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Dynamics of Diseased Prey Predator Model with Nonlinear Feedback Control

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Abstract: In this paper we investigate the dynamics of diseased prey- predator system with nonlinear feedback. A nonlinear feedback mathematical model is proposed and analyzed to study the predator interaction with infected prey. We showed that the continuous time diseased prey-predator system can be asymptotically stabilized using nonlinear feedback control. By constructing Lyapunov function, global asymptotic stability is established. Also we obtained necessary control law for asymptotic stability of this system. Finally, a numerical simulation supports our analytical findings.

Keywords: Global Stability; Nonlinear Feedback Control; Prey-Predator System.

1 Introduction

Ecological population with diseases has been an area of interest for ecologists and mathematicians. Mathematical models of the communities that are afflicted by disease have been investigated on a wide range. This helps in analysis of interaction between the prey and predator populations. Most of the real world systems are non-linear and are described by differential equation. These systems are non-linear because of the reason that they do not abide the principle of superposition. Controlling the output of such systems by modifying its input parameter using feedback is referred as nonlinear feedback control. The general mathematical techniques that describe these systems include Lyapunov theory.

The influence of epidemics on predation was first studied by Anderson and May [1,2]. He examined a modification of Lotka-Volterra prey-predator model with higher predation and no reproduction on infected prey. They established that the invading disease tends to destabilize the prey-predator communities. Venturino [19] studied the dynamics of a system in which only one species get sick. Venturio [18] applied multiple modifications to the Lotka-Volterra model and accounted for the disease that spreaded among one of the species. The dynamics of Holling-Tanner model was investigated by Haque et al [7] with the assumption that the disease spread only among the prey. Kooi et al [14] exaggerated the behavior of Lotka-Volterra model that is infected by non-specified disease using Holling type II functional response to describe the transformation from susceptible to infected. Kundu et al [15] analyzed the prey-predator model with diseased prey and justified the global stability around the interior equilibrium points under specific conditions. The global stability of four prey-predator models was studied by Han et al [6]. Hadeler and Freedman [5] investigated the prey-predator model in which the infected prey is more likely to be predated. Numerous prey-predator models with infected prey have been explained by many researchers [4,13,16,17,20] and the models with infected predator was studied by researchers [8,9,10,11]. They considered that the predators become infected only by predating the infected prey. Haque and Venturino [12] discussed the models of symbiotic communities with disease.

The rest of the paper is structured as follows. In section 2 we present the mathematical model with basic consideration. Positivity and boundedness of the solution of the model are established in section 3. Section 4 deals with all the possible equilibrium points of the model. In section 5, the stability of the model at various at equilibrium points is discussed. Section 6 deals with the prey -predator model with nonlinear feedback control.

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Computer simulations are carried out to illustrate the analytical findings in section 7.

2 Mathematical model

In this paper,a continuous time prey-predator system with two preys viz. susceptible and infected prey and a predator is taken into account. It is assumed that the susceptible prey population is developed on the basis of logistic law and only the infected prey is predated. The disease is inherited only among the prey population and they remain infected and do not recover.

Now to mathematically describe the model of a prey-predator system with diseased prey population, we assume the following:

1. When the predator population vanishes and the absence of disease,susceptible prey population grows logistically with intrinsic growth rate r, carrying capacity k.
2. The occurrence of infection divides the prey population into two groups namely susceptible prey $X(t)$ and infected prey $Y(t)$ and the total population is $P(t) = X(t) + Y(t)$.
3. Only the prey population is diseased and is not genetically inherited and the infected prey populations do not recover or become immune.
4. The disease transmission follows the simple law of mass action given by $aX(t)Y(t)$ where a is the transmission rate.
5. The predator attacks only infected prey which has Beddington-De Angelis functional response which is of the form
   $f(Y,Z) = \dfrac{bZ}{1+bhY+Z}$
   Where b the total is attack rate for predator or predation coefficient and h is the handling time of predator to prey.

