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Existence Results For Impulsive Fractional Neutral Integro-Differential Equations With Nonlocal Conditions

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Abstract: We present a new prospective of the existence of a mild solution for impulsive fractional neutral integro-differential equations with nonlocal conditions in Banach spaces. In light of ideas for Banach contraction principle and Krasnoselskii-Schaefer's fixed point theorem, we build up the existence results with resolvent operator and η-norm. At last, a case is given to represent the acquired outcomes .

Keywords: Fractional calculus, resolvent operator, impulsive fractional differential equation, nonlocal conditions, fixed point theorem.

1 Introduction

In late couple of decades, the idea of fractional calculus has turned into a maximum exciting place for scientists because of its huge applicability in sciences and engineering for example, material sciences, mechanics, in fluid dynamic traffic models, population dynamics, economics, chemical technology, drug and lots of others. One of the big programs of fractional calculus is the hypothesis of fractional evolution equations. certainly, fractional differential equations can be regarded as as an choice version to nonlinear partial differential equations. The fractional derivatives provides a extraordinary instrument for defining the memory and genetic residences of different substances and system that is a major advantage of fractional calculus. For elementary certainties regarding fractional structures, one create relation to the books [4, 15, 18, 24], and the papers [2, 3, 11, 12, 14, 20–22, 25], and the references cited therein.

The investigation of the theoretical nonlocal Cauchy issue can be seen [6, 7]. It has been ascertained that differential equations with nonlocal conditions are extra practical for describing many phenomena and have preferable impacts in applications over the issue without nonlocal conditions. Several researchers generally discussed the differential equations with nonlocal condition and positive outcomes were acquired [3, 6–8].

Alternatively, several real world techniques and phenomena which might be subjected for the duration of their development to quick-time period outside influences may be model as impulsive differential equations. Their length is negligible in comparison with the entire length of the unique manner and phenomena. The perturbations can be moderately well-approximated as being instantaneous changes of state, or inside the form of impulses. The related equations of those phenomena can be model as impulsive differential equations, which allows discontinuities within the evolution of the state. These days, there has been a growing interest within the study of impulsive differential equations as those equations provide a natural framework for mathematical modelling of many real world phenomena, which include mechanics, electrical engineering, medicine, biology, chemistry and control theory and so on. Because of the splendid development inside the idea of impulsive differential equations as well as having wide applications in varies fields. For this reason, lately qualitative idea of impulsive differential equations were considered by means of several authors in literature [5, 9, 13, 16, 19, 23].

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The existence, controllability and alternative subjective and quantitative properties of FDEs are area unit the foremost advancing of pursuit, in particular, see [3, 8, 12, 20–22, 25] and the references therein. In [20–22], the authors investigated different types of fractional neutral integro-differential equations with impulsive conditions and nonlocal conditions in Banach spaces. The outcomes are gotten by utilizing the suitable fixed point theorems. Later, Chadha and Pandy [8] studied the existence of the mild solution for impulsive neutral stochastic fractional integro-differential inclusion with nonlocal conditions in a separable Hilbert spaces. The outcomes are gotten by using Dhage's fixed point techniques for multi-valued operators.

Inspired by the above mentioned works [8, 20] the principle motivation behind this manuscript is to analyze the existence results for the following model

$$
\begin{aligned}
{}^{C}D_t^{\alpha}\left[x(t)-A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\right] \\
= Ax(t)+\int_0^t B(t-s)x(s)ds \\
+A_2\left(t,x(h_3(t)),\int_0^t K_2(t,s,x(h_4(s)))ds\right) \\
+A_3\left(t,x(h_5(t)),\int_0^t K_3(t,s,x(h_6(s)))ds\right), \\
t\in J=[0,T], t\neq t_k, 0<T<\infty, \quad (1.1)
\end{aligned}
$$

$$\Delta x(t_k)=I_k(x(t_k^-)),\ k=1,2,\cdots,m, \quad (1.2)$$

$$x(0)=x_0+g(x)\in\mathbb{X},\quad x'(0)=0, \quad (1.3)$$

where ${}^{C}D_t^{\alpha}(1<\alpha<2)$ denote the Caputo fractional derivative of order α. Assume that A and $B(t), t\geq 0$ are closed, densely linear operators defined on a common domain in a Banach space $\mathbb{X}, I_k:\mathbb{X}\to\mathbb{X}, 0=t_0<t_1<\cdots<t_m<t_{m+1}=T$ are fixed numbers, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ and $x(t_k^+)=\lim_{h\to 0+}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0-}x(t_k+h)$ denote the right and left limits of $x(t)$ at $t=t_k$, respectively. The functions A_i, h_j, K_i, where $i=1,2,3$ and $j=1,\cdots,6$ and g are appropriate continuous functions to be determined later.

