Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110420

On Estimation in the Exponentiated Pareto Distribution in the Presence of Outliers

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Received: 27 Feb. 2016, Revised: 28 May 2017, Accepted: 2 Apr. 2017
Published online: 1 Jul. 2017

Abstract: The maximum likelihood and moment estimations of the probability density function (pdf) and cumulative distribution function (cdf) are derived for the Exponentiated Pareto distribution in the presence of outliers. Also, we calculate the mixture method of ML and moment estimators. It has been shown that this mixture method is better than the others. Further, we have given an actual data from an insurance company.

Keywords: Exponentiated Pareto distribution, Maximum likelihood estimator, Moment estimator, Mixture method, Probability density function, Cumulative distribution function, Outliers, Insurance

1 Introduction

The Pareto distribution was originally used to describe the allocation of wealth among individuals since it seemed to show rather well the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people in that society. It can be shown that from a probability density function (pdf) graph of the population, the probability, or fraction, of pdf that own a small amount of wealth per person, is high. The probability then decreases steadily as wealth increases.
The other application of this distribution is in On-Line Analytical Processing (OLAP) view size estimation. Nadeau and Teorey [10] used Pareto distribution for OLAP aims at gaining useful information quickly from large amounts of data residing in a data warehouse. To improve the quickness of response to queries, pre-aggregation is a useful strategy. However, it is usually impossible to pre-aggregate along all combinations of the dimensions. The multi-dimensional aspects of the data lead to combinatorial explosion in the number and potential storage size of the aggregates. They must selectively pre-aggregate. Cost/benefit analysis involves estimating the storage requirements of the aggregates in question. They presented an original algorithm for estimating the number of rows in an aggregate based on the Pareto distribution model. They tested the Pareto model algorithm empirically against four published algorithms and concluded the Pareto model algorithm is consistently the best of these algorithms for estimating view size. Also, the Pareto distribution is useful for finding the average of annuity and benefit for an insurance problem. In economics, where this distribution is used as an income distribution, the threshold parameter is some minimum income with a known value. Asrabadi [2] derived the uniformly minimum variance unbiased estimator (UMVUE) of the pdf, the cumulative distribution function (cdf) and the r^{th} moment for the Pareto distribution.
Now, if we assume that Y is a Pareto distributed random variable, then we take $X = \ln(Y)$ to have the corresponding exponentiated Pareto distribution as defined by Nadarajah [9]. Usually, Y is defined on the positive side of the real line and so one would hope that models on the basis of the distribution of X would have greater applicability. Nadarajah [9] introduced five exponentiated Pareto distributions and derived several of their properties including the moment generating function, expectation, variance, skewness, kurtosis, Shannon entropy, and the Rényi entropy. Note that another type of exponentiated Pareto distribution was considered by Shawky and Abu-Zinadah [11] and characterized using record values. Afify [1] used Bayes estimators under squared error and LINEX loss functions and classical estimators for the two parameters exponentiated Pareto distribution when a data sample is available from complete, type I and type II censoring scheme. However, in this paper we will restrict to the form defined by Nadarajah [9].

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Let a set of random variables $(Y_1,Y_2,...,Y_n)$ represent the claim amounts of an motor insurance company. So the variables $(X_1,X_2,...,X_n)$ have exponentiated Pareto distribution. It is assumed that claims of some of vehicles (expensive/severe damaged vehicle) are β times higher than normal vehicles.
We know that an application of the Pareto distribution include insurance where it is used to model claims where the minimum claim is also the modal value, but where there is no set maximum (see http://www.brighton-webs.co.uk/distributions/pareto.asp). Also, according to Benktander [3], the Pareto distribution is useful for the automobile insurance problems.
Dixit and Jabbari Nooghabi [5] have given UMVUE and MLE of pdf, cdf and r^{th} moment for the Pareto distribution in the presence of outliers. They have shown that MLE of pdf and cdf are more efficient than their UMVUE. Also, Dixit and Jabbari Nooghabi [6] estimated the parameters of the Pareto distribution in the presence of outliers when all the parameters are unknown. Dixit and Jabbari Nooghabi [5,6] have presented of an actual data from an insurance company as an application of the Pareto distribution in the presence of outliers.
Jabbari Nooghabi and Khaleghpanah Nooghabi [8] have shown that excluding the outliers from data is losing the information. In addition, Jabbari Nooghabi [7] have presented a method to detect outliers in exponentiated Pareto distribution.
Hence, we assume that the random variables $(X_1,X_2,...,X_n)$ are such that k of them are distributed with pdf

$$f_2(x;\alpha,\beta,\theta)=\alpha(\beta\theta)^{\alpha}e^{-\alpha x},\quad \ln(\beta\theta)\leq x,\ \alpha>0,\ \beta>1,\ \theta>0, \tag{1}$$

and remaining $(n-k)$ random variables are distributed as

$$f_1(x;\alpha,\theta)=\alpha\theta^{\alpha}e^{-\alpha x},\quad \ln(\theta)\leq x,\ \alpha>0,\ \theta>0. \tag{2}$$

In this paper, we assume that only θ is known and we have derived the MLE, moment estimator and Mixture of ML and moment estimator for pdf and cdf of exponentiated Pareto distribution in the presence of outliers. At the end, we have given an example of claims in a motor insurance company.

2 Joint distribution of $(X_1,X_2,...,X_n)$ with k outliers

The joint distribution of $(X_1,X_2,...,X_n)$ in the presence of k outliers is given by

$$f(x_1,x_2,..,x_n;\alpha,\beta,\theta)=\frac{\alpha^n\theta^{n\alpha}\beta^{k\alpha}}{C(n,k)}e^{-\alpha\sum_{i=1}^{n}x_i}\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}\prod_{j=1}^{k}\mathbf{I}(x_{A_j}-\ln(\beta\theta)), \tag{3}$$

where $C(n,k)=\frac{n!}{k!(n-k)!}$ and $\mathbf{I}(A)$ represents the indicator function of the set A.
Note that the marginal distribution of X is

$$f(x;\alpha,\beta,\theta)=b\alpha(\beta\theta)^{\alpha}e^{-\alpha x}\mathbf{I}(x-\ln(\beta\theta))+\bar{b}\alpha\theta^{\alpha}e^{-\alpha x}\mathbf{I}(x-\ln(\theta)),\ \alpha>0,\ \beta>1,\ \theta>0, \tag{4}$$

where $b=\frac{k}{n}$, $\bar{b}=1-b$ and $(X_1,X_2,...,X_n)$ are not independent.