Therefore our model becomes:

$$\frac{dX}{dT} = rX\left(1-\frac{X}{k}\right) - aXY$$
$$\frac{dY}{dT} = aXY - \frac{bYZ}{1+bhY+Z} - d_1 Y$$
$$\frac{dZ}{dT} = \frac{ebYZ}{1+bhY+Z} - d_2 Z - \xi Z^2 \qquad (1)$$

With initial data

$X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0$ and the positive coefficients r,k,a,b,e,d_1,d_2,ξ in the model (1). Here the parameters $X(t),Y(t),Z(t)$ denote the susceptible, infected prey and predator population respectively. The parameters r,k,a,e denotes the growth rate of susceptible prey population, the environmental carrying capacity, the rate of transmission from susceptible to infected prey population and the conversion efficiency rate respectively. The parameters d_1,d_2 denotes death rate of infected prey and death rate of predator respectively.

To minimize the number of parameters involved with the model system, it is extremely useful to write the system in non-dimensionalized form. For this purpose introduce the variables X,Y and T as follow

$$x \to \frac{X}{k}, y \to \frac{Y}{k}, z \to \frac{Z}{bhk} \text{ and } t \to Tr \qquad (2)$$

In terms of the non-dimensionalized variables the model system (1) become

$$\frac{dx}{dt} = x(1-x) - \alpha xy$$
$$\frac{dy}{dt} = \alpha xy - \frac{\beta yz}{c+y+z} - \gamma_1 y$$
$$\frac{dz}{dt} = \frac{\delta yz}{c+y+z} - \gamma_2 z - \eta z^2 \qquad (3)$$

Where the relation between the dimensional and non-dimensional parameters are given by:

$$\alpha = \frac{ak}{r}, \beta = \frac{b}{r}, e = \frac{e}{rh}, \gamma_1 = \frac{d_1}{r}, \gamma_2 = \frac{d_2}{r}, \eta = \frac{bhk\xi}{r}$$

Now we will analyze the system of (3) with the following initial conditions:

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \qquad (4)$$

The equation (4) represents the conditions for positivity of susceptible prey, infected prey and predator populations respectively.

3 Positivity and Boundedness of Solution

It is significant to analyze the positivity and boundedness for the system (3). Positivity indicates the survival of population and boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources. The model system (3) can be put into the matrix form $X = F(X)$ where $X = (x,y,z)^T \in R^3$ and $F(X)$ is given by

$$F(X) = \begin{pmatrix} F_1(X_1) \\ F_2(X_2) \\ F_3(X_3) \end{pmatrix} = \begin{pmatrix} x(1-x) - \alpha xy \\ \alpha xy - \dfrac{\beta yz}{c+y+z} - \gamma_1 y \\ -\eta z^2 + \dfrac{\delta yz}{c+y+z} - \gamma_2 z \end{pmatrix}$$

Let $R^3_+ = [0,+\infty)^3$ be the non-negative octant in R^3, the $G : R^3_+ \to R^3$ is locally Lipschitz and satisfy the condition $G_i(X)|_{X_i(t)=0}, X \in R^3_+ \geq 0$
Where $X_1 = x, X_2 = y, X_3 = z$
Due to Lemma in [15] and any solutions of (3) with positive initial conditions exist uniquely and each component of X remains the interval $[0,b)$ for some $b > 0$. Furthermore, if $b < +\infty$, then $\limsup[x(t)+y(t)+z(t)] = +\infty$.

Theorem 3.1.

All the positive solutions of the model system (3) that state in $\Re^3_+$ are uniformly bounded.

Proof.

