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On Bornological Semi Rings

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Abstract: Our main focus in this work is to generalize the theory of algebraic semi rings from the algebraic setting to the framework of bornological sets. More specifically, the concept of a new structure bornological semi ring (BSR) is introduced and some constructions in the class of bornological semi rings are discussed. In particular, the existence of arbitrary projective limits and arbitrary inductive limits of bornological semi rings is ensured. Additionally, the description of the category of bornological semi rings is presented. We also discuss the concept of product, coproduct and fibre product in the category of bornological semi ring. In the context under consideration, general results concerning projective limits and inductive limits as well as an isomorphism theorem are established.

Keywords: Bornological Set, Bornological Ring

1 Introduction

A bornological space is a type of space in functional analysis and it is very useful to solve the problem of boundedness for a set of elements and functions. That means the main goal for bornology is to determine the boundedness, location and setting the boundary for any area. A bornology on a set X is a family β of subsets of X, such that;

–Every singleton set $\{x\}$ is in β;
–If $A \subset B$, and $B \in \beta$ then $A \in \beta$;
–β is stable under a finite union.

The elements of β are called bounded subsets of X. Let X and Y be two bornological sets. A map $f : X \to Y$ is called bounded if the image of every bounded subset of X is bounded in Y.
Starting the year 2012 P.Pombo in [8] began to study some of fundamental constructer of the so-called bornological groups (BG). Thus, the main goal of algebraic bornology is to deal with the problems of boundedness of algebraic structures. In [6] the concept of bornological semigroups was introduced. In [7] conditions to introduce a bornology on a group were discussed. In [1] Bambozzi introduces and studies the boundedness of rings. A bornological ring is a set equipped with both ring and a bornology structures with certain compatibility axioms. In this paper the concept of bornological semi ring (BSR) is introduced and some constructions in the class of bornological semi rings are discussed. In particular, the existence of arbitrary projective and inductive limits of bornological semi ring is ensured. We consider the product, coproduct and fiber product in the category of bornological semi ring. In the context under consideration, general results concerning projective and inductive limits as well as an isomorphism theorem are established.
This research consists of four sections. Section two contains the basic facts on bornological groups. In Section three we study BSR also give some results. In Section four some constructions of BSR are discussed. The study of bornological structures has been carried out in various context, as one may see in the references ([2], [3], [5], [8], [9]).

2 Basic Definition and terminology

In this section, we review some results on bornological sets, groups and semigroups. We start with the following theorems and concepts from [4].

Theorem 21
Let I be a non-empty set, $(X_i, \beta_i)_{i\in I}$ be a family of bornological sets indexed by I and X be a set. Suppose that, for every $i \in I$, a map $v_i : X \longrightarrow X_i$ is given.

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Consider the family β of all subsets B of X such that for any $i \in I$ the image $v_i(B)$ is bounded in X_i. Then β is a bornology on X.

The bornology β on X defined by the theorem above is called the initial bornology on X for the maps $v_i, i \in I$.

Theorem 22

Let I be a non-empty set, $(X_i, \beta_i)_{i\in I}$ be a family of bornological sets and X be a set. Suppose that, for every $i \in I$, a map $u_i : X_i \longrightarrow X$ is given and let β be the bornology on X generated by family $A = \bigcup_{i\in I} u_i(\beta_i)$. Then β is the finest bornology on X for which each map u_i is bounded.

Definition 23

Let (X, β) be a bornological set and $Y \subseteq X$. Then the family of subset of Y defined by $\beta_Y = \{B \bigcap Y : B \in \beta\}$ is a bornology on Y and (Y, β_Y) is said to be a bornological subset of (X, β).

A subfamily β_0 of β is said to be a base for the bornology β, if every element of β is contained in an element of β_0. In other words, β_0 is a base for a bornology on X if and only if β_0 covers X and every finite union of elements of β_0 is contained in a member of β_0.
A bornology β_1 on a set X is a finer bornology than a bornology β_2 on X (or β_2 is a coarser bornology than β_1) if

$$\beta_1 \subset \beta_2.$$

Now we define the notion of a bornological group.

Definition 24

A bornological group is a set G with two structures.

1. *G is a group;*
2. *G is a bornological set such that these two structures are compatible as follows.*
   (i) the product function $f : G \times G \longrightarrow G$ is bounded;
   (ii) the inverse function $h : G \longrightarrow G$ is bounded.