3 Maximum likelihood estimator

Let $X_1,X_2,...,X_n$ be a random sample of size n from the exponentiated Pareto distribution in the presence of outliers. So, ML estimator of $\ln(\beta\theta)$ is $X_{(1)}=\min(X_1,X_2,...,X_n)$, and

$$\tilde{\beta}_{ml}=\frac{\exp(X_{(1)})}{\theta}. \tag{5}$$

For finding the ML estimator of α, we put the ML estimator of β in the likelihood function. Then, the likelihood equation for estimating α is

$$\frac{n}{\alpha}+kx_{(1)}+(n-k)\ln(\theta)-\sum_{i=1}^{n}x_i=0. \tag{6}$$

From (6), we can find the MLE of α and it is given as $\tilde{\alpha}_{ml}$

$$\tilde{\alpha}_{ml}=\frac{n}{\sum_{i=1}^{n}X_i-kX_{(1)}-(n-k)\ln(\theta)}. \tag{7}$$

Therefore by using the property of MLE, we can obtain the estimator of pdf and cdf with replacement of $\tilde{\alpha}_{ml}$ and $\tilde{\beta}_{ml}$ instead of α and β in the pdf and cdf, respectively. So

$$\tilde{f}_{ml}(x)=\tilde{\alpha}_{ml}\theta^{\tilde{\alpha}_{ml}}e^{-\tilde{\alpha}_{ml}x}\left(b\tilde{\beta}_{ml}^{\tilde{\alpha}_{ml}}+\bar{b}\right), \tag{8}$$

and

$$\tilde{F}_{ml}(x)=1-\theta^{\tilde{\alpha}_{ml}}e^{-\tilde{\alpha}_{ml}x}\left(b\tilde{\beta}_{ml}^{\tilde{\alpha}_{ml}}+\bar{b}\right), \tag{9}$$

where $\tilde{\alpha}_{ml}>0$, $\theta>0$ and $\tilde{\beta}_{ml}>1$.
Then, to obtain the expectation of pdf and cdf, we should know the joint pdf of $(X_{(1)},T)$, where $T=\sum_{i=1}^{n}X_i$. The following lemma represents it.
Lemma 3.1. The joint pdf of $(X_{(1)},T)$ is

$$f(x_{(1)},t)=\frac{n\alpha^n\theta^{n\alpha}\beta^{k\alpha}}{\Gamma(n-1)}e^{-\alpha t}[t-nx_{(1)}]^{n-2},\quad t>nx_{(1)},\quad x_{(1)}>\ln(\beta^{k/n}\theta). \tag{10}$$

Proof. We have the joint distribution of $(X_1,X_2,...,X_n)$ in (3). Then, the joint pdf of the order statistics $(X_{(1)},X_{(2)},...,X_{(n)})$ is

$$f_{X_{(1)},X_{(2)},...,X_{(n)}}(x_{(1)},x_{(2)},...,x_{(n)})=n!\,f_{X_1,X_2,...,X_n}(x_{(1)},x_{(2)},...,x_{(n)}). \tag{11}$$

To find the joint pdf of $(X_{(1)},T)$, we set the following transformation:

$$\begin{cases} x_{(1)}=x_{(1)},\\ x_{(2)}=x_{(2)},\\ \quad .\\ \quad .\\ \quad .\\ x_{(n-1)}=x_{(n-1)},\\ x_{(n)}=t-\left(x_{(1)}+x_{(2)}+...+x_{(n-1)}\right).\end{cases}$$

So, we have

$$f(x_{(1)},x_{(2)},\ldots,x_{(n-1)},t) = n!\alpha^n\theta^{n\alpha}\beta^{k\alpha}e^{-\alpha t}. \tag{12}$$

Then, using $(n-2)$ integrations for $x_{(2)},x_{(3)},\ldots,x_{(n)}$, the joint distribution of $(X_{(1)},T)$ is found and the proof is complete.

If we define $W = \tilde{\alpha}$ then by using the joint pdf of $(X_{(1)},T)$, we can find the joint pdf of $(X_{(1)},W)$ as

$$\begin{aligned} f(x_{(1)},w) &= \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!w^2}e^{-\alpha(\frac{n}{w}+kx_{(1)})} \\ &\times \left[\frac{n}{w}-(n-k)x_{(1)}+(n-k)\ln(\theta)\right]^{n-2}, \\ &\quad w>0, x_{(1)}>\ln(\beta^{k/n}\theta). \end{aligned}$$

Theorem 3.2. A) $\tilde{f}_{ml}(x)$ is a biased estimator of $f(x)$ and

$$\begin{aligned} \mathrm{E}(\tilde{f}_{ml}(x)) &= \frac{n^3\alpha^{n-2}\beta^{k\alpha\bar{b}}}{k(n-2)!}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i-1}(\alpha,n) \\ &\times \mathbf{D}_m^{n-2-i}(\ln(\theta))\Big\{b\mathbf{G}_l^j(x)\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta)) \\ &+ \bar{b}(\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m} \\ &\times \mathbf{H}_p^m(k\alpha\ln(\beta^{k/n}\theta))\Big\}, \end{aligned} \tag{13}$$

B) $\tilde{F}_{ml}(x)$ is a biased estimator of $F(x)$ and

$$\begin{aligned} \mathrm{E}(\tilde{F}_{ml}(x)) &= 1-\frac{n^2\alpha^{n-2}\beta^{k\alpha\bar{b}}}{k(n-2)!}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i}(\alpha,n) \\ &\times \mathbf{D}_m^{n-2-i}(\ln(\theta))\Big\{b\,\mathbf{G}_l^j(x)\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta)) \\ &+ \bar{b}(\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m} \\ &\times \mathbf{H}_p^m(k\alpha\ln(\beta^{k/n}\theta))\Big\}, \end{aligned} \tag{14}$$

where

$$\mathbf{A}_i^n(x) = \sum_{i=0}^{n} C(n,i)(x)^{n-i},$$

$$\mathbf{B}_j^i(\alpha,n) = \sum_{j=0}^{i}\frac{n^{j-1}}{j!}\Gamma(i-j+1)\alpha^{j-i},$$

