzy normal subgroup* if $\models_\lambda (xy \in A) \rightarrow (yx \in A) \quad \forall x,y \in G$ where $\lambda \in (0,1]$ is a fixed number.

Theorem 21. *[17] Let $f: G \rightarrow G$ be an homomorphism of a finite group G. Let B be an* implication-based fuzzy normal subgroup *of $f(G)$. Then $A = f \circ B$ is an* implication-based fuzzy normal subgroup *of G.*

Definition 27. [18] Let A and B be two *implication-based fuzzy subgroups* of the finite group $(G,.)$, then the product $A \cdot B$ is defined by

$$\models_\lambda (\exists y,z\{(y \in A) \wedge (z \in B)\}; yz = x; y,z \in G) \rightarrow (x \in A \cdot B), \quad x \in G$$

Definition 28. [18] Let A and B be two *implication-based fuzzy subgroups* of a group G such that $A \leq B$. Then A is called an *implication-based fuzzy normal subgroup* of B if $\models_\lambda (y \in A) \wedge (x \in B) \rightarrow (xyx^{-1} \in A) \quad \forall x,y \in G$

The set of all logic variables be denoted by $\mathscr{V}$ and the set of all combination of these logic variables along with the **0** function by $\mathscr{V}^*$.

Definition 29. [18] Let A be an *implication-based fuzzy subgroup* of G. An *implication-based fuzzy semiautomaton(IBFSA)* over a finite group $(G,\cdot)$ is a triple $\mathscr{S} = (G,\mathscr{V},A)$ where $\mathscr{V}$ denotes the set of all logic variables. $(i.e.) A: G \times \mathscr{V} \times G \rightarrow [0,1]$

Define $A^*: G \times \mathscr{V}^* \times G \rightarrow [0,1]$ by

$$\models_\lambda ((u,\mathbf{0},v) \in A^*) \rightarrow 0\,(Here\ \lambda = 0)$$

$$\models_\lambda (\exists r\{((v,x,r) \in A^*) \wedge ((r,a,u) \in A)\}; r \in G) \rightarrow ((v,x \odot a,u) \in A^*)$$

$$\text{for all} \quad u,v \in G; x \in \mathscr{V}^*; a \in \mathscr{V}$$

Hereafter $\mathscr{S} = (G,\mathscr{V},A)$ be an *implication-based fuzzy semiautomaton (IBFSA)*.

Definition 210. [20] A triangular norm is a real continuous function $t: [0,1] \times [0,1] \rightarrow [0,1]$ fulfilling the following properties, for every $a,b,c,d \in [0,1]$

(i)$t(0,a) = 0,\ t(a,1) = a$ [boundary conditions],
(ii)$t(a,b) \leq t(c,d)$ if $a \leq c$ and $b \leq d$ [monotonicity],
(iii)$t(a,b) = t(b,a)$ [commutativity],
(iv)$t\,(t\,(a,b),c) = t\,(a,t(b,c))$ [associativity]

Lemma 22. *[21] Generalised Associative Law Let $T: [0,1] \times [0,1] \rightarrow [0,1]$ be a t-norm then $T(T(a,b),T(c,d)) = T(T(a,c),T(b,d)) \quad \forall a,b,c,d \in [0,1]$*

Definition 211. Let A be a *fuzzy* subset of G. For any $x,y \in G$ if

(i)$\models_\lambda (T((x \in A),(y \in A))) \rightarrow (xy \in A)$
(ii)$\models_\lambda (x \in A) \rightarrow (x^{-1} \in A)$

Then A is called an *implication-based T-fuzzy subgroup* of G.

Example for *implication-based T-fuzzy subgroup* of a finite group. Consider the group $G = \{e,a,b,c\}$ along with the binary operation $'*'$ whose closure table is as follows.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

For the *fuzzy set* $A : G \to [0,1]$ defined by $A(e) = 1, A(a) = .25, A(b) = .5, A(c) = .75$ with $\lambda = .2$ and the *implication* operator is that of Łukasiewicz, with the t-norm defined by $T(a,b) = ab$ we have

T	$[e]$	$[a]$	$[b]$	$[c]$
$[e]$	1	.25	.5	.75
$[a]$	.25	.0625	.125	.1875
$[b]$	.5	.125	.25	.375
$[c]$	.75	.1875	.375	.5625

Then A is an *implication-based T-fuzzy subgroup* of G.

Theorem 23. *Let f be a homomorphism of the group G and B be an implication-based T-fuzzy subgroup of $f(G)$ then $A = f \circ B$ is an implication-based T-fuzzy subgroup of G.*

Theorem 24. *Let A be an implication-based T-fuzzy subgroup of G and f be an homomorphism on G. Then B the image of A under f is also an implication-based T-fuzzy subgroup of f(G).*

Hereafter $(G,\cdot)$ is a group with $'e'$ as its identity element. And $\lambda \in (0,1]$ is a fixed number. $T : [0,1] \to [0,1]$ is a t-norm. Throughout this paper, we shall denote
$T(a_1, T(a_2, T(\ldots, T(a_{n-1}, a_n))))$ by $T(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n \in [0,1]$

3 Main Results

Definition 31. Let $\mathscr{V}$ denotes the set of all logic variables and $\mathscr{V}^*$, the set of all combination of these logic variables along with the $\mathbf{0}$ function. Let A be an *implication-based fuzzy subgroup* i.e. $A : G \times \mathscr{V} \times G \to [0,1]$ and let $\mathscr{S} = (G, \mathscr{V}, A)$ be an $IB-FSA$ over a finite group G.

Define $A^* : G \times \mathscr{V}^* \times G \to [0,1]$ by

$$\models_\lambda ((p,\mathbf{0},q) \in A^*) \to 0 \qquad (Here\ \lambda = 0)$$

$$\models_\lambda (\exists q_i\{T(((p,a_1,q_1) \in A),((q_1,a_2,q_2) \in A),\ldots,((q_{n-1},a_n,q) \in A))\}; q_i \in G, i = 1,2,\ldots,n-1) \to ((q, a_1 \odot a_2 \odot \ldots \odot a_n, p) \in A^*)$$

for all $p,q \in G$, $a_1, a_2, \ldots, a_n \in \mathscr{V}$.

$\mathscr{S}$ is called an *Implication-Based T-Fuzzy Semiautomaton* over a finite group G which is the generalisation of the *implication-based fuzzy semiautomaton* over a finite group. Hereafter *Implication-Based T-Fuzzy Semiautomaton* over a group G shall be denoted as $IB-T-FSA$.

Definition 32. A fuzzy subset B of G is called as an *implication-based-T-fuzzy kernel* of an $IB-T-FSA$, $\mathscr{S} = (G, \mathscr{V}, A)$ if for all $p,q,r,k \in G$ and $x \in \mathscr{V}$

(i)B is an *implication-based T-fuzzy normal subgroup* of G.