In this paper, we present an appropriate idea of mild solution for new class of framework (1.1)-(1.3) in Section 2. Based on fractional calculus, the resolvent operators with semigroup theory, we study the existence of mild solution of framework (1.1)-(1.3) under Banach contraction and Krasnoselskii-Schaefer's fixed point hypothesis in Section 3.

2 Basic Tools

Below some basic definitions of fractional calculus, theorems, lemma and notations about α-resolvent operators are given.

Let $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ denote a Banach space. Let notation $C(J,\mathbb{X}) : J\to\mathbb{X}$ with supremum norm i.e., $\|x\|_C=\sup_{t\in J}\|x(t)\|$ and $L^1(J,\mathbb{X})$ means the Banach space of functions $x:J\to\mathbb{X}$ which are Bochner integrable normed by $\|x\|_{L^1}=\int_0^T\|y(t)\|dt$, for every $x\in L^1(J,\mathbb{X})$. A measurable function $x:J\to\mathbb{X}$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. Let notation $B(\mathbb{X}):\mathbb{X}\to\mathbb{X}$ having norm

$$\|\mathscr{F}\|_{B(\mathbb{X})}=\sup\{\|\mathscr{F}(x)\| : \|x\|\leq 1\}.$$

For $0<\eta\leq 1$, assume that $(-A)^{\eta}$ denote the fractional power of the operator $-A$ with dense domain $D((-A)^{\eta})$ in $\mathbb{X}$. It is simple to verify that $D((-A)^{\eta})$ is a Banach space have the norm

$$\|x\|_{\eta}=\|(-A)^{\eta}x\|.$$

For greater info on the fractional powers of closed linear operator, see [17].

Definition 1.[15] *The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I_t^{\alpha}F(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{F(s)}{(t-s)^{1-\alpha}}ds,\quad t>0,\quad \alpha>0,$$

provided the right hand side is point wise defined on $[0,+\infty)$, where Γ is the gamma function.

Definition 2.*The Riemann-Liouville fractional derivative is given as*

$$D_t^{\alpha}F(t)=D_t^mJ_t^{m-\alpha}F(t), m-1<\alpha<m, m\in\mathbb{N},$$

where $D_t^m=\frac{d^m}{dt^m}, F\in L^1((0,T);\mathbb{X}), J_t^{m-\alpha}F\in W^{m,1}((0,T);\mathbb{X})$. Here the notation $W^{m,1}((0,T);\mathbb{X})$ stands for the sobolev space defined as

$$W^{m,1}((0,T);\mathbb{X})=\Big\{x\in\mathbb{X}:\exists z\in L^1((0,T);\mathbb{X}):$$
$$x(t)=\sum_{k=0}^{m-1}d_k\frac{t^k}{k!}+\frac{t^{m-1}}{(m-1)!}*z(t), t\in(0,T)\Big\}.$$

We recall that $z(t)=y^m(t), d_k=y^k(0)$.

Definition 3.*The Caputo fractional derivative is given by*

$${}^{C}D_t^{\alpha}F(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}F^m(s)ds,$$

where $m-1<\alpha<m$, $F\in C^{m-1}((0,T);\mathbb{X})\cap L^1((0,T);\mathbb{X})$ and the following holds

$$J_t^{\alpha}({}^{C}D_t^{\alpha}F(t))=F(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}F^k(0).$$

To take a look at the impulsive differential equation, we characterize the space $PC(J,\mathbb{X}_\eta) : J \to \mathbb{X}_\eta$. A function $x : J \to \mathbb{X}$ is a normalized piecewise continuous function if x is piecewise continuous and continuous in J. In similarly, we define the space $PC : J \to \mathbb{X}_\eta$ such that $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1,\cdots,m$. Throughout the paper, PC is assumed to be endowed with the norm $\|x\|_{PC} = \sup_{t\in J}\|x(t)\|_\eta$. Thus, $(PC,\|\cdot\|_{PC})$ denotes that Banach space. For a function $x \in PC(J,\mathbb{X})$ and $k = \{0,1,\cdots,m\}$, we characterize the function $y_k \in C([t_k,t_{k+1}],\mathbb{X})$ ensure that

$$y_k(t) = \begin{cases} x(t), \text{ for } t \in (t_k,t_k+1], \\ x(t_k^+), \text{ for } t = t_k. \end{cases}$$

Now, we present α- resolvent operator which appeared in [1].