The boundedness conditions above are equivalent to the following condition: the function $f : G \times G \longrightarrow G$ such that $f(g_1, g_2) = g_1 g_2^{-1}$ for each $g_1, g_2 \in G$ is bounded.

Definition 25

A bornological semigroup is a nonempty set S with two structures:

1. *S is a semigroup with a binary operation f;*
2. *S is a bornological set.*

Such that the function $f : S \times S \longrightarrow S$ is bounded.

We illustrate this by example.

Example 26

Let us consider $S = (\mathbb{N}, +)$ and β be the family of all finite subsets of $\mathbb{N}$. Then $\mathbb{N}$ is a bornological semigroup.

3 Bornological Semi Rings

In this section, we introduce the concept of bornological semi ring and study the basic properties of the bornological semi rings. As we mention earlier, the concept of bornological ring is a set equipped with two structures ring and bornology such that

1. $(R, +, \beta)$ is a bornological group;
2. $(R, \cdot, \beta)$ is a bornological semigroup.

Definition 31

A semi ring $(A, +, \cdot)$ and a bornology β on A is said to be a bornological semi ring if

1. *$(A, +, \beta)$ is a bornological semigroup;*
2. *$(A, \cdot, \beta)$ is a bornological semigroup.*

Example 32

1. *Consider the commutative ring $(\mathbb{Z}, +, \cdot)$ and a bornology $\beta = \{B \subset \mathbb{Z} : B \subset (-\infty, b],$ for $b \in \mathbb{Z}\}$ on $\mathbb{Z}$ generated by the half line*

   $$\{n \in \mathbb{Z} : n \leq a, \text{ for } a \in \mathbb{Z}\}.$$

   $(\mathbb{Z}, +, \cdot)$ is bornological semi ring, i.e., the product map with respect to the addition and the multiplication are bounded

   $$f : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}.$$

   In other words, for any two elements $B_1, B_2 \in \beta$ we have $B_1 \subset (-\infty, a], B_2 \subset (-\infty, b]$. Then $B_1 + B_2 \subset (-\infty, a+b]$ and therefore $B_1 + B_2 \in \beta$. In the same way it can be proved that $B_1 \cdot B_2 \in \beta$. However, $(\mathbb{Z}, \beta)$ is not a bornological ring since $(\mathbb{Z}, +, \beta)$ is not bornological group but it is a bornological semi ring.
2. *Consider the ring $(\mathbb{R}, +, \cdot)$ let β be the bornology on $\mathbb{R}$ generated by the sets $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ where $a, b \in \mathbb{R}$ and $a < b$. The addition and multiplication on $\mathbb{R}$ are bounded and $(\mathbb{R}, \beta)$ is a bornological semi ring.*

Example 33

1. *Let A be any semi ring, the bornology consisting of all finite subsets of A is a semi ring bornology.*
2. *If R is a Hausdorff topological ring, then the bornology consisting of all relatively compact subsets of R is a semi ring bornology.*

Proposition 34

Let A be a semi ring and β a bornology on A. Then β is a semi ring bornology if and only if there exists a basis $\{B_i : i \in I\}$ for β such that for every $i, j \in I$ $\exists k_1, k_2 \in I$, $B_i + B_j \subset B_{i1} \cup \ldots \cup B_{in} = B_{k_1}$ and $B_i \cdot B_j \subset B_{i1} \cup \ldots \cup B_{in} = B_{k_2}$.

Proof.
If (A,β) is a bornological semi ring then

$$B_i + B_j$$

and

$$B_i \cdot B_j$$

are bounded and hence must be contained in a finite union of the elements of the basis by the concept base of the bornology.
On the other hand, if (A,β) has such a basis, then for any two bounded subsets $B_1, B_2 \in \beta$ we have that

$$B_1 \subset B_{i_1}$$

and

$$B_2 \subset B_{i_2}$$

for some $i_1, i_2 \in I$. Then

$$B_1 + B_2 \subset B_{i_1} + B_{i_2} \subset B_{k_1}$$

$$B_1 \cdot B_2 \subset B_{i_1} \cdot B_{i_2} \subset B_{k_2}$$

for some k_1, k_2, which shows that additive and multiplication are bounded maps.

Remark 35
Any finite semi rings is a bornological semi ring with the collection of all subsets of A.

However, we need the following property to clarify the property of base for this structure.

Proposition 36
Let β_0 be a base for a semi ring bornology β on A and let β_0^* be a family of bounded sets containing β_0, i.e., $\beta_0 \subset \beta_0^* \subset \beta$. Then that β_0^* is also a base of β.