(ii)$\models_\lambda (T(((qk,x,p) \in A),((q,x,r) \in A),(k \in B))) \to (pr^{-1} \in B)$

Definition 33. A fuzzy subset B of G is called as *implication-based-T-fuzzy subsemiautomaton* of an $IB-T-FSA$, $\mathscr{S} = (G, \mathscr{V}, A)$ if for all $p,q \in G$ and $x \in \mathscr{V}$

(i)B is an *implication-based T-fuzzy subgroup* of G.

(ii)$\models_\lambda (T((q,x,p) \in A),(q \in B))) \to (p \in B)$

Definition 34. Let $\mathscr{S}_1 = (G_1, \mathscr{V}_1, A_1)$ and $\mathscr{S}_2 = (G_2, \mathscr{V}_2, A_2)$ be two $IB-T-FSA$ over finite groups G_1 and G_2 respectively. A pair of mappings $< f,g >$ where $f : G_1 \to G_2$ and $g : \mathscr{V}_1 \to \mathscr{V}_2$ is called an *homomorphism* from $\mathscr{S}_1$ to $\mathscr{S}_2$ and is written as $< f,g >: \mathscr{S}_1 \to \mathscr{S}_2$ if

(i)$f : G_1 \to G_2$ is a group homomorphism.

(ii)$\models_\lambda ((p,x,q) \in A_1) \to ((f(p),g(x),f(q)) \in A_2) \quad \forall p,q \in G_1, x \in \mathscr{V}_1$

The homomorphism $< f,g >$ is called an isomorphism if f and g are both one-to-one and onto.

Definition 35. The pair $< f,g >$ is called a *strong homomorphism* from $\mathscr{S}_1$ to $\mathscr{S}_2$ if
$\forall p,q \in G_1 ; x \in \mathscr{V}_1$

(i)$f : G_1 \to G_2$ is a group homomorphism.

(ii)$\models_\lambda ((f(p),g(x),f(q)) \in A_2) \to (\exists s,t\{((s,y,t) \in A_1)\}; f(s) = f(p), f(t) = f(q), g(y) = g(x); s,t \in G_1, y \in \mathscr{V}_1)$

Theorem 31. *Let $\mathscr{S} = (G, \mathscr{V}, A)$ be an IB-T-FSA then* $\models_\lambda (\exists r\{T(((p,x,r) \in A^*),((r,y,q) \in A^*))\}; r \in G) \to ((p, x \odot y, q) \in A^*)$ *for all $p,q \in G$ and $x,y \in \mathscr{V}^*$*

*Proof.*Let $p,q \in G$
Case (i) : If $x = \mathbf{0}$ or $y = \mathbf{0}$ then
$\models_\lambda ((p, x \odot y, q) \in A^*) \to ((p,\mathbf{0},q) \in A^*) \to 0$
And if $x = \mathbf{0}$
$\models_\lambda (\exists r\{T(((p,x,r) \in A^*),((r,y,q) \in A^*))\} r \in G)$
$\to (\exists r\{T(((p,\mathbf{0},r) \in A^*),((r,y,q) \in A^*))\} r \in G)$
$\to (\exists r\{T(0,((r,y,q) \in A^*))\} r \in G)$
$\to 0$ here $\lambda = 0$
Therefore
$\models_\lambda (\exists r\{T(((p,x,r) \in A^*),((r,y,q) \in A^*))\}; r \in G) \to ((p, x \odot y, q) \in A^*)$
Case (ii) : Let $x = a_1 \odot a_2 \odot \ldots \odot a_n$ and $y = b_1 \odot b_2 \odot \ldots \odot b_m$
$\models_\lambda (\exists r\{T(((p,x,r) \in A^*),((r,y,q) \in A^*))\}; r \in G)$
$\to (\exists r\{T(((p, a_1 \odot a_2 \odot \ldots \odot a_n, r) \in A^*),$

$((r,b_1 \odot b_2 \odot \ldots \odot b_m,q) \in A^*))\};r \in G)$
$\rightarrow (\exists r\{T((\exists q_i\{T(((p,a_1,q_1) \in A),((q_1,a_2,q_2) \in A),$
$((q_{n-1},a_n,r) \in A)\};q_i \in G,\ i=1,2,\ldots n-1)$
$\ldots,(\exists q_j\{T(((r,b_1,q_n) \in A),((q_n,b_2,q_{n+1}) \in A),$
$\ldots,((q_{n+m-1},b_m,q) \in A))\};q_j \in G,$
$j=n,n+1,\ldots n+m-1))\};r \in G)$
$\rightarrow (\exists q_i,r\{T(((p,a_1,q_1) \in A),((q_1,a_2,q_2) \in A),$
$\ldots,((q_{n-1},a_n,r) \in A),((r,b_1,q_n) \in A),$
$((q_n,b_2,q_{n+1}) \in A),\ldots,((q_{n+m-1},b_m,q) \in A))\};$
$q_i,r \in G, i=1,2,\ldots n+m-1)$
$\rightarrow ((p,a_1 \odot a_2 \odot \ldots \odot a_n \odot b_1 \odot b_2 \odot \ldots \odot b_m,q) \in A^*)$
$\rightarrow ((p,x \odot y,q) \in A^*)$ □

Theorem 32. *Let $\mathscr{S}_1=(G_1,\mathscr{V}_1,A_1)$, $\mathscr{S}_2=(G_2,\mathscr{V}_2,A_2)$ and $\mathscr{S}_3=(G_3,\mathscr{V}_3,A_3)$ be $IB-T-FSA$ over finite groups G_1,G_2,G_3 respectively. (i) If $<f,g>: \mathscr{S}_1 \rightarrow \mathscr{S}_2$ and $<f',g'>: \mathscr{S}_2 \rightarrow \mathscr{S}_3$ are homomorphisms. Then $<f' \circ f, g' \circ g>: \mathscr{S}_1 \rightarrow \mathscr{S}_3$ is a homomorphism. (ii) If f and g are onto and also if $<f,g>$ and $<f',g'>$ are strong homomorphism then $<f' \circ f, g' \circ g>$ is also a strong homomorphism.*