$$\mathbf{D}_m^n(x) = \sum_{m=0}^{n} C(n,m)(-1)^m(x)^{n-m},$$

$$\mathbf{G}_l^j(x) = \sum_{l=0}^{j} C(j,l)(-x)^{j-l}\Gamma(l+m+1)(k\alpha)^{-l-m},$$

and

$$\mathbf{H}_p^m(x) = \sum_{p=0}^{m}\frac{x^p}{p!}.$$

Proof. A) We have

$$\begin{aligned} \mathrm{E}(\tilde{f}_{ml}(x)) &= \int_{x_{(1)}}\int_{w}\tilde{f}_{ml}(x)f(x_{(1)},w)dwdx_{(1)} \\ &= \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\times\Big\{b\int_{\ln(\beta^{k/n}\theta)}^{\infty}e^{-k\alpha x_{(1)}}\int_0^{\infty}\frac{e^{-w(x-x_{(1)})}}{w}e^{-\frac{\alpha n}{w}} \\ &\times \left[\frac{n}{w}+(n-k)\ln(\theta)-(n-k)x_{(1)}\right]^{n-2}dwdx_{(1)} \\ &+ \bar{b}\int_{\ln(\beta^{k/n}\theta)}^{\infty}e^{-k\alpha x_{(1)}}\int_0^{\infty}\frac{e^{-wx}\theta^w}{w}e^{-\frac{\alpha n}{w}} \\ &\times \left[\frac{n}{w}+(n-k)\ln(\theta)-(n-k)x_{(1)}\right]^{n-2}dwdx_{(1)}\Big\}. \end{aligned}$$

Then for calculating these integrals, we put

$$e^{-w(x-x_{(1)})} = \sum_{j=0}^{\infty}\frac{w^j[x_{(1)}-x]^j}{j!},$$

$$\theta^w e^{-wx} = e^{w[\ln(\theta)-x]} = \sum_{j=0}^{\infty}\frac{w^j[\ln(\theta)-x]^j}{j!},$$

and set the transformation $z=\frac{n}{w}$, hence

$$\begin{aligned} \mathrm{E}(\tilde{f}_{ml}(x)) &= \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\int_{\ln(\beta^{k/n}\theta)}^{\infty}e^{-k\alpha x_{(1)}}\sum_{j=0}^{\infty}\frac{n^j}{j!}\Big[b(x_{(1)}-x)^j \\ &+ \bar{b}(\ln(\theta)-x)^j\Big]\int_0^{\infty}z^{-j-1}e^{-\alpha z}[z+(n-k)\ln(\theta) \\ &- (n-k)x_{(1)}]^{n-2}dzdx_{(1)}. \end{aligned}$$

So, we put

$$\begin{aligned} &[z+(n-k)\ln(\theta)-(n-k)x_{(1)}]^{n-2} \\ &= \sum_{i=0}^{n-2}C(n-2,i)z^i[(n-k)\ln(\theta)-(n-k)x_{(1)}]^{n-2-i}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{E}(\tilde{f}_{ml}(x)) &= \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\Big[b\sum_{i=0}^{n-2}C(n-2,i)(n-k)^{n-2-i}\sum_{j=0}^{i-1}\frac{n^j}{j!}\Gamma(i-j) \\ &\times \alpha^{j-i}\int_{\ln(\beta^{k/n}\theta)}^{\infty}(\ln(\theta)-x_{(1)})^{n-2-i}(x_{(1)}-x)^je^{-k\alpha x_{(1)}}dx_{(1)} \\ &+ \bar{b}\sum_{i=0}^{n-2}C(n-2,i)(n-k)^{n-2-i}\sum_{j=0}^{i-1}\frac{n^j(\ln(\theta)-x)^j}{j!}\Gamma(i-j) \\ &\times \alpha^{j-i}\int_{\ln(\beta^{k/n}\theta)}^{\infty}(\ln(\theta)-x_{(1)})^{n-2-i}e^{-k\alpha x_{(1)}}dx_{(1)}\Big]. \end{aligned}$$

For calculating the above integrals, we can put

$$(\ln(\theta)-x_{(1)})^{n-2-i} = \sum_{m=0}^{n-2-i}C(n-2-i,m)(-1)^m{x_{(1)}}^m(\ln(\theta))^{n-2-i-m},$$

and

$$(x_{(1)}-x)^j = \sum_{l=0}^{j}C(j,l)(-x)^{j-l}(x_{(1)})^l.$$

Then by using some elementary algebra, the proof is complete.

B) We have

$$\mathrm{E}(\tilde{F}_{ml}(x)) = \int_{x_{(1)}}\int_{w}\tilde{F}_{ml}(x)f(x_{(1)},w)dwdx_{(1)}.$$

To calculate the integral similarly to case A, we have

$$\begin{aligned} \mathrm{E}(\tilde{F}_{ml}(x)) &= 1-\frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\sum_{i=0}^{n-2}C(n-2,i)(n-k)^{n-2-i} \\ &\times \sum_{j=0}^{i}\frac{n^{j-1}}{j!}\Gamma(i-j+1)\alpha^{j-i-1}\Big[b\int_{\ln(\beta^{k/n}\theta)}^{\infty}[\ln(\theta) \\ &- x_{(1)}]^{n-2-i}(x_{(1)}-x)^je^{-k\alpha x_{(1)}}dx_{(1)}+\bar{b}[\ln(\theta)-x]^j \\ &\times \int_{\ln(\beta^{k/n}\theta)}^{\infty}[\ln(\theta)-x_{(1)}]^{n-2-i}e^{-k\alpha x_{(1)}}dx_{(1)}\Big], \end{aligned}$$

and the proof is obvious.

4 MSE of ML estimator

In the previous section, we found the MLE of $f(x)$ and $F(x)$. Now, we try to find the MSE of them.