Proof.
Let B be a bounded subset of (A,β). Since β_0 is a base for (A,β), then B is contained in an element of β_0 by the concept of base, i.e. $B \subset B_0$ where $B_0 \in \beta_0$. But $\beta_0 \subset \beta_0^* \subset \beta$; then each $B_0 \in \beta_\circ$ also belongs to β_0^*. So B is contained in an element of β_0^*, and hence β_0^* is also a base for (A,β).

Theorem 37
Let (A,β) be a bornological semi ring and $f : A' \to A$ be a homomorphism of semi rings. Then the bornology $\beta_f = \{B \subset A' : f(B) \in \beta\}$ is a semi ring bornology on A'.

Proof.
Let $B_1, B_2 \in \beta_f$ i.e., $f(B_1), f(B_2) \in \beta$. Since β is a bornological semi ring and f is a homomorphism, one has

$$f(B_1 + B_2) = f(B_1) + f(B_2) \in \beta, \text{ hence } B_1 + B_2 \in \beta_f,$$

and

$$f(B_1 \cdot B_2) = f(B_1) \cdot f(B_2) \in \beta, \text{ hence } B_1 \cdot B_2 \in \beta_f.$$

Thus β_f is a semi ring bornology on A'.

Furthermore, we give a sufficient condition to the codomain of bornological semi ring to be bornological semi ring.

Theorem 38
Let (A,β) be a bornological semi ring and $f : A \to A'$ be an epimorphism of semi rings. Then

$$\beta_f = \{B \subset A' : f^{-1}(B) \in \beta\}$$

is a semi ring bornology on A'.

Proof.
Let $B_1, B_2 \in \beta_f$. Then there exist $B_1^A, B_2^A \in \beta$ such that $f(B_1^A) = B_1$ and $f(B_2^A) = B_2$ so $f(B_1^A + B_2^A) = f(B_1^A) + f(B_2^A) = B_1 + B_2$ and $f(B_1^A \cdot B_2^A) = f(B_1^A) \cdot f(B_2^A) = B_1 \cdot B_2$, therefore β_f is a semi ring bornology on A'.

Proposition 39
Let $(A,\beta_i)_{i\in I}$ be a collection of a bornological semi ring, then $\bigcap\limits_{i\in I} \beta_i$ is a semi ring bornology on A.

Proof.
It is known that $\beta = \bigcap\limits_{i\in I} \beta_i$ is a bornology, we must show that it also is a semi ring bornology. Let $B_1, B_2 \in \beta$, then $B_1, B_2 \in \beta_i$ for any $i \in I$, hence $B_1 + B_2 \in \beta_i$ and $B_1 \cdot B_2 \in \beta_i$ since β_i are semi rings bornology. Therefore $B_1 + B_2 \in \beta$ and $B_1 \cdot B_2 \in \beta$, i.e., the additive and multiplication operation are bounded.

A morphism of bornological semi rings $\psi : (A,\beta_A) \to (A',\beta_{A'})$ is defined to be a bounded semi ring homomorphism if $\psi(B) \in \beta_{A'}$ for all $B \in \beta_A$. It is clear that the composition of two bounded semi rings homomorphisms is bounded, so we can define the category of bornological semi rings denote **BornSR**, with objects bornological semi rings and arrows bounded semi ring homomorphisms.
The category of bornological semi rings admits limits, colimits given as follows. Let $(A_i,\beta_i)_{i\in I}$ be a family of bornological semi rings, then the direct product $\prod\limits_{i\in I}(A_i,\beta_i)_{i\in I}$ is defined to be the semi ring $\prod\limits_{i\in I} A_i$ equipped with the bornology generated by the family

$$\prod_{i\in I} \beta_i = \{B : B \subset \prod_{i\in I} B_i, B_i \in \beta_i\}.$$

This bornology is the coarser bornology on $\prod\limits_{i\in I} A_i$ such that all the projections

$$\prod_j : \prod_{i\in I} A_i \to A_j$$

are bounded. Moreover we can describe the coproduct in **BornSR**, by endowing $\bigoplus\limits_{i\in I} A_i$ with the bornology

$$\bigoplus_{i\in I} \beta_i = \{B : B \subset \bigoplus_{i\in I} B_i,\ B_i \in \beta_i,\ B_i = 0 \; for\, almost\, all\; i \in I\},$$

we get the coproduct in **BornSR**. In this setting we have that the canonical projections

$$\prod_j : \prod_{i\in I} A_i \to A_j$$

and the canonical injections

$$i_j : A_j \to \bigoplus_{i\in I} A_i$$

are morphisms of bornological semi rings.