Since the densities of population can never be negative, obviously the solutions $x(t), y(t)$ and $z(t)$ are positive for all $t \geq 0$.
From the first equation of model (1), we have

$$\frac{dx}{dt} \leq x(1-x)$$

This gives $\lim_{t\to\infty} \sup x(t) \leq 1$

Consider $L = x + y + z$
Then

$$\frac{dL}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \tag{5}$$

Substituting (3) in equation (5), we get

$$\frac{dL}{dt} + mL \leq x(1+m) + (m-\gamma_1)y + (m-\gamma_2)z$$

If $m < \gamma_1, m < \gamma_2$ then

$$\frac{dL}{dt} + mL \leq (1+m)$$
$$\leq \phi$$

Applying Lemma on differential inequalities Birkoff [3], we obtain

$$0 \leq L(x,y,z) \leq \frac{\phi}{m}(1-e^{-mt}) + \frac{L(x(0),y(0),z(0))}{e^{mt}}$$

Thus for $t \to \infty$ we have
$$0 \leq L(x,y,z) \leq \frac{\phi}{m}$$

Thus all solutions of system (3) enter into the region

$$\Gamma = \left\{(x,y,z) \in R^3_+ : 0 \leq x \leq 1, 0 \leq L \leq \frac{\phi}{m} + \varepsilon, \forall \varepsilon > 0\right\}$$

This completes the proof.

4 Existence of Equilibrium Points

The existence of equilibrium points for the system (3) as follows:

1. The trivial equilibrium point $E_0(0,0,0)$ always exists.
2. In no predation condition, since the prey population expands to the carrying capacity, the fixed point $E_1(1,0,0)$ always occurs.
3. The survival of both prey species is ensured in the absence of predator species. Hence the fixed point $E_2(\bar{x},\bar{y},0)$ exists in the interior of positive quadrant of xy plane, where $\bar{x},\bar{y}$ are given as follows: $\bar{x} = \frac{\gamma_1}{\alpha}, \bar{y} = \frac{\alpha-\gamma_1}{\alpha^2}$, provided that $\alpha > \gamma_1$.
4. The positive stationary point $E_3(x^*,y^*,z^*)$ occurs in the interior of the first octant if and only if the following algebraic non-linear system yields a positive solution.

$$1 - x - \alpha y = 0$$
$$\alpha x - \gamma_1 - \frac{\beta z}{c+y+z} = 0$$
$$\frac{\delta y}{c+y+z} - \eta z - \gamma_2 = 0 \tag{6}$$

Solving the system of equation (6) we get
If x^* is fixed (independent) and y^*, z^* becomes
$y^* = \frac{1-x^*}{\alpha}$ with $1 > x^*$
$$z^* = \frac{(\alpha x^* - \gamma_1)(1 + c\alpha - x^*)}{\alpha(\beta + \gamma_1 - \alpha x^*)}$$
provided with the conditions $\alpha x^* > \gamma_1, \beta + \gamma_1 > \alpha x^*$ and $1 + c\alpha > x^*$.

5 Local and Global Stability Analysis

5.1 Local Stability Analysis

In this section the stability of the system (3) is examined by constructing the variational matrix relating to every equilibrium point.

Lemma 5.1.

The stationary point E_0 is asymptotically stable in the $y-z$ direction and is unstable in x direction.

Proof.

The variational matrix for the equilibrium point at $E_0(0,0,0)$ is

$$V(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\gamma_1 & 0 \\ 0 & 0 & -\gamma_2 \end{pmatrix}$$

The eigenvalues of E_0 are
$\lambda_1 = 1, \lambda_2 = -\gamma_1, \lambda_3 = -\gamma_2$
Since λ_2, λ_3 are negative, E_0 is asymptotically stable in the $y-z$ direction and since λ_1 is positive, E_0 is unstable in x direction. This completes the proof.

Lemma 5.2.

The equilibrium point E_1 is asymptotically stable in the $x-y-z$ plane if $\alpha < \gamma_1$. But if $\alpha > \gamma_1$ in this case it is stable in $y-z$ direction and is unstable y direction.

Proof.