Proof.(i) Clearly $f' \circ f: G_1 \rightarrow G_3$ and $g' \circ g: \mathscr{V}_1 \rightarrow \mathscr{V}_3$.
Since f and f' are group homomorphisms.
$\Rightarrow f' \circ f$ is also a group homomorphism.
Let $p,q \in G_1$ and $x \in \mathscr{V}_1$
$\vDash_\lambda ((p,x,q) \in A_1)$
$\rightarrow ((f(p),g(x),f(q)) \in A_2)$
$\rightarrow ((f'(f(p)),g'(g(x)),f'(f(q))) \in A_3)$
$\rightarrow (((f' \circ f)(p),(g' \circ g)(x),(f' \circ f)(q)) \in A_3)$
Therefore $<f' \circ f, g' \circ g>: \mathscr{S}_1 \rightarrow \mathscr{S}_3$ is a homomorphism.
(ii)Let $<f,g>$ and $<f',g'>$ be strong homomorphisms.
Clearly $f' \circ f$ is also a group homomorphism.
$\vDash_\lambda (((f' \circ f)(p),(g' \circ g)(x),(f' \circ f)(q)) \in A_3)$
$\rightarrow ((f'(f(p)),g'(g(x)),f'(f(q))) \in A_3)$
Since $<f',g'>$ is a strong homomorphism.
$\rightarrow \Big(\exists u,v\{((u,y,v) \in A_2)\}; f'(u)=f'(f(p)),$
$f'(v)=f'(f(q)),g'(y)=g'(g(x));u,v \in G_2,y \in \mathscr{V}_2\Big)$
Since f and g are onto, there exists $r,s \in G_1$ and $z \in \mathscr{V}_1$ such that $f(r)=u$, $f(s)=v$ and $g(z)=y$.
$\rightarrow \Big(\exists r,s\{((f(r),g(z),f(s)) \in A_2)\}; f'(f(r))=f'(f(p)),$
$f'(f(s))=f'(f(q)),g'(g(z))=g'(g(x));r,s \in G_1,z \in \mathscr{V}_1\Big)$
Since $<f,g>$ is a strong homomorphism.
$\rightarrow (\exists r,s\{\exists a,b\{((a,t,b) \in A_1)\}; f(a)=f(r),$
$f(b)=f(s),g(t)=g(z);a,b \in G_1,t \in \mathscr{V}_1\};$
$f'(f(r))=f'(f(p)),f'(f(s))=f'(f(q)),$
$g'(g(z))=g'(g(x));r,s \in G_1,z \in \mathscr{V}_1\Big)$
$\rightarrow \Big(\exists a,b\{((a,t,b) \in A_1)\}; f'(f(a))=f'(f(p)),$
$f'(f(b))=f'(f(q)),g'(g(t))=g'(g(x));$
$a,b \in G_1,t \in \mathscr{V}_1\Big)$
$<f' \circ f, g' \circ g>$ is a strong homomorphism. □

Lemma 33. *Let $<f,g>: \mathscr{S}_1 \rightarrow \mathscr{S}_2$ be a homomorphism where $\mathscr{S}_1=(G_1,\mathscr{V}_1,A_1)$ and $\mathscr{S}_2=(G_2,\mathscr{V}_2,A_2)$ are $IB-T-FSA$ over the finite groups G_1 and G_2 respectively. Define $g^*: \mathscr{V}_1{}^* \rightarrow \mathscr{V}_2{}^*$ such that $g^*(\mathbf{0})=\mathbf{0}'$ and $g^*(a_1 \odot a_2 \odot \ldots \odot a_n)=g(a_1) \odot g(a_2) \odot \ldots \odot g(a_n)$ where $a_1,a_2,\ldots,a_n \in \mathscr{V}_1$ then $g^*(x \odot y)=g^*(x) \odot g^*(y)$ for all $x,y \in \mathscr{V}_1{}^*$*

*Proof.*Let $x,y \in \mathscr{V}_1{}^*$ such that
$x=a_1 \odot a_2 \odot \ldots \odot a_n$ and $y=b_1 \odot b_2 \odot \ldots \odot b_m$
where $a_i,b_j \in \mathscr{V}_1$ and $i=1,2,\ldots,n$; $j=1,2,\ldots,m$
By the definition of g^* we have
$g^*(x)=g^*(a_1 \odot a_2 \odot \ldots \odot a_n)$
$=g(a_1) \odot g(a_2) \odot \ldots \odot g(a_n)$
$g^*(y)=g^*(b_1 \odot b_2 \odot \ldots \odot b_m)$
$=g(b_1) \odot g(b_2) \odot \ldots \odot g(b_m)$

$$\begin{aligned} g^*(x \odot y) &= g^*(a_1 \odot a_2 \odot \ldots \odot a_n \odot b_1 \odot b_2 \odot \ldots \odot b_m) \\ &= [g(a_1) \odot g(a_2) \odot \ldots \odot g(a_n)] \odot \\ &\qquad [g(b_1) \odot g(b_2) \odot \ldots \odot g(b_m)] \\ &= g^*(x) \odot g^*(y) \end{aligned}$$

□

Theorem 34. *Let $\mathscr{S}_1=(G_1,\mathscr{V}_1,A_1)$ and $\mathscr{S}_2=(G_2,\mathscr{V}_2,A_2)$ be $IB-T-FSA$ over the finite groups G_1 and G_2 respectively. Let $<f,g>: \mathscr{S}_1 \rightarrow \mathscr{S}_2$ be a strong homomorphism. If f is one-to-one and onto, g is one-to-one then $\vDash_\lambda ((f(p),g^*(x),f(q)) \in A_2{}^*) \rightarrow ((p,x,q) \in A_1{}^*)$ for all $p,q \in G_1$, $x \in \mathscr{V}_1{}^*$*

*Proof.*Given f is one-to-one and onto, g is one-to-one.
Case(i): If $x=\mathbf{0}$ and let $p,q \in G_1$. Here $\lambda=0$
$\vDash_\lambda ((f(p),g^*(\mathbf{0}),f(q)) \in A_2{}^*)$
$\rightarrow ((f(p),\mathbf{0}',f(q)) \in A_2{}^*) \rightarrow 0$
Also $\vDash_\lambda ((p,x,q) \in A_1{}^*) \rightarrow ((p,\mathbf{0},q) \in A_1{}^*) \rightarrow 0$
Therefore
$\vDash_\lambda ((f(p),g^*(x),f(q)) \in A_2{}^*) \rightarrow ((p,x,q) \in A_1{}^*)$ if $x=\mathbf{0}$
Case (ii) Let $x=a_1 \odot a_2 \odot \ldots \odot a_n$ where $a_1,a_2,\ldots,a_n \in \mathscr{V}_1$ then
$\vDash_\lambda ((f(p),g^*(x),f(q)) \in A_2{}^*)$
$\rightarrow ((f(p),g^*(a_1 \odot a_2 \odot \ldots \odot a_n),f(q)) \in A_2{}^*)$
$\rightarrow ((f(p),g(a_1) \odot g(a_2) \odot \ldots \odot g(a_n),f(q)) \in A_2{}^*)$
$\rightarrow (\exists r_i\{T(((f(p),g(a_1),r_1) \in A_2),((r_1,g(a_2),r_2) \in A_2),$
$\ldots,((r_{n-1},g(a_n),f(q)) \in A_2))\};r_i \in G_2,i=1,2,\ldots n-1)$
$\rightarrow (\exists q_i\{T(((f(p),g(a_1),f(q_1)) \in A_2),$
$((f(q_1),g(a_2),f(q_2)) \in A_2),\ldots,$
$((f(q_{n-1}),g(a_n),f(q)) \in A_2))\};q_i \in G_1,$
$i=1,2,\ldots n-1)$
Since f is onto
$\rightarrow (\exists q_i\{T(((p,a_1,q_1) \in A_1),((q_1,a_2,q_2) \in A_1),$
$\ldots((q_{n-1},a_n,q) \in A_1))\};q_i \in G_1,$
$i=1,2,\ldots n-1)$
Since f,g are one-to-one and $<f,g>$ is a strong homomorphism.
$\rightarrow ((p,a_1 \odot a_2 \odot \ldots \odot a_n,q) \in A_1{}^*)$
$\rightarrow ((p,x,q) \in A_1{}^*)$ □