Theorem 4.1. A)

$$
\begin{aligned}
MSE(\tilde{f}_{ml}(x)) = & \frac{n^3\alpha^{n-1}\beta^{k\alpha\bar{b}}}{k(n-2)!}\Big\{\frac{n}{\alpha}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i-2}(\alpha,n) \\
& \times \mathbf{D}_m^{n-2-i}(\ln(\theta))\Big\{[b^2 2^j\mathbf{G}_l^j(x)+2b\bar{b}\mathbf{G}_l^j \\
& \times (\ln(\theta)-2x)]\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta))\bar{b}^2 2^j \\
& \times (\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m}\mathbf{H}_p^m \\
& \times (k\alpha\ln(\beta^{k/n}\theta))\Big\}-2\theta^\alpha e^{-\alpha x}(b\beta^\alpha+\bar{b}) \\
& \times \mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i-1}(\alpha,n)\mathbf{D}_m^{n-2-i}(\ln(\theta)) \\
& \times \Big\{b\mathbf{G}_l^j(x)\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta)) \\
& + \bar{b}(\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m}\mathbf{H}_p^m \\
& \times (k\alpha\ln(\beta^{k/n}\theta))\Big\}\Big\} + \alpha^2\theta^{2\alpha}e^{-2\alpha x} \\
& \times (b\beta^\alpha+\bar{b})^2.
\end{aligned} \tag{15}
$$

B)

$$
\begin{aligned}
MSE(\tilde{F}_{ml}(x)) = & \frac{n^2\alpha^{n-2}\beta^{k\alpha\bar{b}}}{k(n-2)!}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i}(\alpha,n)\mathbf{D}_m^{n-2-i} \\
& \times (\ln(\theta))\Big\{\Big[b^2 2^j-2b\theta^\alpha e^{-\alpha x}(b\beta^\alpha+\bar{b})\Big] \\
& \times \mathbf{G}_l^j(x)\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta))+\Big[\bar{b}^2 2^j-2\bar{b}\theta^\alpha \\
& \times e^{-\alpha x}(b\beta^\alpha+\bar{b})\Big](\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m} \\
& \times \mathbf{H}_p^m(k\alpha\ln(\beta^{k/n}\theta))+2b\bar{b}\mathbf{G}_l^j(\ln(\theta)-2x) \\
& \times \mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta))\Big\}+\theta^{2\alpha}e^{-2\alpha x} \\
& \times \Big(b\beta^\alpha+\bar{b}\Big)^2.
\end{aligned} \tag{16}
$$

Proof. In case A, we must find $\mathrm{E}(\tilde{f}_{ml}(x))^2$. So

$$
\begin{aligned}
\mathrm{E}(\tilde{f}_{ml}(x))^2 = & \int_{x_{(1)}}\int_w(\tilde{f}_{ml}(x))^2 f(x_{(1)},w)dwdx_{(1)} \\
= & \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\times\Big\{b^2\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}} \\
& \times \int_0^\infty e^{2w(x_{(1)}-x)}e^{-\frac{\alpha n}{w}}\Big[\frac{n}{w}+(n-k)\ln(\theta) \\
& -(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)}+\bar{b}^2\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}} \\
& \times \int_0^\infty e^{-2wx}\theta^{2w}e^{-\frac{\alpha n}{w}}\Big[\frac{n}{w}+(n-k)\ln(\theta) \\
& -(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)}+2b\bar{b}\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}} \\
& \times \int_0^\infty e^{w(x_{(1)}-2x)}\theta^w e^{-\frac{\alpha n}{w}}\Big[\frac{n}{w}+(n-k)\ln(\theta) \\
& -(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)}\Big\},
\end{aligned}
$$

and to find these integrals similarly to the previous theorem, we put

$$
e^{2w(x_{(1)}-x)}=\sum_{j=0}^\infty\frac{2^j w^j[x_{(1)}-x]^j}{j!},
$$

$$
\theta^{2w}e^{-2wx}=e^{2w[\ln(\theta)-x]}=\sum_{j=0}^\infty\frac{2^j w^j[\ln(\theta)-x]^j}{j!},
$$

and

$$
\theta^w e^{w(x_{(1)}-2x)}=e^{w[x_{(1)}+\ln(\theta)-2x]}=\sum_{j=0}^\infty\frac{w^j[x_{(1)}+\ln(\theta)-2x]^j}{j!}.
$$

Hence by using the same transformation as before, we have

$$
\begin{aligned}
\mathrm{E}(\tilde{f}_{ml}(x))^2 = & \frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!}\sum_{i=0}^{n-2}C(n-2,i)(n-k)^{n-2-i} \\
& \times \sum_{j=0}^{i-2}\frac{n^{j+1}}{j!}\Gamma(i-j-1)\alpha^{j-i+1}\Big[b^2 2^j \\
& \times \int_{\ln(\beta^{k/n}\theta)}^\infty(\ln(\theta)-x_{(1)})^{n-2-i}(x_{(1)}-x)^j e^{-k\alpha x_{(1)}}dx_{(1)} \\
& + \bar{b}^2 2^j(\ln(\theta)-x)^j\int_{\ln(\beta^{k/n}\theta)}^\infty(\ln(\theta)-x_{(1)})^{n-2-i} \\
& \times e^{-k\alpha x_{(1)}}dx_{(1)}+2b\bar{b}\int_{\ln(\beta^{k/n}\theta)}^\infty(\ln(\theta)-x_{(1)})^{n-2-i} \\
& \times (\ln(\theta)+x_{(1)}-2x)^j e^{-k\alpha x_{(1)}}dx_{(1)}\Big].
\end{aligned}
$$

Similarly to the previous theorem, we obtain

$$
\begin{aligned}
\mathrm{E}(\tilde{f}_{ml}(x))^2 = & \frac{n^4\alpha^{n-2}\beta^{k\alpha\bar{b}}}{k(n-2)!}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^{i-2}(\alpha,n)\mathbf{D}_m^{n-2-i}(\ln(\theta)) \\
& \times \Big\{\Big[b^2 2^j\mathbf{G}_l^j(x)+2b\bar{b}\mathbf{G}_l^j(\ln(\theta)-2x)\Big]\mathbf{H}_p^{m+l} \\
& \times (k\alpha\ln(\beta^{k/n}\theta))\bar{b}^2 2^j(\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m} \\
& \times \mathbf{H}_p^m(k\alpha\ln(\beta^{k/n}\theta))\Big\}.
\end{aligned} \tag{17}
$$

So by using elementary algebra, we can get the MSE of $\tilde{f}_{ml}(x)$.