The variational matrix for the equilibrium point at $E_1(1,0,0)$ is

$$V(E_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \alpha-\gamma_1 & 0 \\ 0 & 0 & -\gamma_2 \end{pmatrix}$$

The eigenvalues of E_1 are
$\lambda_1 = -1, \lambda_2 = \alpha - \gamma_1, \lambda_3 = -\gamma_2$.
If $\alpha < \gamma_1$, in this case all the eigenvalues are negative. Hence E_1 is asymptotically stable in the $x-y-z$ direction. But if $\alpha > \gamma_1$ in this case two of the eigenvalues λ_1, λ_3 are negative so it is stable in $x-z$ direction and unstable y direction. This completes the proof.

Lemma 5.3.

The equilibrium point E_2 is asymptotically stable in the $x-y-z$ plane if $\alpha > \gamma_1$ and $c\alpha^2+\alpha < \gamma_1$.

Proof.

The variational matrix for the equilibrium point at E_2 is

$$V(E_2) = \begin{pmatrix} \frac{-\gamma_1}{\alpha} & -\gamma_1 & 0 \\ 1-\frac{\gamma_1}{\alpha} & 0 & \frac{\beta(\alpha-\gamma_1)}{c\alpha^2+\alpha-\gamma_1} \\ 0 & 0 & \frac{\delta(\alpha-\gamma_1)}{c\alpha^2+\alpha-\gamma_1}-\gamma_2 \end{pmatrix}$$

The corresponding characteristic equation for E_2 is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{7}$$

Where

$$p_1 = \frac{\gamma_1}{\alpha} + \frac{\delta(\alpha-\gamma_1)}{c\alpha^2+\alpha-\gamma_1} + \gamma_2$$

$$p_2 = \frac{\gamma_1\gamma_2}{\alpha} - \frac{\delta(\gamma_1-\alpha)}{c\alpha^2+\alpha-\gamma_1} + \gamma_1\left(1-\frac{\gamma_1}{\alpha}\right)$$

$$p_3 = \frac{\delta(\gamma_1-\alpha)\gamma_1^2}{\alpha(c\alpha^2+\alpha-\gamma_1)} - \frac{\delta(\alpha-\gamma_1)\gamma_1}{c\alpha^2+\alpha-\gamma_1} - \frac{\gamma_1^2\gamma_2}{\alpha} + \gamma_1\gamma_2$$

By using Routh-Hurwitz criteria

If $p_1 > 0, p_3 > 0$ and $p_1p_2 - p_3 > 0$ then E_2 is locally asymptotically stable.

straight forward calculation gives if $\alpha > \gamma_1$ and $c\alpha^2+\alpha < \gamma_1$ then E_2 is locally asymptotically stable.

Theorem 5.1.

The positive equilibrium point E_3 is locally asymptotically stable if satisfy the following conditions

(i) $1-\alpha y^* < 2x^*$.

(ii) $\alpha x^* < \gamma_1 + \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}$.

(iii) $\delta(cy^*+y^{*^2}) < 2\eta z^*(c+y^*+z^*)^2 + \gamma_2(c+y^*+z^*)^2$.

Proof.

The variational matrix of (3) at E_3 is given below:

$$E_3 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

Where

$a_{11} = 1-2x^*-\alpha y^*; a_{12} = -\alpha x^*;$

$$a_{21} = \alpha y^*; a_{22} = \alpha x^* - \gamma_1 - \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2};$$

$$a_{23} = \frac{\beta(cy^*+y^{*^2})}{(c+y^*+z^*)^2};$$

$$a_{32} = \frac{\delta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}; a_{33} = \frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2} - 2\eta z^* - \gamma_2$$