Theorem 35. *Let $\mathscr{S}_1 = (G_1, \mathscr{V}_1, A_1)$ and $\mathscr{S}_2 = (G_2, \mathscr{V}_2, A_2)$ be $IB-T-FSA$ over the finite groups G_1 and G_2 respectively. Let $f : \mathscr{S}_1 \to \mathscr{S}_2$ be a homomorphism then (i) If B is an* implication-based T-fuzzy normal subgroup *of $\mathscr{S}_2$ then $A = f \circ B$ is also an* implication-based T-fuzzy normal subgroup *of $\mathscr{S}_1$. (ii)If f is onto and B is an* implication-based T-fuzzy normal subgroup *of $\mathscr{S}_1$, then C the image of B in $f(G)$ is an* implication-based T-fuzzy normal subgroup *of $\mathscr{S}_2$.*

*Proof.*By theorem 23 and theorem 24 we have that the image and pre-image of an *implication-based T-fuzzy subgroup* of a finite group is also an *implication-based T-fuzzy subgroup*. And by theorem 21, pre-image of an *implication-based fuzzy normal subgroup* is an *implication-based fuzzy normal subgroup*.
It is enough to prove that the image of an *implication-based fuzzy normal subgroup* is an *implication-based fuzzy normal subgroup*.
Let B be an *implication-based T-fuzzy normal subgroup* of $\mathscr{S}_1$.
Let $x, y \in C$

$$\begin{aligned}\models_\lambda (xy \in C) &\to (f(p)f(q) \in C) \text{ since f is onto}\\ &\to (f(pq) \in C) \text{ f is an homomorphism}\\ &\to (pq \in B)\\ &\to (qp \in B)\\ &\text{since B is an implication-based}\\ &\qquad \text{fuzzy normal subgroup}\\ &\to (f(qp) \in C)\\ &\to (f(q)f(p) \in C)\\ &\to (yx \in C)\end{aligned}$$

Therefore C the image of B in $f(G)$ is an *implication-based T-fuzzy normal subgroup* of $\mathscr{S}_2$. □

Theorem 36. *Let $\mathscr{S} = (G, \mathscr{V}, A)$ be an IB-T-FSA over a finite group G. Let B be an* implication-based T-fuzzy normal subgroup *of G. Then B is an* implication-based T-fuzzy kernel *of $\mathscr{S}$ if and only if $\models_\lambda (T(((qk,x,p) \in A^*), ((q,x,r) \in A^*), (k \in B))) \to (pr^{-1} \in B)$ for all $p,q,r,k \in G$ and $x \in \mathscr{V}^*$.*

*Proof.*Let B be an *implication-based T-fuzzy kernel* of $\mathscr{S}$.
The theorem is proved by the method of induction on $ord(x) = n$.
Case(i): Let $p,q,r,k \in G$ and $x = \mathbf{0}$
$\models_\lambda (T(((qk,\mathbf{0},p) \in A^*), ((q,\mathbf{0},r) \in A^*), (k \in B)))$
$\to (T(0,0,(k \in B)))$
$\to 0$
$\to (pr^{-1} \in B)$ here $\lambda = 0$
Therefore the result holds for $n = 0$.
Case(ii): Assume that the result hold for all $x \in \mathscr{V}^*$ such that $ord(x) = n-1, n > 0$. Now let $x = x_1 \odot x_2 \odot \ldots \odot x_n$ where $x_1, x_2, \ldots, x_n \in \mathscr{V}$.

$$\begin{aligned}&\models_\lambda (T(((qk,x,p) \in A^*), ((q,x,r) \in A^*), (k \in B)))\\ &\to (T(((qk,x_1 \odot x_2 \odot \ldots \odot x_n, p) \in A^*),\\ &\quad ((q,x_1 \odot x_2 \odot \ldots \odot x_n, r) \in A^*), (k \in B)))\\ &\to (T((\exists u_i\{(T((qk,x_1,u_1) \in A), ((u_1,x_2,u_2) \in A),\\ &\quad \ldots((u_{n-1},x_n,p) \in A))\}; u_i \in G, i = 1,2,\ldots,n-1),\\ &\quad (\exists v_i\{T(((q,x_1,v_1) \in A), ((v_1,x_2,v_2) \in A), \ldots,\\ &\quad ((v_{n-1},x_n,r) \in A))\}; v_i \in G, i = 1,2,\ldots n-1), (k \in B)))\\ &\to (T(\exists v_i,u_i\{T(((qk,x_1,u_1) \in A), ((u_1,x_2,u_2) \in A),\\ &\quad \ldots,((u_{n-1},x_n,p) \in A), ((q,x_1,v_1) \in A),\\ &\quad ((v_1,x_2,v_2) \in A), \ldots, ((v_{n-1},x_n,r) \in A), (k \in B))\};\\ &\quad u_i,v_i \in G, i = 1,2,\ldots n-1))\\ &\to (\exists v_i,u_i\{T(((qk,x_1,u_1) \in A), ((q,x_1,v_1) \in A), (k \in B),\\ &\quad ((u_1,x_2,u_2) \in A), \ldots, ((u_{n-1},x_n,p) \in A),\\ &\quad ((v_1,x_2,v_2) \in A), \ldots, ((v_{n-1},x_n,r) \in A))\};\\ &\quad u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\to (\exists v_i,u_i\{T(((u_1 {v_1}^{-1}) \in B), ((u_1,x_2,u_2) \in A), \ldots,\\ &\quad ((u_{n-1},x_n,p) \in A), ((v_1,x_2,v_2) \in A), \ldots,\\ &\quad ((v_{n-1},x_n,r) \in A))\}; u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\to (\exists v_i,u_i\{T(((u_1,x_2,u_2) \in A), ((v_1,x_2,v_2) \in A),\\ &\quad (u_1 {v_1}^{-1}) \in B), ((u_2,x_3,u_3) \in A), \ldots, ((u_{n-1},x_n,p) \in A),\\ &\quad ((v_2,x_3,v_3) \in A), \ldots, ((v_{n-1},x_n,r) \in A))\};\\ &\quad u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\to (\exists v_i,u_i\{T((u_2 {v_2}^{-1}) \in B), ((u_2,x_3,u_3) \in A), \ldots,\\ &\quad ((u_{n-1},x_n,p) \in A), ((v_2,x_3,v_3) \in A), \ldots,\\ &\quad ((v_{n-1},x_n,r) \in A))\}; u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\vdots\\ &\to (\exists v_i,u_i\{T((u_{n-1} {v_{n-1}}^{-1}) \in B), ((u_{n-1},x_n,p) \in A),\\ &\quad ((v_{n-1},x_n,r) \in A))\}; u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\to (\exists v_i,u_i\{T((u_{n-1} {v_{n-1}}^{-1}) \in B),\\ &\quad ((v_{n-1}.{v_{n-1}}^{-1} u_{n-1}, x_n, p) \in A),\\ &\quad ((v_{n-1},x_n,r) \in A))\}; u_i,v_i \in G, i = 1,2,\ldots n-1)\\ &\to (\exists v_i,u_i\{T(((v_{n-1}.{v_{n-1}}^{-1} u_{n-1}, x_n, p) \in A),\\ &\quad ((v_{n-1},x_n,r) \in A), ({v_{n-1}}^{-1} u_{n-1}) \in B))\};\\ &\quad u_i,v_i \in G, i = 1,2,\ldots n-1)\end{aligned}$$

since B is an implication-based T - fuzzy normal subgroup
$\to (pr^{-1} \in B)$
The converse is trivial. □