In case B, we can easily find $\mathrm{E}(\tilde{F}_{ml}(x))^2$. For this purpose, We have

$$
\begin{aligned}
\mathrm{E}(\tilde{F}_{ml}(x))^2 = & \int_{x_{(1)}}\int_w(\tilde{F}_{ml}(x))^2 f(x_{(1)},w)dwdx_{(1)} \\
= & 1-2[\mathrm{E}(\tilde{F}_{ml}(x))-1]+\frac{n^2\alpha^n\theta^{k\alpha}\beta^{k\alpha}}{(n-2)!} \\
& \times \Big\{b^2\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}}\int_0^\infty\frac{1}{w^2}e^{2w(x_{(1)}-x)}e^{-\frac{n\alpha}{w}} \\
& \times \Big[\frac{n}{w}+(n-k)\ln(\theta)-(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)} \\
& + \bar{b}^2\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}}\int_0^\infty\frac{1}{w^2}e^{-2wx}\theta^{2w}e^{-\frac{n\alpha}{w}} \\
& \times \Big[\frac{n}{w}+(n-k)\ln(\theta)-(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)} \\
& + 2b\bar{b}\int_{\ln(\beta^{k/n}\theta)}^\infty e^{-k\alpha x_{(1)}}\int_0^\infty\frac{1}{w^2}e^{2w(x_{(1)}-2x)}e^{-\frac{n\alpha}{w}} \\
& \times \Big[\frac{n}{w}+(n-k)\ln(\theta)-(n-k)x_{(1)}\Big]^{n-2}dwdx_{(1)}\Big\}.
\end{aligned}
$$

So similarly to case A and using some elementary algebra, we obtain

$$\begin{aligned}
\mathrm{E}(\tilde{F}_{ml}(x))^2 &= 1 + \frac{n^2\alpha^{n-2}\beta^{k\alpha\bar{b}}}{k(n-2)!}\mathbf{A}_i^{n-2}(n-k)\mathbf{B}_j^i(\alpha,n)\mathbf{D}_m^{n-2-i} \\
&\times (\ln(\theta))\Big\{\left[b^2 2^j - 2b\right]\mathbf{G}_l^j(x)\mathbf{H}_p^{m+l}(k\alpha\ln(\beta^{k/n}\theta)) \\
&+ \left[\bar{b}^2 2^j - 2\bar{b}\right](\ln(\theta)-x)^j\Gamma(m+1)(k\alpha)^{-m}\mathbf{H}_p^m \\
&\times \left(k\alpha\ln(\beta^{k/n}\theta)\right) + 2b\bar{b}\mathbf{G}_l^j(\ln(\theta)-2x)\mathbf{H}_p^{m+l} \\
&\times \left(k\alpha\ln(\beta^{k/n}\theta)\right)\Big\}. \qquad (18)
\end{aligned}$$

Therefore, we can get the MSE of $\tilde{F}_{ml}(x)$ and the proof is complete.

5 Method of moment

In this section, we use the expectations of X and $\exp(X)$ to find the moment estimator of parameters α and β. By using some elementary algebra, we can easily find these expectation as

$$m_1 = \mathrm{E}(X) = \frac{1}{\alpha} + b\ln(\beta) + \ln(\theta), \qquad (19)$$

and

$$m_2 = \mathrm{E}(e^X) = \frac{\alpha}{\alpha-1}\theta(b\beta+\bar{b}). \qquad (20)$$

So from (20), we have

$$\alpha = \frac{1}{m_1 - b\ln(\beta) - \ln(\theta)}. \qquad (21)$$

Then by using (21), we can get moment estimator of β as

$$\tilde{\beta}_{mm} = \frac{m_2}{\theta}LambertW\left(\frac{\theta}{m_2}\exp\left\{\frac{1}{b}(m_1 - \ln(\theta) - 1 - \frac{\bar{b}\theta}{m_2})\right\}\right), \qquad (22)$$

where LambertW(.) is the LambertW function (see Corless et al., 1996) satisfies

$$LambertW(x)\exp\{LambertW(x)\} = x. \qquad (23)$$

Therefore, the moment estimator of α is

$$\tilde{\alpha}_{mm} = \frac{1}{m_1 - b\ln(\tilde{\beta}_{mm}) - \ln(\theta)}. \qquad (24)$$

It is easy to show that the moment estimator of parameters α and β are asymptotically consistent. Now for finding the moment estimator of pdf and cdf, we can replace $\tilde{\alpha}_{mm}$ and $\tilde{\beta}_{mm}$ instead of α and β in the pdf and cdf, respectively. So

$$\tilde{f}_{mm}(x) = \tilde{\alpha}_{mm}\theta^{\tilde{\alpha}_{mm}}e^{-\tilde{\alpha}_{mm}x}\left(b\tilde{\beta}_{mm}^{\tilde{\alpha}_{mm}} + \bar{b}\right), \qquad (25)$$

and

$$\tilde{F}_{mm}(x) = 1 - \theta^{\tilde{\alpha}_{mm}}e^{-\tilde{\alpha}_{mm}x}\left(b\tilde{\beta}_{mm}^{\tilde{\alpha}_{mm}} + \bar{b}\right). \qquad (26)$$

Finding the expectation and the MSE of moment estimator of pdf and cdf by using the mathematical methods are not possible.

6 Mixture method of moment and ML estimators

In this section, we get the estimator of β from ML method as

$$\tilde{\beta}_{mix} = \frac{\exp(x_{(1)})}{\theta}. \qquad (27)$$

If we put this estimator to (20), we can easily find the mixture estimator of α as following

$$\tilde{\alpha}_{mix} = \frac{n}{m_1 - bx_{(1)} - b\ln(\theta)}. \qquad (28)$$

Therefore, the mixture estimator of pdf and cdf are respectively of

$$\tilde{f}_{mix}(x) = \tilde{\alpha}_{mix}\theta^{\tilde{\alpha}_{mix}}e^{-\tilde{\alpha}_{mix}x}\left(b\tilde{\beta}_{mix}^{\tilde{\alpha}_{mix}} + \bar{b}\right), \qquad (29)$$

and

$$\tilde{F}_{mix}(x) = 1 - \theta^{\tilde{\alpha}_{mix}}e^{-\tilde{\alpha}_{mix}x}\left(b\tilde{\beta}_{mix}^{\tilde{\alpha}_{mix}} + \bar{b}\right). \qquad (30)$$

7 Comparison of ML, moment and mixture estimators and an example

In order to get the idea of efficiency between the three types of estimator i.e. ML, moment and mixture estimators, we have generated a sample of size 4(1)25 from the exponentiated Pareto distribution in the presence of outliers with k=1, 2, 3, α=0.5(0.5)2, β=1.5, 2 and θ=0.5, 1, 5, using **R** software. We have given graphs based on one thousand independent replication of each experiments.
Figures 1 to 12 show that the mixture method of ML and moment estimation of pdf and cdf are more efficient than the others.