Then corresponding characteristic equation becomes

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{8}$$

Where

$$A_1 = -(a_{11}+a_{22}+a_{33})$$

$$= -(1-2x^*-\alpha y^*) + \left(\alpha x^* - \gamma_1 - \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right) + \left(\frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2} - 2\eta z^* - \gamma_2\right)$$

$$= \left(2x^* + \alpha y^* + 2\eta z^* + \gamma_2 + \gamma_1 + \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right) - \left(\frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2} + 1 + \alpha x^*\right)$$

$$A_2 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{32}$$

$$= \left[(1-2x^*-\alpha y^*)\left(\alpha x - \gamma_1 - \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right)\right] + \left[\left(\alpha x^* - \gamma_1 - \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right) \times \left(\frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2} - 2\eta z^* - \gamma_2\right)\right] + \left[(1-2x^*-\alpha y^*)\left(\frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2} - 2\eta z^* - \gamma_2\right)\right] + [\alpha^2 x^* y^*] - \left[\frac{\left(\beta\delta(cy^*+y^{*^2})(cz^*+z^*)^2\right)}{(c+y^*+z^*)^4}\right]$$

$$A_3 = -[(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - a_{12}(a_{21}a_{32})]$$

$$= a_{11}a_{23}a_{32} + a_{12}a_{21}a_{32} - a_{11}a_{22}a_{33}$$

$$= \left[(1-2x^*-\alpha y^*)\left[\frac{(\beta\delta(cy^*+y^{*^2})(cz^*+z^{*^2}))}{(c+y^*+z^*)^4}\right]\right] + \left[\alpha x^*\left(\frac{-\alpha\delta y^*(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right)\right] - \left[(1-2x^*-\alpha y^*)\left(\alpha x^* - \gamma_1 - \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}\right) \times \left(\frac{\delta(cy^*+y^{*^2})}{(c+y^*+z^*)^2}\right)\right]$$

According to Routh Hurwitz criterion, for all values of the parameters, we notice that

$A_1 > 0, A_3 > 0$ and $\Delta = A_1A_2 - A_3 > 0$.

A straight forward calculation gives

If $a_{11} < 0$ gives

$$1-\alpha y^* < 2x^*$$

If $a_{22} < 0$ gives

$$\alpha x^* < \gamma_1 + \frac{\beta(cz^*+z^{*^2})}{(c+y^*+z^*)^2}$$

If $a_{33} < 0$ gives Therefore, the positive interior equilibrium point is locally asymptotically stable. Hence, the theorem is proved.

5.2 Global Stability Analysis

In this section, we shall study the global dynamics of the system (3) around the equilibrium point E_2 and the positive equilibrium point $E_3(x^*,y^*,z^*)$.

Theorem 5.2.
The interior equilibrium E_2 is globally asymptotically stable in the interior of the quadrant of $x-y$ plane.

Proof.

$$\text{Let } H(x,y) = \frac{1}{xy} \tag{9}$$

Clearly $H(x,y)$ is positive in the interior of the positive quadrant of $x-y$ plane.
$h'(x,y) = x(1-x) - \alpha xy$
$h''(x,y) = \alpha xy - \gamma_1 y$

$$\text{Then } \Delta(x,y) = \frac{\partial}{\partial x}(h'H) + \frac{\partial}{\partial y}(h''H)$$
$$= -\frac{1}{y} < 0 \tag{10}$$

By using Bendixson-Dulac criteria, it is brought to light that $\Delta(x,y)$ remains with the same sign and is not identically zero in the interior of the positive quadrant of the $x-y$ plane. This completes the proof.

Theorem 5.3.
The co-existence equilibrium point $E_3(x^*,y^*,z^*)$ is globally asymptotically stable with respect to all solutions initiating from the interior of Γ, satisfying the following conditions $x < x^*$ and $zy^* > z^*y$.