Theorem 37. *Let $\mathscr{S} = (G, \mathscr{V}, A)$ be an IB-T-FSA over a finite group G. Let B be an* implication based-T-fuzzy subgroup *of G. Then B is an* implication-based T-fuzzy subsemiautomaton *of $\mathscr{S}$ if and only if $\models_\lambda (T(((q,x,p) \in A^*), (q \in B))) \to (p \in B)$ for all $p,q \in G$ and $x \in \mathscr{V}^*$.*

*Proof.*Let B be an *implication-based T-fuzzy subsemiautomaton* of $\mathscr{S}$.
We prove the theorem by the method of induction on $ord(x) = n$.
Case(i): Let $p,q \in G$ and $x = \mathbf{0}$
$\models_\lambda (T(((q,\mathbf{0},p) \in A^*), (q \in B)))$
$\to (T(0,(q \in B)))$
$\to 0$
$\to (p \in B)$ here $\lambda = 0$
Case(ii): Assume that the result hold for all $x \in \mathscr{V}^*$ such that $ord(x) = n-1$ and $n > 0$. Now let $x = x_1 \odot x_2 \odot \ldots \odot x_n$ where $x_1, x_2, \ldots, x_n \in \mathscr{V}$.
$\models_\lambda (T(((q,x,p) \in A^*), (q \in B)))$

$\rightarrow (T(((q,x_1 \odot x_2 \odot \ldots \odot x_n, p) \in A^*),(q \in B)))$
$\rightarrow (\exists u_i\{T(((q,x_1,u_1) \in A),((u_1,x_2,u_2) \in A),\ldots, ((u_{n-1},x_n,p) \in A),(q \in B))\};$
$u_i \in G, i=1,2,\ldots n-1)$
$\rightarrow (\exists u_i\{T(((q,x_1,u_1) \in A),(q \in B),((u_1,x_2,u_2) \in A), \ldots,((u_{n-1},x_n,p) \in A))\};$
$u_i \in G, i=1,2,\ldots n-1)$
$\rightarrow (\exists u_i\{T(((u_1 \in B),((u_1,x_2,u_2) \in A),\ldots, ((u_{n-1},x_n,p) \in A))\}; u_i \in G, i=1,2,\ldots n-1)$
$\vdots$
$\rightarrow (\exists u_i\{T(((u_{n-1} \in B),((u_{n-1},x_n,p) \in A))\};$
$u_i \in G, i=1,2,\ldots n-1)$
$\rightarrow (\exists u_i\{T((((u_{n-1},x_n,p) \in A),(u_{n-1} \in B))\};$
$u_i \in G, i=1,2,\ldots n-1)$
$\rightarrow (p \in B)$
The converse is trivial. □

Theorem 38. *Let $\mathscr{S}_1 = (G_1,\mathscr{V}_1,A_1)$ and $\mathscr{S}_2 = (G_2,\mathscr{V}_2,A_2)$ be $IB-T-FSA$ over the finite groups G_1 and G_2 respectively. Let $< f,g >: \mathscr{S}_1 \rightarrow \mathscr{S}_2$ be an homomorphism. Then (i) If B is an* implication-based T-fuzzy kernel *of $\mathscr{S}_2$ then $A = f \circ B$ is an* implication-based T-fuzzy kernel *of $\mathscr{S}_1$. (ii) If B is an* implication-based T-fuzzy subsemiautomaton *of $\mathscr{S}_2$ then $A = f \circ B$ is an* implication-based T-fuzzy subsemiautomaton *of $\mathscr{S}_1$.*

Proof.(i) Let B be an *implication-based T-fuzzy kernel* of $\mathscr{S}_2$ then $A = f \circ B$ is an *implication-based T-fuzzy normal subgroup* of $\mathscr{S}_1$ by theorem 35.
Let $p_1,q_1,r_1,k_1 \in G_1$ and $x_1 \in \mathscr{V}_1$
Now $< f,g >: \mathscr{S}_1 \rightarrow \mathscr{S}_2$ is an homomorphism. Therefore $f(p_1) = p_2$, $f(q_1) = q_2$, $f(r_1) = r_2$, $f(k_1) = k_2$ and $g(x_1) = x_2$ where $p_2,q_2,r_2,k_2 \in G_2$ and $x_2 \in \mathscr{V}_2$.
$\models_\lambda (T(((q_1k_1,x_1,p_1) \in A_1),((q_1,x_1,r_1) \in A_1),(k_1 \in A)))$
$\rightarrow (T(((q_2k_2,x_2,p_2) \in A_2),((q_2,x_2,r_2) \in A_2),(k_2 \in B)))$
$\rightarrow (p_2{r_2}^{-1} \in B)$
since B is an implication-based T-fuzzy kernel
$\rightarrow (f(p_1)f({r_1}^{-1}) \in B)$
$\rightarrow (f(p_1{r_1}^{-1}) \in B)$
$\rightarrow (p_1{r_1}^{-1} \in A)$
Therefore $A = f \circ B$ is an *implication-based T-fuzzy kernel* of $\mathscr{S}_1$.
(ii)Let B be an *implication-based T-fuzzy subsemiautomaton* of $\mathscr{S}_2$ then $A = f \circ B$ is an *implication-based T-fuzzy subgroup* of $\mathscr{S}_1$ by theorem 23.
Let $p_1,q_1 \in G_1$ and $x_1 \in \mathscr{V}_1$
$\models_\lambda (T(((q_1,x_1,p_1) \in A_1),(q_1 \in A)))$
$\rightarrow (T(((q_2,x_2,p_2) \in A_2),(q_2 \in B)))$
$\rightarrow (p_2 \in B)$
since B is an implication-based T-fuzzy subsemiautomaton
$\rightarrow (f(p_1) \in B)$
$\rightarrow (p_1 \in A)$
Therefore $A = f \circ B$ is an *implication-based T-fuzzy subsemiautomaton* of $\mathscr{S}_1$. □

Theorem 39. *Let $\mathscr{S}_1 = (G_1,\mathscr{V},A_1)$ and $\mathscr{S}_2 = (G_2,\mathscr{V},A_2)$ be $IB-T-FSA$ over the finite groups G_1 and G_2 respectively and let f be an onto strong homomorphism from $\mathscr{S}_1$ to $\mathscr{S}_2$. If B is an* implication-based T-fuzzy subsemiautomaton *of $\mathscr{S}_1$, then C the image of B in $f(G)$ is an* implication-based T-fuzzy subsemiautomaton *of $\mathscr{S}_2$.*