Example 1. In an insurance company, we have the motor insurance service. A claim can be made of at least 500,000 Rials as compensation for the motor insurance. The vehicles involved are of different cost of which some of them may be very expensive. Claim amounts varies according to the damage occurred to the vehicles. It has been observed that claims of these vehicles (expensive/severe damaged vehicle) are β times higher than normal vehicles. In this paper, we have drawn 50 random samples of size 20 of the claim amounts. It is observed that such natural logarithm of claims follow exponentiated Pareto distribution in the presence of outliers with parameters α, β and θ, where α and β are unknown, θ=500,000 and the number of outliers (k) is unknown. One should note that for normal vehicles claims bellow 500,000 are not entertained.
The natural logarithm of a random data of size 20 of claims from the Iranian insurance company for the year 2009 is given below:

Table 1: The estimation results for $\theta = 500,000$

Method of Estimation	k	$\tilde{\alpha}$	$\tilde{\beta}$
Maximum Likelihood		0.9877135	1.1600000
Moment Method	1	1.010251	16.534750
Mixture Method		0.8907057	1.1600000
Maximum Likelihood		0.9950067	1.1600000
Moment Method	2	1.010251	4.066294
Mixture Method		0.8966324	1.1600000
Maximum Likelihood		1.002408	1.160000
Moment Method	3	1.010251	2.547608
Mixture Method		0.9026385	1.1600000

Table 2: The values of likelihood functions for $\theta = 500,000$

k	$\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{ml}, \tilde{\beta}_{ml})$	$\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{mm}, \tilde{\beta}_{mm})$	$\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{mix}, \tilde{\beta}_{mix})$
1	1.609644e-09	2.345693e-08	1.451694e-09
2	1.864792e-09	2.345693e-08	1.679404e-09
3	2.162741e-09	2.345693e-08	1.944893e-09

Table 3: The pdf and cdf estimates for $n = 20$, $k = 1$ and $\theta = 500,000$

Method of Estimation	$\tilde{f}(x)$	$\tilde{F}(x)$
Maximum Likelihood	0.7567518	0.2338347
Moment Method	1.3743551	0.3604096
Mixture Method	0.7004814	0.2135659

13.4000, 13.5008, 13.2708, 14.0346, 14.1186
14.8271, 15.5056, 15.1373, 13.7747, 13.6762
14.3041, 14.4986, 14.6085, 14.0144, 14.4521
14.3283, 13.9553, 14.1802, 13.7429, 13.5144.

So $\tilde{\alpha}_{ml}$, $\tilde{\beta}_{ml}$, $\tilde{\alpha}_{mm}$, $\tilde{\beta}_{mm}$, $\tilde{\alpha}_{mix}$, and $\tilde{\beta}_{mix}$ for k=1, 2, 3 are shown in Table 1. Also from the likelihood function corresponding to k, $\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{ml}, \tilde{\beta}_{ml})$, $\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{mm}, \tilde{\beta}_{mm})$ and $\mathbf{L}(\mathrm{x}; \tilde{\alpha}_{mix}, \tilde{\beta}_{mix})$ for k=1, 2, 3 are shown in Table 2. Table 2 shows that the likelihood function is maximized for $k = 1$, $\tilde{\alpha}_{ml} = 0.9877135$, $\tilde{\beta}_{ml} = 1.1600000$, $\tilde{\alpha}_{mm} = 1.010251$, $\tilde{\beta}_{mm} = 16.534750$, $\tilde{\alpha}_{mix} = 0.8907057$, and $\tilde{\beta}_{mix} = 1.1600000$.
Therefore, for $n = 20$, $k = 1$ and $\theta = 500,000$ the final result of MLE, moment estimator and mixture estimator of $f(x)$ and $F(x)$ corresponding to first observation are given in Table 3.

Acknowledgement

The authors are grateful to the anonymous editor and referee for a careful checking of the details and for helpful comments that improved this paper.

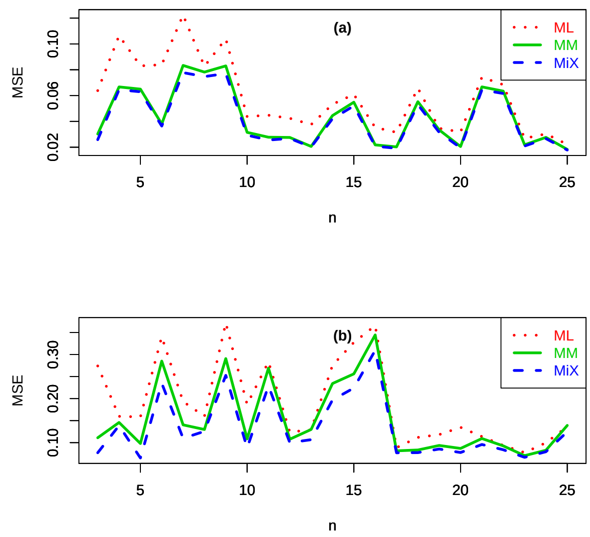

Fig. 1: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=1, α=0.5, β=1.5 and θ=0.5 and (b) for k=1, α=1, β=1.5 and θ=1

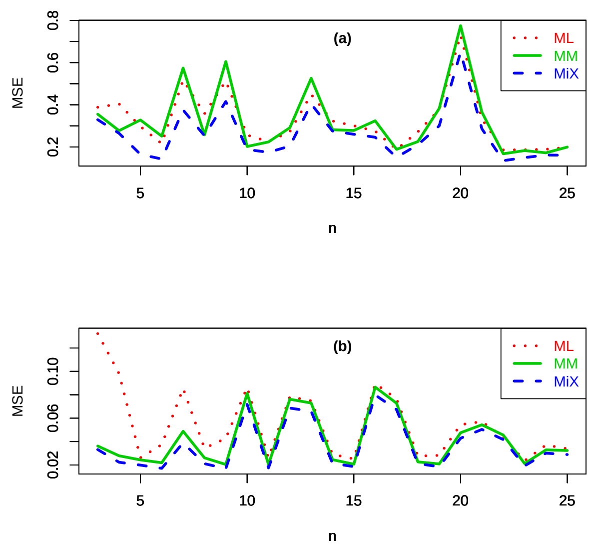

Fig. 2: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=1, α=1.5, β=1.5 and θ=1 and (b) for k=1, α=0.5, β=2 and θ=5