Proof.
The sufficient conditions for proving the theorem is given by Lyapunov stability theorem.
Now let us define

$$L = S\left[x - x^* - x^* ln\left(\frac{x}{x^*}\right)\right] + T\left[y - y^* - y^* ln\left(\frac{y}{y^*}\right)\right]$$
$$+U\left[z - z^* - z^* ln\left(\frac{z}{z^*}\right)\right] \tag{11}$$

Where S, T, U are positive constant to be chosen later

$$\frac{dL}{dt} = S\left[\frac{x-x^*}{x}\right]\frac{dx}{dt} + T\left[\frac{y-y^*}{y}\right]\frac{dy}{dt} + U\left[\frac{z-z^*}{z}\right]\frac{dz}{dt} \tag{12}$$

$$= S[(1-x) - \alpha y](x-x^*)$$
$$+T\left[\frac{-\beta z}{c+y+z} - \gamma_1 + \alpha x\right](y-y^*)$$
$$+U\left[\frac{\delta y}{c+y+z} - \gamma_2 - \eta z\right](z-z^*)$$
$$= S[(x-x^*) - \alpha(y-y^*)](x-x^*)$$
$$+T\left[\alpha(x-x^*) - \left[\frac{\beta z}{c+y+z} - \frac{\beta z^*}{c+y^*+z^*}\right]\right](y-y^*)$$
$$+U(z-z^*)\left[\frac{\delta y}{c+y+z} - \frac{\delta y^*}{c+y^*+z^*} - \eta(z-z^*)\right]$$
$$= S[-(x-x^*)](x-x^*)$$
$$+T\left[-\left[\frac{\beta(c(z-z^*) + (zy^* - z^*y))}{(c+y+z)(c+y^*+z^*)}\right]\right](y-y^*)$$
$$+U(z-z^*)\left[\frac{\delta(c(y-y^*) + (yz^* - y^*z))}{(c+y+z)(c+y^*+z^*)} - \eta(z-z^*)\right] \tag{13}$$

We choose the parameters $T = \frac{U\delta}{\beta}, U = \frac{1}{2}, S = 1, T = 1, \delta = 1$ and substitute in (13) then we get

$$\frac{dL}{dt} = -(x-x^*)^2 - \eta(z-z^*)^2$$
$$-\frac{(y^*z - yz^*)}{(c+y+z)(c+y^*+z^*)}(y-y^*)(z-z^*) \tag{14}$$

Then using the given condition, we see that $\frac{dL}{dt}$ is negative definite. L is a Lyapunov function with respect to all solutions in the interior of the positive octant which proves the theorem.

6 A Prey Predator Model with Nonlinear Feedback Controls

Theorem 6.1.
With the following nonlinear controllers, at the fixed point, the system (3) is found to be stable
$u_1 = \alpha xy - 2x + x^2$

$$u_2 = \frac{\beta yz}{c+y+z} - \alpha xy$$
$$u_3 = \eta z^2 - \frac{\delta yz}{c+y+z} \tag{15}$$

Proof.
The biological model with diseased prey (3) controlled by nonlinear feedback is described by

$$\frac{dx}{dt} = x(1-x) - \alpha xy + u_1$$
$$\frac{dy}{dt} = \alpha xy - \gamma_1 y - \frac{\beta yz}{c+y+z} + u_2$$
$$\frac{dz}{dt} = \frac{\delta yz}{c+y+z} - \gamma_2 z - \eta z^2 + u_3 \tag{16}$$

Where x, y, z are the state variables and $\alpha, \beta, \delta, \gamma_1, \gamma_2, \eta$ are positive parameters and u_1, u_2, u_3 are feedback controllers which are the functions of the state variables.

These control feedbacks stabilize the system (16) and converges it to zero as the time goes to infinity.
(i.e.) $\lim_{t\to\infty} ||x(t)|| = 0$

The Lyapunov function is taken as

$$F(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \quad (17)$$

Differentiating (17) along the trajectories of the system (3) gives

$$\dot{F}(x,y,z) = x(x(1-x) - \alpha xy + u_1) + y\left(\alpha xy - \gamma_1 y - \frac{\beta yz}{c+y+z} + u_2\right) + z\left(\frac{\delta yz}{c+y+z} - \eta z^2 - \gamma_2 z + u_3\right) \quad (18)$$

Substituting the equations (15) in (18), then we get $\dot{F}(x,y,z) = -x^2 - \gamma_1 y^2 - \gamma_2 z^2$ which is a negative definite function.