*Proof.*Let B is an *implication-based T-fuzzy subsemiautomaton* of $\mathscr{S}_1$. Therefore B is an *implication-based T-fuzzy subgroup* of G_1. Let $p,p_2,q_2 \in G_2$. and $x \in \mathscr{V}$
$\models_\lambda (T((p_2 \in C),(q_2 \in C)))$
$\rightarrow (T((\exists p_1\{(p_1 \in B)\}; f(p_1) = p_2, p_1 \in G_1), (\exists q_1\{(q_1 \in B)\}; f(q_1) = q_2, q_1 \in G_1)))$
$\rightarrow (\exists p_1,q_1\{T((p_1 \in B),(q_1 \in B))\}; f(p_1) = p_2, f(q_1) = q_2, p_1,q_1 \in G_1)$
$\rightarrow (\exists p_1,q_1\{(p_1q_1 \in B\}; f(p_1) = p_2, f(q_1) = q_2, p_1,q_1 \in G_1)$
Since B is an *implication-based T-fuzzy subgroup* of G_1.
$\rightarrow (\exists z\{(z \in B)\}; f(z) = f(p_1q_1) = f(p_1)f(q_1) = p_2q_2, z \in G_1)$
$\rightarrow (p_2q_2 \in C)$
Therefore $\models_\lambda (T((p_2 \in C),(q_2 \in C))) \rightarrow (p_2q_2 \in C)$
Now $\models_\lambda (p^{-1} \in C)$
$\rightarrow \left(\exists q\{(q \in B)\}; f(q) = p^{-1}, q \in G_1\right)$
$\rightarrow \left(\exists q\{(q \in B)\}; f(q^{-1}) = p, q \in G_1\right)$
$\rightarrow \left(\exists q\{(q^{-1} \in B)\}; f(q^{-1}) = p, q \in G_1\right)$
Since B is an *implication-based T-fuzzy subgroup* of G_1.
$\rightarrow \exists r\{(r \in B)\}; f(r) = p, r \in G_1$
$\rightarrow (p \in C)$
Therefore $\models_\lambda (p^{-1} \in C) \rightarrow (p \in C)$.
Similarly we can prove that $\models_\lambda (p \in C) \rightarrow (p^{-1} \in C)$.
Therefore C is an *implication-based T-fuzzy subgroup* of G_2.
$\models_\lambda (T(((q_2,x,p_2) \in A_2),(q_2 \in C)))$
$\rightarrow (T(((q_2,x,p_2) \in A_2),(\exists q_1\{(q_1 \in B)\}; f(q_1) = q_2, q_1 \in G_1)))$
$\rightarrow (\exists q_1\{T(((q_2,x,p_2) \in A_2),(q_1 \in B))\}; f(q_1) = q_2, q_1 \in G_1)$
Let $p_1,q_1 \in G_1$ be such that $f(p_1) = p_2$ and $f(q_1) = q_2$
Now $\models_\lambda (T(((q_2,x,p_2) \in A_2),(q_1 \in B)))$
$\rightarrow (T(((f(q_1),x,f(p_1)) \in A_2),(q_1 \in B)))$
$\rightarrow (T((\exists u,v\{((u,x,v) \in A_1)\}; f(u) = f(q_1) = q_2, f(v) = f(p_1) = p_2, u,v \in G_1),(q_1 \in B)))$
$\rightarrow (\exists u,v\{T(((u,x,v) \in A_1),(q_1 \in B))\}; f(u) = f(q_1) = q_2, f(v) = f(p_1) = p_2, u,v \in G_1)$
$\rightarrow (\exists u,v\{T(((u,x,v) \in A_1),(u \in B))\}; f(u) = f(q_1) = q_2, f(v) = f(p_1) = p_2, u,v \in G_1)$
$\rightarrow (\exists v\{(v \in B)\}; f(v) = p_2, v \in G_1)$
Since B is an *implication-based T-fuzzy subsemiautomaton* of G_1.
$\rightarrow (p_2 \in C)$
Hence $\models_\lambda (T(((q_2,x,p_2) \in A_2),(q_2 \in C)))$
$\rightarrow (\exists q_1\{(p_2 \in C)\}; f(q_1) = q_2, q_1 \in G_1)$
$\rightarrow (p_2 \in C)$
Therefore C is an *implication-based T-fuzzy subsemiautomaton* of $\mathscr{S}_2$. □

Theorem 310. *Let $\mathscr{S}_1 = (G_1, \mathscr{V}, A_1)$ and $\mathscr{S}_2 = (G_2, \mathscr{V}, A_2)$ be $IB-T-FSA$ over the finite groups G_1 and G_2 respectively and let f be an onto strong homomorphism from $\mathscr{S}_1$ to $\mathscr{S}_2$ satisfying the following conditions. If $\models_\lambda ((f(qk), x, f(p)) \in A_2) \rightarrow ((q'k', x, p') \in A_1)$ and $\models_\lambda ((f(q), x, f(r)) \in A_2) \rightarrow ((q'', x, r') \in A_1)$ then $q'k' = q''k$ where $p, p', q, q', q'', r, r', k, k' \in G_1$ and $x \in \mathscr{V}$. If B is an* implication-based T-fuzzy kernel *of $\mathscr{S}_1$, then C the image of B is also an* implication-based T-fuzzy kernel *in $\mathscr{S}_2$*

*Proof.*Let B be an *implication-based T-fuzzy kernel* of $\mathscr{S}_1$.
By theorem 39, C the image of B is an *implication-based T-fuzzy subgroup* of G_2.
By theorem 35, C is an *implication-based T-fuzzy normal subgroup* of G_2.
Let $p_2, q_2, r_2, k_2 \in G_2$ and $x \in \mathscr{V}$.

$$\models_\lambda (T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (k_2 \in C)))$$
$$\rightarrow (T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (\exists k_1\{(k_1 \in B)\}; f(k_1) = k_2, k_1 \in G_1)))$$
$$\rightarrow (\exists k_1\{T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (k_1 \in B)\}; f(k_1) = k_2, k_1 \in G_1)))$$

Now let $p_1, q_1, r_1 \in G_1$ be such that $f(p_1) = p_2$, $f(q_1) = q_2$, $f(r_1) = r_2$.

$$\models_\lambda (T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (k_1 \in B)))$$
$$\rightarrow (T(((f(q_1k_1), x, f(p_1)) \in A_2), ((f(q_1), x, f(r_1)) \in A_2), (k_1 \in B)))$$
$$\rightarrow \Big(T\Big((\exists p', q', k'\{((q'k', x, p') \in A_1)\}; f(p') = f(p_1), f(q') = f(q_1), f(k') = f(k_1), p', q', k' \in G_1), (\exists q'', r'\{((q'', x, r') \in A_1)\}; f(q'') = f(q_1), f(r') = f(r_1), q'', r' \in G_1), (k_1 \in B)\Big)\Big)$$
$$\rightarrow \Big(\exists p', q', q'', k', r'\{T(((q'k', x, p') \in A_1), ((q'', x, r') \in A_1), (k_1 \in B))\}; f(p') = f(p_1), f(q') = f(q_1) = f(q''), f(k') = f(k_1), f(r') = f(r_1), p', q', q'', k', r' \in G_1\Big)$$
$$\rightarrow \Big(T(((q'k', x, p') \in A_1), ((q'', x, r') \in A_1), (k_1 \in B))\Big) \text{ for some } p', q', q'', k', r' \in G_1$$
$$\rightarrow \Big(T(((q''k_1, x, p') \in A_1), ((q'', x, r') \in A_1), (k_1 \in B))\Big) \text{ by (1)}$$
$$\rightarrow (p'(r')^{-1} \in B) \text{ since } B \text{ is an IB-T-fuzzy kernel of } G_1$$
$$\rightarrow (p_2'(r_2')^{-1} \in C)$$