Definition 4.*A one-parameter family of bounded linear operators $S_\alpha(t), t \geq 0$ on $\mathbb{X}$ is said to be an α-resolvent operator for*

$$^{C}D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \tag{2.1}$$
$$x(0) = x_0, x'(0) = 0, \tag{2.2}$$

if

(a)*The function $S_\alpha(\cdot) : [0,\infty) \to L(\mathbb{X})$ is strongly continuous;*
(b)*$S_\alpha(0)x = x$, for all $x \in \mathbb{X}$ and $\alpha \in (0,1)$;*
(c)*For $x \in D(A), S_\alpha(\cdot)x \in C([0,\infty);[D(A)]) \cap C^1((0,\infty);\mathbb{X})$ and*

$$^{C}D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x + \int_0^t B(t-s)S_\alpha(s)xds,$$
$$= S_\alpha(t)Ax + \int_0^t S_\alpha(t-s)f(s)xds, t \geq 0.$$

As a way to see the existence of α- resolvent operator for system (2.1)-(2.2), we have taken consideration the following conditions

(P1)Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a closed, densely linear operator. Let $\alpha \in (0,2)$. For a few $\phi_0 \in (0,\frac{\pi}{2}]$ for each $\phi < \phi_0$, there exists a $C_0 = C_0(\phi) > 0$ ensure that $\lambda \in \rho(A)$ for every

$$\lambda \in \sum_{0,\alpha\eta} = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \alpha\eta\},$$

here $\eta = \phi + \frac{\pi}{2}$ and $\|R(\lambda,A)\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\alpha\eta}$.

(P2)For each $t \geq 0, f(t) : D(f(t)) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator with $D(A) \subseteq D(f(t))$ and $f(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. For $t > 0$ and $x \in D(A)$, there exists $d(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\overline{d}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|f(t)x\| \leq d(t)\|x\|_1$. Moreover, the operator valued function $\overline{f} : \Sigma_{0,\frac{\pi}{2}} \to \mathscr{L}([D(A)],\mathbb{X})$ has an analytic extension which is denoted by $\overline{f}$ to $\Sigma_{0,\eta}$ ensure that $\|\overline{f}(\lambda)x\| \leq \|\overline{f}(\lambda)\|\|x\|_1$ for all $x \in D(A)$ and $\|\overline{f}(\lambda)\| = 0\left(\frac{1}{|\lambda|}\right)$ as $\lambda \to \infty$.

(P3)There exist positive constants $C_i, i = 1,2$ and subspace $\overline{D} \subseteq D(A)$ dense in $[D(A)]$ such that $A(\overline{D}) \subset D(A), \overline{f}(\lambda)(\overline{D}) \subset D(A)$ and $\|A\overline{f}(\lambda)x\| \leq C_1\|x\|$ for every $x \in \overline{D}, \lambda \in \Sigma_{0,\eta}$.

Besides, we conclude that for $\theta \in (\frac{\pi}{2},\eta)$ and $r > 0$,

$$\sum_{r,\theta} = \Big\{\lambda \in \mathbb{C} : \lambda \neq 0, r < |\lambda|, 0 > |arg(\lambda)|\Big\}.$$

and for $\Gamma_{r,\theta}$

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\},$$
$$\Gamma^2_{r,\theta} = \{re^{i\zeta} : -\theta \leq \zeta \leq \theta\},$$
$$\Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$$

where $\Gamma^i_{r,\theta}, i = 1,2,3$ are the ways with the end goal that $\Gamma_{r,\theta} = \cup_{i=1}^3 \Gamma^i_{r,\theta}$ situated counterclockwise. In addition, we present after sets $\rho(G_\alpha)$ as

$$\rho(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda)$$
$$= \lambda^{\alpha-1}(\lambda^\alpha I - A - A\overline{f}(\lambda))^{-1} \in \mathscr{L}(\mathbb{X})\}.$$

Define the operator family $S_\alpha(t), t \geq 0$ by

$$S_\alpha(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}G_\alpha(\lambda)d\lambda, t > 0, \\ I, \qquad t = 0. \end{cases} \tag{2.3}$$

Lemma 1.*[2] Assume that conditions $(P1)-(P3)$ are fulfilled. Then there exists a unique α-resolvent operator for problem (2.1)-(2.2).*

Lemma 2.*[2] The function $S_\alpha : [0,\infty) \to \mathscr{L}(\mathbb{X})$ is strongly continuous and $S_\alpha : (0,\infty) \to \mathscr{L}(\mathbb{X})$ is uniformly continuous.*