Now we give the details of the descriptions of the fiber product (pullback) in **BornSR**. Given two bounded semi ring homomorphism of bornological semi rings $\varphi : (A,\beta_A) \to (A_1,\beta_{A_1})$, $\psi : (A_2,\beta_{A_2}) \to (A_1,\beta_{A_1})$ the fiber product $(A,\beta_A) \times_{(A_1,\beta_{A_1})} (A_2,\beta_{A_2})$ is defined as the fiber product of semi rings

$$A \times_{A_1} A_2 = \{(a,b) \in A \times A_2 : \varphi(a) = \psi(b)\}$$

equipped with the bornology generated by the family $\{B_1 \times_{A_1} B_2\}$ such that B_1 is bounded in A and B_2 is bounded in A_2.

4 Some Constructions on Bornological Semi Rings.

In this section, some constructions in the class of bornological semi rings are discussed. In particular, the existence of arbitrary projective limits and inductive limits of bornological semi rings is ensured.

Theorem 41
Let (A,β) be a bornological semi ring and A' be subsemi ring of A. Then the collection $\beta_{A'} = \{B \cap A' : B \in \beta\}$ is a semi ring bornology on A' induced from (A,β).

Proof.
It is obvious that $\beta_{A'}$ is a bornology on A' by Definition 23. We have to prove that for every $B_1,B_2 \in \beta_{A'}$ implies $B_1+B_2 \in \beta_{A'}$ and $B_1 \cdot B_2 \in \beta_{A'}$. Indeed, since $B_1 = B \cap A'$, $B \in \beta$ then $B_1 \subseteq B$, thus $B_1 \in \beta$. And the same reason $B_2 \in \beta$. Hence $B_1+B_2 \in \beta$ and $B_1 \cdot B_2 \in \beta$ since A' is a subsemi ring of A therefore $B_1+B_2 \subseteq A'$. Also $B_1 \cdot B_2 \subseteq A'$. Moreover, $B_1+B_2 = (B_1+B_2)\cap A'$ and $B_1 \cdot B_2 = (B_1 \cdot B_2)\cap A'$, then $B_1+B_2 \in \beta_{A'}$ and $B_1 \cdot B_2 \in \beta_{A'}$.

Example 42
Let us consider $(\mathbb{Z},+,\cdot)$ where $\mathbb{Z}$ is the set of all integer numbers with the collection of all finite subset of $\mathbb{Z}$ is a bornological subsemi ring of $(\mathbb{R},+,\cdot)$ with the usual bornology which is a bornological semi ring.

Proposition 43
Let $(A',\beta_{A'})$ be a bornological subsemi ring of bornological semi ring (A,β) and let β_0 be the base of the semi ring bornology β. Then $\beta_0' = \{B_0 \cap C : B_0 \in \beta_0\}$ is a base of the semi ring bornology β_A'.

Proof.
Let B be a bounded subset of $\beta_{A'}$, i.e. $B = V \cap C$, $V \in \beta$, C subsemi ring of A, since β_0 is a base for β (by hypothesis), that implies $V \subseteq B_0$ for some $B_0 \in \beta_0$ by the concept of base. Therefor, $B = V \cap A' \subseteq B_0 \cap A'$. Then, β_0' is a base for β_A'.

Proposition 44
Let I be a non-empty set, (A_i,β_i) be a family of bornological semi rings indexed by I and A be a semi ring. Suppose that, for every $i \in I$, a map $v_i : A \to A_i$ is given. Then the initial bornology β is a semi ring bornology on A.

Proof.
Let $B_1,B_2 \in \beta$. To prove that the initial bornology β on A is a semi ring bornology we must to show that $B_1+B_2 \in \beta$ and $B_1 \cdot B_2 \in \beta$. Since $v_i(B_1)$, $v_i(B_2)$ are bounded in A_i by Theorem 21, for every $i \in I$, and β_i is semi ring bornology on A_i such that v_i is semi ring homomorphism, so we get $v_i(B_1) + v_i(B_2) = v_i(B_1 + B_2) \in \beta_i$. Therefor, $B_1+B_2 \in \beta$. The same hold true for $B_1 \cdot B_2$, i.e., the initial bornology β on A is semi ring bornology.

Remark 45
Let (A_1,β_1) be a bornological semi ring and A be endowed with the initial bornology for every $i \in I$ for the maps v_i. Then a map $v : (A_1,\beta_1) \longrightarrow (A,\beta)$ is bounded if and only if $v_i \circ v$ is bounded.