Thus the prey-predator system with Beddington-De-Angelis functional response having competition in predator alone and mortality in both predator and infected prey is proved to be globally asymptotically stable.

7 Numerical Solution

Analytical studies justify the novelty of the theoretical results. A qualitative analysis of the main features in the system is described by numerical simulations. The analytical results are gathered for three sets of parameter. It is found thatunique results are obtained for eachunique set of parameter gives.

The numerical simulation of phase portraits and the corresponding time series graph of the system (3) gives its complex dynamical behavior. The value assumed are $\beta = 0.9, \delta = 0.5, \gamma_1 = 0.3, \gamma_2 = 0.13, \eta = 0.36, c = 0.5$ and the initial densities are $x(0) = 1, y(0) = 0.8, z(0) = 0.6$. The figure (1), shows the dynamic behavior of the uncontrolled system (3) when $\alpha = 0.293$ and figure (2) gives the corresponding phase plot when the system approaches the equilibrium point E_2. The numerical results for the same value of parameters $\beta = 0.9, \delta 1.478, \gamma_1 = 0.4, \gamma_2 = 0.4, \eta = 0.23, c = 0.2$ and same initial densities $x(0) = 1, y(0) = 0.8, z(0) = 0.6$ altering only the disease transmission rate α.

In figure (3), it is visualized that when the disease transmission rate is $\alpha = 3$ the density of infected prey decreases with an increase in density of the susceptible prey. Figure (4) gives the corresponding phase plot. When the disease transmission rate is further reduced to $\alpha = 1.9$, in figure (5), there is a deflation in the density of infected prey indicating a deflation in predator population and an inflation in the density of susceptible prey. Its corresponding phase plot is given in figure (6).

In figure (7) the population density approaches the stable point quickly. All the population densities in the system (16) approach approximately zero for the above disease transmission values.

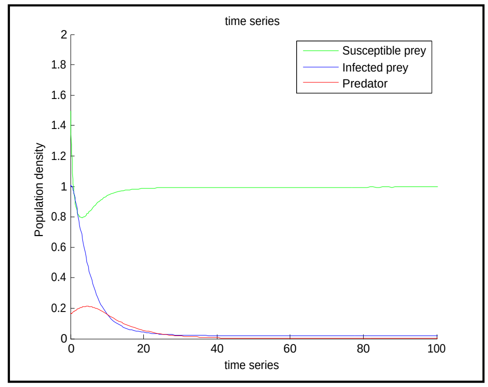

Fig. 1: Time series of the system (3) approaches asymptotically to (E_2) for $\alpha = 0.293$

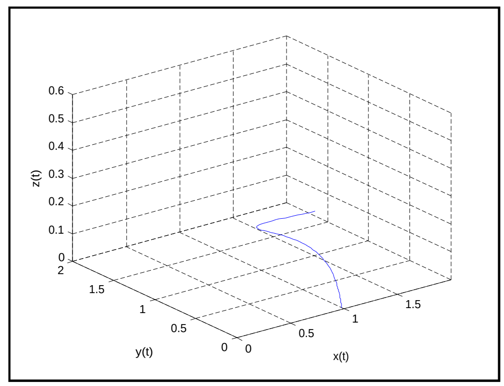

Fig. 2: Phase portrait when $\alpha = 0.293$ for the system (3) approaches asymptotically to E_2

This proves the theoretical work by Lyapunov stability theory (Hahn, 1967), the dynamics (16) is globally asymptotically stable and hence the condition $\lim_{t\to\infty} ||x(t)|| = 0$ will be satisfied for all initial conditions $x(0) \in \Re^n$.