Hence

$$\models_\lambda (T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (k_2 \in C)))$$
$$\rightarrow (\exists k_1\{T(((q_2k_2, x, p_2) \in A_2), ((q_2, x, r_2) \in A_2), (k_1 \in B)\}; f(k_1) = k_2, k_1 \in G_1)))$$
$$\rightarrow (p_2'(r_2')^{-1} \in C)$$

Therefore C is an *implication-based T-fuzzy kernel of* $\mathscr{S}_2$. ☐

Definition 36. Let A and B be two implication-based T-fuzzy subgroups of a finite group G such that $A \leq B$. Then A is called an implication-based T-fuzzy normal subgroup of B if $\models_\lambda (T((y \in A), (x \in B))) \rightarrow (xyx^{-1} \in A)$ for all $x, y \in G$.

Definition 37. Let B be an implication-based T-fuzzy subsemiautomaton of an IB-T-FSA $\mathscr{S} = (G, \mathscr{V}, A)$. An IB-T-fuzzy subgroup C is called an implication-based T-fuzzy kernel of B if

(i) $C \leq B$ and C is an implication-based T-fuzzy normal subgroup of B.

(ii)$\models_\lambda (T(((qk, x, p) \in A), ((q, x, r) \in A), (k \in C))) \rightarrow (pr^{-1} \in C)$ for all $p, r, k \in G$, $x \in \mathscr{V}$ and $q \in Supp(B)$.

Definition 38. Let A and B be two implication-based T-fuzzy subgroups of a finite group G then the product $A \cdot B$ is defined by $\models_\lambda (\exists y, z\{T((y \in A), (z \in B))\}; yz = x, y, z \in G) \rightarrow (x \in A \cdot B)$ for all $x \in G$.

Lemma 311. *Let A and B be two implication-based T-fuzzy subgroups of a finite group G then the product $A \cdot B$ is an implication-based T-fuzzy subgroup of G.*

*Proof.*Let $x, y \in G$.

$$\models_\lambda (T((x \in A \cdot B), (y \in A \cdot B)))$$
$$\rightarrow (T((\exists u, v\{T((u \in A), (v \in B))\}; uv = x; u, v \in G), (\exists l, m\{T((l \in A), (m \in B))\}; lm = y; l, m \in G)))$$
$$\rightarrow (\exists u, v, l, m\{T((T((u \in A), (v \in B))), (T((l \in A), (m \in B))))\}; uv = x, lm = y; u, v, l, m \in G)$$
$$\rightarrow (\exists u, v, l, m\{T((T((u \in A), (l \in A))), (T((v \in B), (m \in B))))\}; uv = x, lm = y; u, v, l, m \in G)$$
by lemma 22
$$\rightarrow (\exists u, v, l, m\{T((ul \in A), (vm \in B))\}; ulvm = xy; u, v, l, m \in G)$$
$$\rightarrow (xy \in A \cdot B)$$

And $\models_\lambda (x \in A \cdot B)$

$$\rightarrow (\exists u, v\{T((u \in A), (v \in B))\}; uv = x; u, v \in G)$$
$$\rightarrow (\exists u, v\{T((v \in B), (u \in A))\}; uv = x; u, v \in G)$$
$$\rightarrow \big(\exists u, v\{T((v^{-1} \in B), (u^{-1} \in A))\}; v^{-1}u^{-1} = uv^{-1} = x^{-1}; u, v \in G\big)$$
$$\rightarrow \big(\exists u, v\{T((u^{-1} \in A), (v^{-1} \in B))\}; v^{-1}u^{-1} = x^{-1}; u, v \in G\big)$$
$$\rightarrow (x^{-1} \in A \cdot B)$$

Therefore $\models_\lambda (x \in A \cdot B) \rightarrow (x^{-1} \in A \cdot B)$
Hence $A \cdot B$ is an *implication-based T-fuzzy subgroup* of G. ☐

Lemma 312. *Let A and B be two implication-based T-fuzzy normal subgroups of a finite group G then the product $A \cdot B$ is an implication-based T-fuzzy normal subgroup of G.*

Lemma 313. *Let B be an implication-based T-fuzzy normal subgroup and C be an implication-based T-fuzzy subgroup of a finite group G then $B \cdot C = C \cdot B$.*

Theorem 314. *Let $\mathscr{S} = (G, \mathscr{V}, A)$ be an IB-T-FSA.(i) Let B be an implication-based T-fuzzy kernel and C be an implication-based T-fuzzy subsemiautomaton of $\mathscr{S}$ then $B \cdot C$ is an implication-based T-fuzzy subsemiautomaton of $\mathscr{S}$. (ii) Let B and C be an implication-based T-fuzzy kernels of $\mathscr{S}$ then $B \cdot C$ is an implication-based T-fuzzy kernel of $\mathscr{S}$*

Proof.(i) By lemma 311, $B\cdot C$ is an implication-based T-fuzzy subgroup of G.
Let $p,q\in G$ and $x\in\mathscr{V}$.
$\models_{\lambda} (T(((q,x,p)\in A),(q\in B\cdot C)))$
$\to (T(((q,x,p)\in A),(\exists a,b\{T((a\in B),(b\in C))\};$
$ab=q;a,b\in G)))$
$\to (T(((ab,x,p)\in A),(\exists a,b\{T((a\in B),(b\in C))\};$
$ab=q;a,b\in G)))$
$\to (\exists a,b\{T(((ab,x,p)\in A),(T((a\in B),(b\in C))))\};$
$ab=q;a,b\in G)$
$\to (\exists a,b\{T(((ab,x,p)\in A),(a\in B),(b\in C))\};$
$ab=q;a,b\in G)$
Since $\models_{\lambda} ((ab,x,p)\in A)\to((a,x,r)\in A)$
$\to (\exists a,b\{T((T(((ab,x,p)\in A),((a,x,r)\in A),(a\in B))),$
$(T(((a,x,r)\in A),(b\in C))))\};ab=q;a,b\in G)$
$\to \left(\exists p,r\{T(((pr^{-1})\in B),(r\in C))\};\right.$
$\left.(pr^{-1})r=p;p,r\in G\right)$
$\to (p\in B\cdot C)$
Therefore $B\cdot C$ is an implication-based T-fuzzy subsemiautomaton of $\mathscr{S}$.
(ii) By lemma 312, $B\cdot C$ is an implication-based T-fuzzy normal subgroup of G.
Let $p,q,k,r\in G$ and $x\in\mathscr{V}$.
$\models_{\lambda} (T(((qk,x,p)\in A),((q,x,r)\in A),(k\in B\cdot C)))$
$\to (T(((qk,x,p)\in A),((q,x,r)\in A),$
$(\exists b,c\{T((b\in B),(c\in C))\};bc=k;b,c\in G)))$
$\to (\exists b,c\{T(((qbc,x,p)\in A),((q,x,r)\in A),$
$(b\in B),(c\in C))\};bc=k;b,c\in G)$
$\to (\exists b,c\{T((T(((qbc,x,p)\in A),((qb,x,u)\in A),$
$(b\in B))),(T(((qb,x,u)\in A),((q,x,r)\in A),$
$(c\in C)))))\};bc=k;b,c\in G)$
Since $\models_{\lambda} ((qab,x,p)\in A)\to((qb,x,u)\in A)$
$\to \left(\exists pu^{-1},ur^{-1}\{T((pu^{-1}\in B),(ur^{-1}\in C))\};\right.$
$\left.(pu^{-1})(ur^{-1})=pr^{-1},p,u,r\in G\right)$
$\to (pr^{-1}\in B\cdot C)$
Therefore $B\cdot C$ is an implication-based T-fuzzy kernel of $\mathscr{S}$.
□