The most important particular case of the initial bornologies is product bornology. Let us discuses briefly a bit more general operation called bornological product of arbitrary family of bornological semi rings.
The next theorem shows that the product of a collection of bornological semi rings is also bornological semi ring.

Theorem 46
Suppose that $\{(A_i,\beta_i), i \in I\}$ is a family of bornological semi rings, $A = \prod_{i\in I} A_i$ is the product of the semi rings A_i with the product bornology $\beta = \prod_{i\in I} \beta_i$. Then (A,β) is a bornological semi ring.

Proof.
The proof follows immediately from Proposition 44

4.1 Projective and Inductive Systems and limits.

Projective Systems and Limits
Let $(A_i)_{i\in I}$ be a family of semi rings. Suppose that for every pair $(i,j) \in I \times I$ such that $i \leq j$, there exists a semi ring homomorphisms $p_{ij} : A_j \to A_i$ such that the system of homomorphisms (p_{ij}) satisfies the following conditions:

(i)For every $i \in I$, $p_{ii} : A_i \to A_i$ is the identity homomorphism of A_i;

(ii)For every i, j, k elements of I such that $i \leq j < k$ we have $p_{ik} = p_{ij} \circ p_{jk}$.

Then the system (A_i, p_{ij}) is called a projective system of semi rings.

The notion of the projective system of semi rings can be extended for the bornological semi rings case as follows.

Let I be a non-empty set and (A_i, p_{ij}) be a projective system of semi rings indexed by I, such that for every $i \in I$, (A_i, β_i) is a bornological semi rings. The system $(A_i, \beta_{ij}, p_{ij})$ is called an projective system of bornological semi rings if the homomorphisms $p_{ij} = (A_j, \beta_j) \to (A_i, \beta_i)$ are bounded whenever $i \leq j$.

Let us conceder the projective limits. The projective limit of semi rings is given as follows.
Let (A_i, p_{ij}) be a projective system of semi rings. There exists a semi ring A and, for every $i \in I$, a homomorphisms $p_i : A \to A_i$, such that:

(i)$p_i = p_{ij} \circ p_j$ wherever $i \leq j$;
(ii)For every semi ring A' and family of homomorphisms $q_i : A' \to A_i$ such that $q_i = p_{ij} \circ q_j$ for $i \leq j$, there exists a unique homomorphism $q : A' \to A$ satisfying $q_i = p_i \circ q$.

The semi ring A is unique up to isomorphism and is called projective limit of the projective system (A_i, p_{ji}). For every $i \in I$, the map $p_i : A \to A_i$ is called the canonical projection of A onto A_i.

Here is an extension of the notion of Projective limit to bornological semi rings case.
Let (A_i, β_i, p_{ij}) be a projective system of bornological semi rings and A be the semi ring which is the projective limit of the system (A_i, p_{ij}). For every $i \in I$, denote by β_i the semi rings bornology of A_i and by p_i the homomorphism of A into A_i. The projective limit semi ring bornology on A with respect to semi rings bornologies β_i is the initial semi ring bornology on A for the maps p_i and A.

Theorem 47
Let A be a semi ring, for every $i \in I$, (A_i, β_i) be bornological semi rings and the map u_i be homomorphism. Then β is the semi ring bornology on A.

Proof.
It is clear that β is a bornology on A by Theorem 22. We have to show that it is a semi ring bornology. Let $B_1, B_2 \in \beta$, then $B_1 = \bigcup\limits_{i \in I} u_i(B_{i1})$ and $B_2 = \bigcup\limits_{i \in I} u_i(B_{i2})$. Therefore,

$$\bigcup_{i \in I} u_i(B_{i1} + B_{i2}) = \bigcup_{i \in I} u_i(B_{i1}) + \bigcup_{i \in I} u_i(B_{i2}) = B_1 + B_2 \in \beta.$$

Definition 48
The bornology β on A constructed in Theorem 47 is called final semi ring bornology.

Remark 49
Let (A_1, β_1) be a bornological semi ring and A be endowed with the final semi ring bornology for the map u_i, $i \in I$. Then a map $u : (A, \beta) \longrightarrow (A_1, \beta_1)$ is bounded if and only if $u \circ u_i$ is bounded for every $i \in I$.