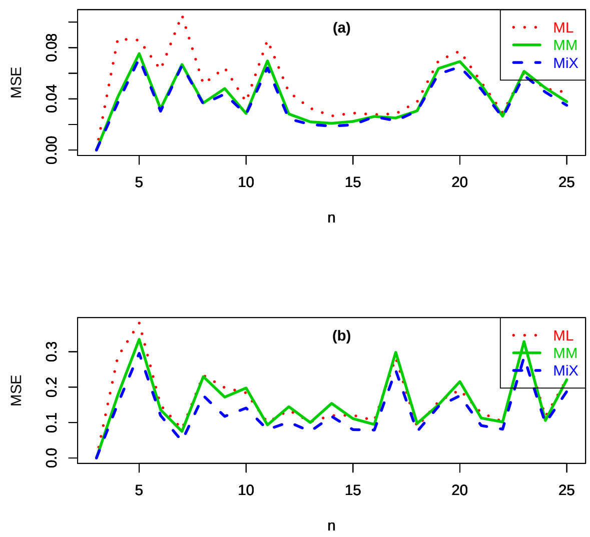

Fig. 3: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=2, α=0.5, β=1.5 and θ=0.5 and (b) for k=2, α=1, β=1.5 and θ=1

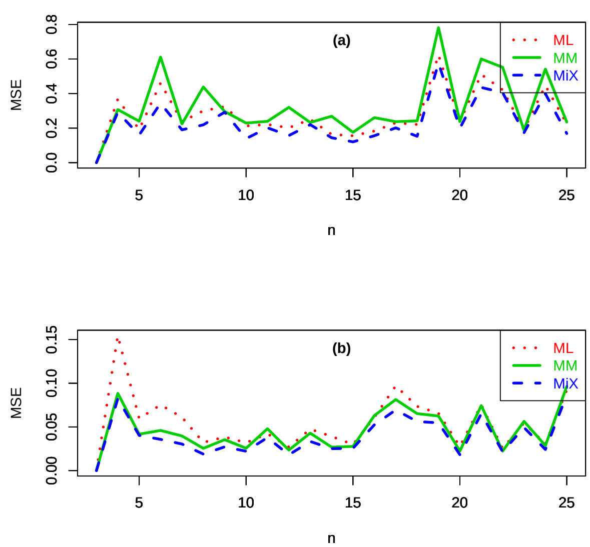

Fig. 4: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=2, α=1.5, β=1.5 and θ=1 and (b) for k=2, α=0.5, β=2 and θ=5

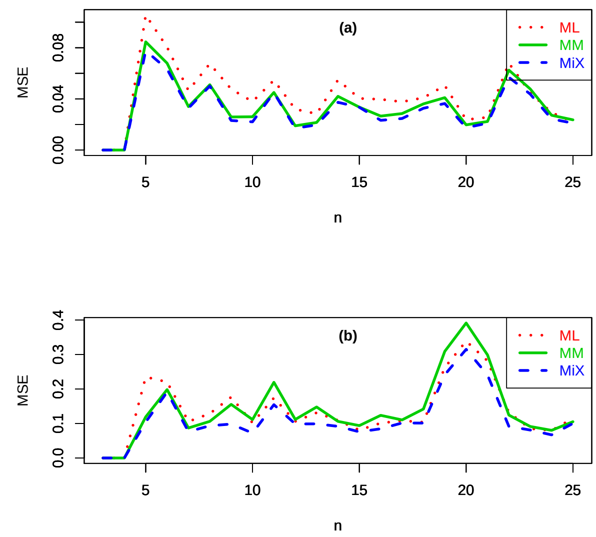

Fig. 5: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=3, α=0.5, β=1.5 and θ=0.5 and (b) for k=3, α=1, β=1.5 and θ=1

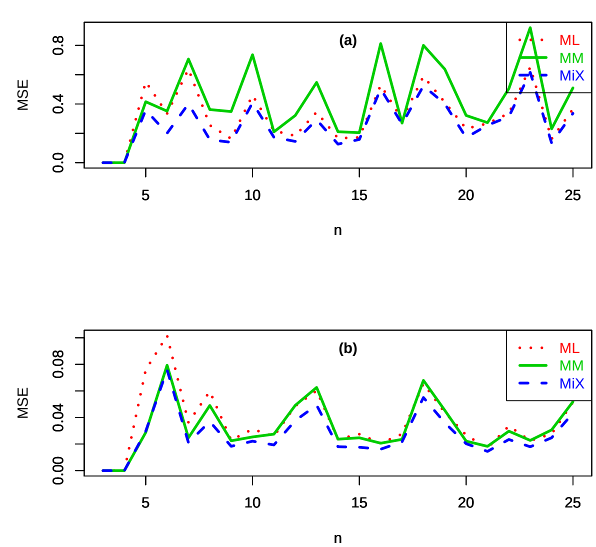

Fig. 6: Comparison the MSE of the estimators of pdf based on simulation results, (a) for k=3, α=1.5, β=1.5 and θ=1 and (b) for k=3, α=0.5, β=2 and θ=5

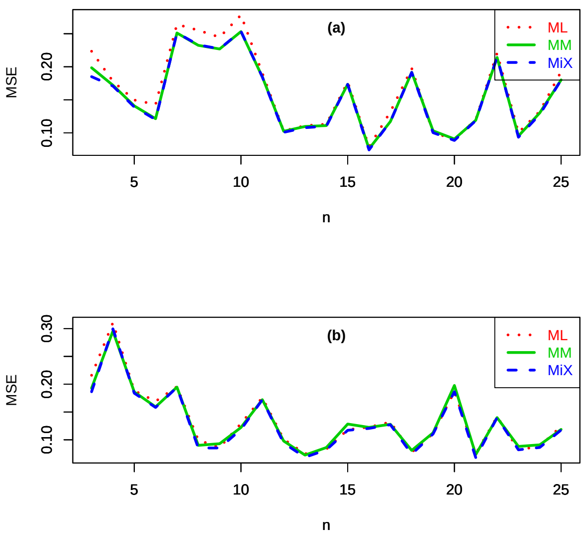

Fig. 7: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=1, α=0.5, β=1.5 and θ=0.5 and (b) for k=1, α=1, β=1.5 and θ=1