Lemma 3.*[2] If the function $S_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}([D(A)])$, then $R_\alpha(\cdot)$ is exponentially bounded in $\mathscr{L}([D(A)])$.*

Definition 5.*[10] Let $\alpha \in (1,2)$, we define the family $R_\alpha(t), t \geq 0$ by*

$$R_\alpha(t)x = \int_0^t h_{\alpha-1}(t-s)S_\alpha(s)xds, t \geq 0.$$

Lemma 4.*[2] The operator families $S_\alpha(t)$ and $R_\alpha(t)$ are compact for all $t \geq 0$ if $R(\lambda_0^\alpha,A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$.*

Theorem 1.[2, 10] *Suppose that the conditions $(P1)-(P3)$ are satisfied. Let $\alpha \in (1,2)$ and $\eta \in (0,1)$ such that $\alpha\eta \in (0,1)$, then there exists a positive number C_η such that*

$$\|(-A)^\eta S_\alpha(t)\| \le C_\eta e^{rt} t^{-\alpha\eta},$$
$$\|(-A)^\eta R_\alpha(t)\| \le C_\eta e^{rt} t^{\alpha(1-\eta)-1},$$

for all $t>0$. If $y \in [D((-A)^\eta)]$, then

$$(-A)^\eta S_\alpha(t)x = S_\alpha(t)(-A)^\eta x,$$
$$(-A)^\eta R_\alpha(t)x = R_\alpha(t)(-A)^\eta x, \text{ for all } t>0.$$

Lemma 5. *A set $\mathbb{X} \subset PC$ is relatively compact in PC if and only if the set $\overline{\mathbb{X}}_k$ is relatively compact in $C([t_k,t_{k+1}];\mathbb{X}_\eta)$, for every $k=0,1,\cdots,m$.*

Definition 6.[8] *A continuous function $x: J \to \mathbb{X}_\eta$ is said to be a mild solution of system (1.1) − (1.3) if $x(0) = x_0 + g(x), x'(0) = 0$, the function $s \to AR_\alpha(t-s)A_1\Big(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\Big)$ and $s \to \int_0^s B(s-\tau)R_\alpha(t-s)A_1\Big(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\Big)d\tau$ are integrable on $t \in (0,T]$ and $\Delta x(t_k) = I_k(x(t_k^-)), k=1,\cdots,m$ and $x(\cdot)$ satisfies the following integral equation*

$$x(t) = \begin{cases} S_\alpha(t)[x_0 + g(x) - A_1(0,x(h_1(0)),0)] \\ +A_1\Big(t, x(h_1(t)), \int_0^t K_1(t,s,x(h_2(s)))ds\Big) \\ +\int_0^t AR_\alpha(t-s) \\ A_1\Big(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\Big)ds \\ +\int_0^t\int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\Big(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\Big)d\tau ds \\ +\int_0^t R_\alpha(t-s) \\ A_2\Big(s, x(h_3(s)), \int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\Big)ds \\ +\int_0^t R_\alpha(t-s) \\ A_3\Big(s, x(h_5(s)), \int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\Big)ds, \\ t \in [0,t_1], \\ S_\alpha(t-t_1)\Big[x(t_1^-) + I_1(x(t_1^-)) \\ -A_1\Big(t_1, x(h_1(t_1^-)), \int_0^{t_1} K_1(t_1,s,x(h_2(s)))ds\Big)\Big] \\ +A_1\Big(t, x(h_1(t)), \int_0^t K_1(t,s,x(h_2(s)))ds\Big) \\ +\int_{t_1}^t AR_\alpha(t-s) \\ A_1\Big(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\Big)ds \\ +\int_{t_1}^t\int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\Big(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\Big)d\tau ds \\ +\int_{t_1}^t R_\alpha(t-s) \\ A_2\Big(s, x(h_3(s)), \int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\Big)ds \\ +\int_{t_1}^t R_\alpha(t-s) \\ A_3\Big(s, x(h_5(s)), \int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\Big)ds, \\ t \in (t_1,t_2], \\ \vdots \qquad \vdots \qquad \vdots \\ S_\alpha(t-t_m)\Big[x(t_m^-) + I_m(x(t_m^-)) \\ -A_1\Big(t_m, x(h_1(t_m^-)), \int_0^{t_m} K_1(t_m,s,x(h_2(s)))ds\Big)\Big] \\ +A_1\Big(t, x(h_1(t)), \int_0^t K_1(t,s,x(h_2(s)))ds\Big) \\ +\int_{t_m}^t AR_\alpha(t-s) \\ A_1\Big(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\Big)ds \\ +\int_{t_m}^t\int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\Big(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\Big)d\tau ds \\ +\int_{t_m}^t R_\alpha(t-s) \\ A_2\Big(s, x(h_3(s)), \int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\Big)ds \\ +\int_{t_m}^t R_\alpha(t-s) \\ A_3\Big(s, x(h_5(s)), \int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\Big)ds, \\ t \in (t_m,T]. \end{cases}$$