8 Conclusion

The eco-epidemiological interactions in the predator-prey model with disease in prey species have been investigated and its dynamical behavior has been analyzed. The three different classes of populations viz the susceptible prey, the infected prey and the predator have been described by unique ordinary differential equations. The continuous time diseased prey-predator system has been asymptotically stabilized using nonlinear feedback control. These stability conditions at various equilibrium

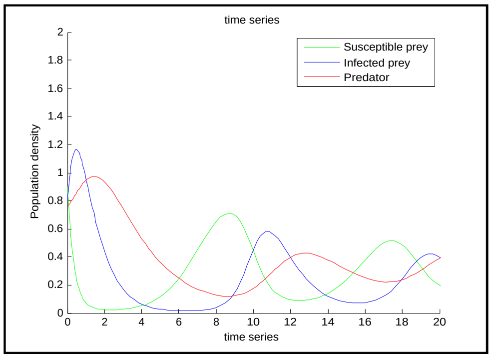

Fig. 3: Time series of the system (3) for $\alpha = 3$

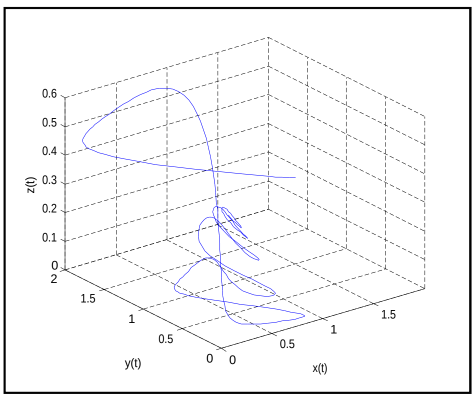

Fig. 4: Phase portrait of the system(3) when $\alpha = 3$

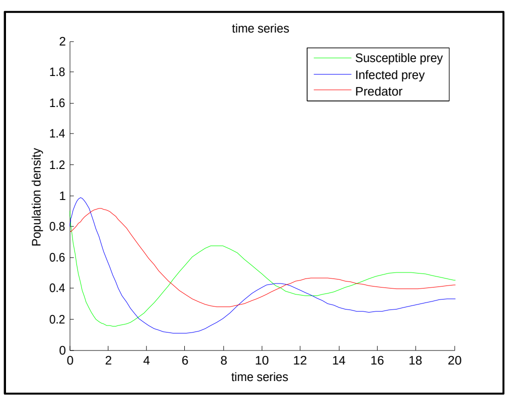

Fig. 5: Time series of the system (3) approaches to (E_3) for $\alpha = 1.9$

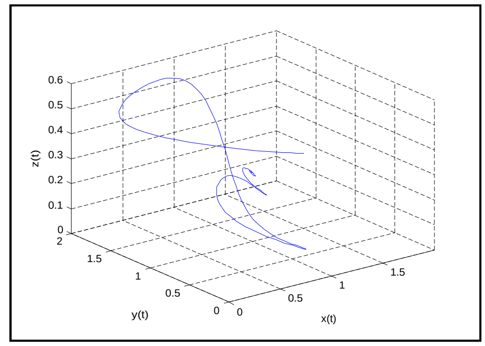

Fig. 6: Phase portrait of the system (3) for $\alpha = 1.9$

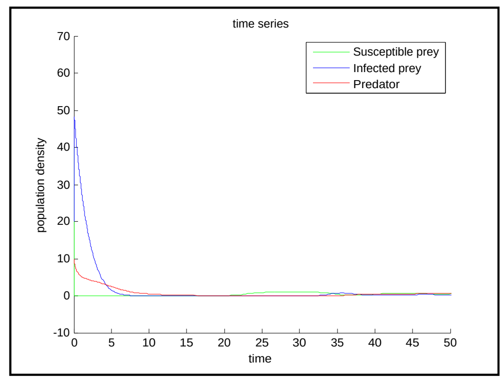

Fig. 7: Time series of the system (16) approaches asymptotically to $(0,0,0)$ for $\alpha = 1.9$

points and the boundedness of the solution have been examined. Lyapunov function is used to illustrate the asymptotic stability of the controlled system. The results obtained from the numerical simulations conclude that a decrease in the contact rate between susceptible prey and infected prey increases the density of susceptible prey but decreases the density of predator.

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