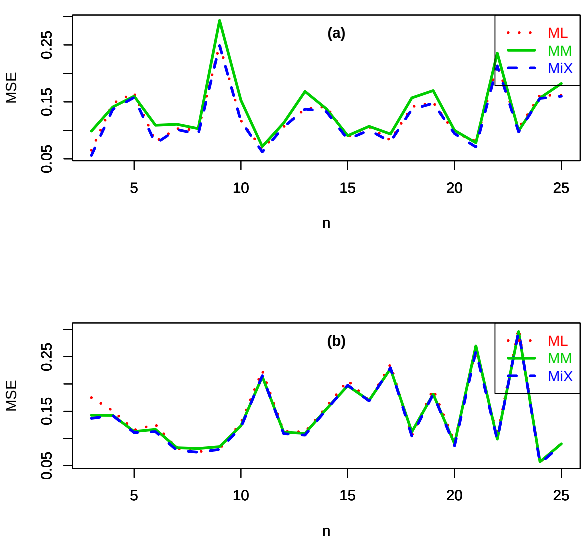

Fig. 8: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=1, α=1.5, β=1.5 and θ=1 and (b) for k=1, α=0.5, β=2 and θ=5

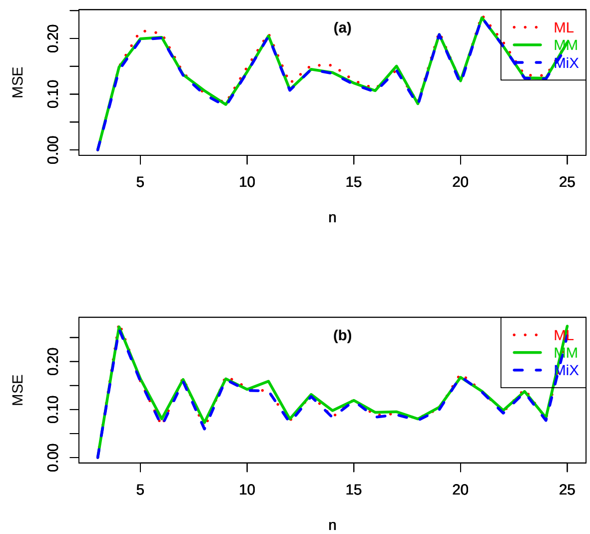

Fig. 9: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=2, α=0.5, β=1.5 and θ=0.5 and (b) for k=2, α=1, β=1.5 and θ=1

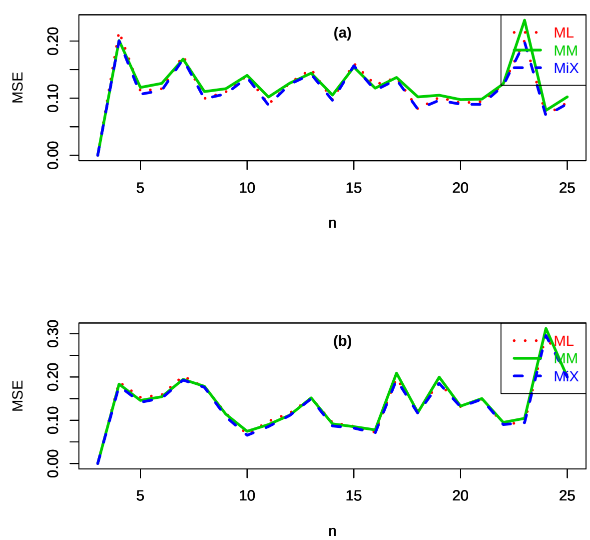

Fig. 10: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=2, α=1.5, β=1.5 and θ=1 and (b) for k=2, α=0.5, β=2 and θ=5

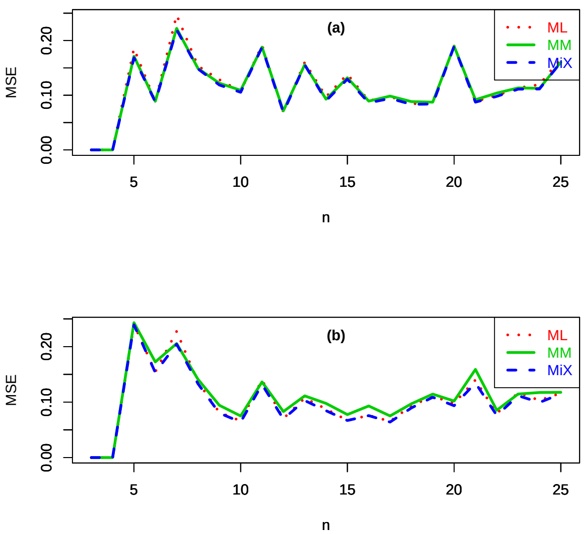

Fig. 11: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=3, α=0.5, β=1.5 and θ=0.5 and (b) for k=3, α=1, β=1.5 and θ=1

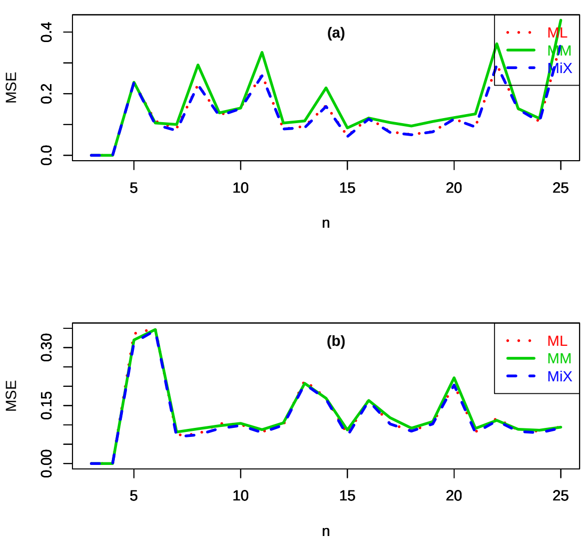

Fig. 12: Comparison the MSE of the estimators of cdf based on simulation results, (a) for k=3, α=1.5, β=1.5 and θ=1 and (b) for k=3, α=0.5, β=2 and θ=5

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