Theorem 315. *Let $\mathscr{S}=(G,\mathscr{V},A)$ be an IB-T-FSA. Let B be an implication-based T-fuzzy subsemiautomaton of $\mathscr{S}$ and C be an implication-based T-fuzzy kernel of $\mathscr{S}$ such that $\models_{\lambda} (e\in B)\to(e\in C)$. Then (i)$C$ is an implication-based T-fuzzy kernel of $B\cdot C$. (ii) $B\cap C$ is an implication-based T-fuzzy kernel of B.*

Proof.(i) Let $x\in G$
$\models_{\lambda} (x\in C)$
$\to (ex\in C)$
$\to (T((e\in C),(x\in C)))$
$\to (T((e\in B),(x\in C));ex=x)$
$\to (\exists x\{T((e\in B),(x\in C))\};ex=x;x\in G)$
$\to (x\in B\cdot C)$
Let $x,y\in G$
$\models_{\lambda} (T((y\in C)),(x\in B\cdot C))$
$\to (T((y\in C)),(x\in C))$
$\to (T((y\in C)),T((x\in C),(x^{-1}\in C)))$
Since $\models_{\lambda} (x\in C)\to(x^{-1}\in C)$
$\to (T((y\in C),(x\in C),(x^{-1}\in C)))$
$\to (T((x\in C),(y\in C),(x^{-1}\in C)))$
$\to (xyx^{-1}\in C)$
Therefore C is an *implication-based T-fuzzy normal subgroup* of $B\cdot C$.
$\Rightarrow C$ is an *implication-based T-fuzzy kernel* of $\mathscr{S}$.
$\Rightarrow C$ is an *implication-based T-fuzzy kernel* of $B\cdot C$.
(ii) Let $x,y\in G \models_{\lambda} (T((x\in B\cap C),(y\in B\cap C)))$
$\to (T((T((x\in B),(x\in C))),(T((y\in B),(y\in C)))))$
$\to (T((T((x\in B),(y\in B))),(T((x\in C),(y\in C)))))$
by lemma 22
$\to (T((xy\in B),(xy\in C)))$
$\to (xy\in B\cap C)$
And $\models_{\lambda} (x\in B\cap C)$
$\to (T((x\in B),(x\in C)))$
$\to (T((x^{-1}\in B),(x^{-1}\in C)))$
$\to (x^{-1}\in B\cap C)$
Therefore $B\cap C$ is an *implication-based T-fuzzy subgroup* of G.
$\models_{\lambda} (T((y\in B\cap C),(x\in B)))$
$\to (T((y\in B),(x\in B)))$
Since $\models_{\lambda} (y\in B\cap C)\to(y\in B)$
$\to (T((y\in B),(T((x\in B),(x^{-1}\in B)))))$
Since $\models_{\lambda} (x\in C)\to(x^{-1}\in C)$
$\to (T((y\in B),(x\in B),(x^{-1}\in B)))$
$\to (T((x\in B),(y\in B),(x^{-1}\in B)))$
$\to (xyx^{-1}\in B)$
Therefore $B\cap C$ is an *implication-based T-fuzzy normal subgroup* of B.
Let $p,r,k\in G$, $x\in\mathscr{V}$ and $q\in Supp(B)$.
$\models_{\lambda} (T(((qk,x,p)\in A),((q,x,r)\in A),(k\in B\cap C)))$
$\to (T(((qk,x,p)\in A),((q,x,r)\in A),(k\in B),(k\in C)))$
$\to (T((T(((qk,x,p)\in A),(k\in B))),(T(((qk,x,p)\in A),$
$((q,x,r)\in A),(k\in C)))))$
$\to (T((T(((k,x,pr^{-1})\in A),(k\in B))),$
$(T(((qk,x,p)\in A),((q,x,r)\in A),(k\in C)))))$
Since $\models_{\lambda} (qk,x,p)\in A)\to$
$((k,x,pr^{-1})\in A)$
$\to (T((pr^{-1}\in B),(pr^{-1}\in C)))$
Since C is an implication-based T-fuzzy kernel and
B is an implication-based T-fuzzy subsemiautomaton.
$\to (pr^{-1}\in B\cap C)$
Therefore $B\cap C$ is an *implication-based T-fuzzy kernel* of B. □

4 Conclusion

In this paper, we have characterized a subsystem namely *implication-based T-fuzzy semiautomaton* (IB-T-FSA) of a finite group. Further using the concept of *implication-based T-fuzzy subgroup* and *implication-based T-fuzzy normal subgroup* of a finite group, *implication-based T-fuzzy kernel* and *implication-based T-fuzzy subsemiautomaton* of an IBFSA are defined. The necessary and sufficient conditions for *implication-based fuzzy kernel* and *implication-based fuzzy subsemiautomaton* of an IBFSA are proved.

The concept of Fuzzy finite State Machine is applied in various fields as in recurrent neural networks, gesture recogntion, vision based hand gesture recognition, fuzzy signal processing and many more. Fuzzy Logic Finite State Machine Models are used in Real Time Systems to tune the event - driven behaviour of controllers. For a fuzzy based vehicle control application where online tuning is not advisable, the performance of the controller could be improved by *implication-based T-fuzzy semiautomaton*. This ensures better control of the vehicle. Hence the work on *implications-based fuzzy semiautomaton* and *implication-based T-fuzzy subsemiautomaton* can be further studied to apply in real time systems and in medical field.

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Michael Anna Spinneli.J is working as an Assistant Professor of Mathematics at L.R.G. Government Arts College for Women, Tirupur, India. She received the Ph.D., degree in Mathematics from Karunya University, India. Her research interest is in the field of Fuzzy Automata.

Selva Rathi. M received M.Sc., and M.Phil., degree in Mathematics from Manonmaniam Sundaranar University, India. She is currently working as an Assistant Professor of Mathematics in Karunya University, India. She is doing her research in the area of Fuzzy Automata.