Inductive Systems and Limits.
For a family of semi rings the concept of inductive system is defined as follows.
Let $(A_i)_{i \in I}$ be a family of semi rings. Suppose that for every pair $(i, j) \in I \times I$ such that $i \leq j$, there exists a semi ring homomorphism $u_{ji} : A_i \to A_j$ such that the system of homomorphisms (u_{ji}) satisfies the following conditions:

(i)For every $i \in I$, $u_{ii} : A_i \to A_i$ is the identity homomorphism of A_i;
(ii)For every i, j, k elements of I such that $i \leq j < k$ we have $u_{ki} = u_{kj} \circ u_{ji}$.

Then the system (A_i, u_{ji}) is called an inductive system of semi rings.

We give the extension of the inductive system to the bornology case as follows.
Let I be a direct set and (A_i, u_{ji}) be an inductive system of semi rings, indexed by I, such that for every $i \in I$, (A_i, β_i) is a bornological semi rings. The system (A_i, u_{ji}) is called an inductive system of bornological semi rings if the homomorphisms $u_{ji} : A_i \to A_j$ are bounded whenever $i \leq j$.

Let (A_i, u_{ji}) be an inductive system of semi rings. There exists a semi ring A and for every $i \in I$, a homomorphisms $u_i : A_i \to A$, such that:

(i)$u_i = u_j \circ u_{ji}$ wherever $i \leq j$;
(ii)For every semi ring A, and family of homomorphisms $v_i : A_i \to A$, such that $v_i = v_j \circ u_{ji}$ for $i \leq j$, there exists a unique homomorphism $v : A \to A'$ satisfying $v_i = v \circ u_i$.

Then the semi ring A is unique up to isomorphism and is called inductive limit of the inductive system (A_i, u_{ij}).
Here we consider the concept of inductive limit to bornology case.
Let (A_i, β_i, u_{ji}) be an inductive system of bornological semi rings and (A, β) be the bornological semi ring which is the inductive limit of the system (A_i, β_i, u_{ji}). For every $i \in I$, denote by β_i the semi rings bornology of A_i and by u_i the homomorphisms of A_i into A. The inductive limit semi ring bornology on A with respect to semi ring bornologies β_i is the final semi rings bornology on A for the maps u_i. endowed with such a semi ring bornology is called the bornological inductive limit of the bornological inductive system (A_i, u_{ij}).

Theorem 410

Let (A,β) and (H,β') be two bornological semi rings and $v:(A,\beta)\longrightarrow(H,\beta')$ is bounded homomorphism, with v surjective. Then, there exists a (necessarily unique) bornological semi ring isomorphism $v'=(A/ker(v),\beta'')\longrightarrow(H,\beta')$ making the diagram.

commutative(where q is the canonical surjection and β'' is the quotient bornology), it is necessary and sufficient that for each $B\in\beta'$ there exists a $B_1\in\beta$ such that $v^{-1}(B)\subset B_1ker(v)$.

Proof.

In order to prove the necessity of the condition, let us suppose the existence of a function v' as in the statement of the theorem. If $B\in\beta'$ is arbitrary, the boundedness of $(v')^{-1}$ implies the existence of a $B_1\in\beta$ so that $(v')^{-1}(B)=q(\beta)$, thus $v^{-1}(B)=B_1ker(v)$. Let us establish the sufficiency of the condition. We first suppose that $\beta'=v(\beta)$, the inclusion $v(\beta)\subset\beta'$ being obvious. On the other hand, if $B\in\beta'$ is arbitrary, there is a $B_1\in\beta$ so that $v^{-1}(B)\subset B_1ker(v)$, hence $B=v(v^{-1}(B))\subset v(B_1ker(v))=v(B_1)$ thus $B\in v(\beta)$. Let v' be the unique semi ring isomorphism from $A/ker(v)$ onto H such that. $v=v'\circ q$. Obviously, $v'=(A/ker(v),\beta'')\longrightarrow(H,\beta')$. Moreover, $(v')^{-1}=(H,\beta')\longrightarrow(A/ker(v),\beta'')$ is a bounded homomorphism, in view of the equalities $\beta'=v(\beta)$ and $(v')^{-1}\circ v=q$.

5 Conclusion

This work is concerned with the idea determine the boundedness of algebraic structure (ring). New structure, namely bornological semi ring are defined to solve the problem of boundedness for ring. More specifically, the category of bornological semi ring are described.

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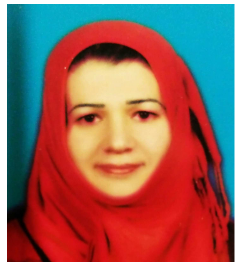

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