3 Existence of mild solutions

In this area, we exhibit and set up our fundamental come about by utilizing α-resolvent semigroup hypothesis. Let $\eta \in (0,1)$. Presently, accept the assumption to establish to set up the required outcome:

(H1)Let $S_\alpha(t), t>0$ and $R_\alpha(t), t>0$ be compact and there exist constants $M_S>0$ and $M_R>0$ such that $\|S_\alpha(t)\|_{L(\mathbb{X})} \le M_S$ and $\|R_\alpha(t)\|_{L(\mathbb{X})} \le M_R$ for each $t>0$ and

$$\|(-A)^\eta R_\alpha(t)\| \le M_R t^{\alpha(1-\eta)-1}, t\in(0,T].$$

(H2)For each $x\in[D((-A)^{1-\eta})]$, $B(\cdot)x \in C(J;\mathbb{X})$ and there is a function $\mu(\cdot)\in L^1(J;\mathbb{R}^+)$ and a constant M_B such that

$$\|B(s)R_\alpha(t)\|_{L([D((-A)^\eta)];\mathbb{X})} \le M_B\mu(s)t^{\alpha\eta-1}, 0\le s<t\le T.$$

(H3)For $0<\beta<1$, The function $A_1 : J\times\mathbb{X}_\eta\times\mathbb{X}_\eta \to \mathbb{X}_\beta$ is a continuous function and we can find constants $,N_1>0$ and $N_1^*>0$ ensure that $(-A)^{\beta+\eta}A_1$ satisfies the Lipschitz condition:

$$\begin{aligned}&\| (-A)^{\beta+\eta}A_1(s_1,x_1,x_2)-(-A)^{\beta+\eta}A_1(s_2,\overline{x}_1,\overline{x}_2) \| \\ &\le N_1\Big(|s_1-s_2|+\|x_1-\overline{x}_1\|_\eta+\|x_2-\overline{x}_2\|_\eta\Big),\end{aligned}$$

for any $0\le s_1,s_2\le T, x_i,\overline{x}_i\in\mathbb{X}_\eta,\ i=1,2;$ and the inequality $\max_{t\in J}\|(-A)^{\beta+\eta}A_1(t,0,0)\|\le N_1^*$.

(H4)$A_2 : J\times\mathbb{X}_\eta\times\mathbb{X}_\eta\to\mathbb{X}_\eta$ is continuous and we can observe the positive constants N_2,N_2^* ensure that the function fulfills the Lipschitz condition:

$$\begin{aligned}&\| A_2(s_1,x_1,x_2)-A_2(s_2,\overline{x}_1,\overline{x}_2) \| \\ &\le N_2\Big(|s_1-s_2|+\|x_1-\overline{x}_1\|_\eta+\|x_2-\overline{x}_2\|_\eta\Big),\end{aligned}$$

for any $0\le s_1,s_2\le T, x_i,\overline{x}_i\in\mathbb{X}_\eta,\ i=1,2;$ and the inequality $\max_{t\in J}\|A_2(t,0,0)\|\le N_2^*$.

(H5)$A_3 : J\times\mathbb{X}_\eta\times\mathbb{X}_\eta\to\mathbb{X}_\eta$ is continuous and we can find positive constants N_3,N_3^* ensure that the function fulfills the Lipschitz condition:

$$\begin{aligned}&\| A_3(s_1,x_1,x_2)-A_3(s_2,\overline{x}_1,\overline{x}_2) \| \\ &\le N_3\Big(|s_1-s_2|+\|x_1-\overline{x}_1\|_\eta+\|x_2-\overline{x}_2\|_\eta\Big),\end{aligned}$$

for any $0\le s_1,s_2\le T, x_i,\overline{x}_i\in\mathbb{X}_\eta,\ i=1,2;$ and the inequality $\max_{t\in J}\|A_3(t,0,0)\|\le N_3^*$.

(H6)$K_i : J\times J\times\mathbb{X}_\eta\to\mathbb{X}_\eta, i=1,2,3;$ are continuous and we can find constants $N_{K_i}>0$ and $N_{K_i}^*>0$ to such that

$$\left\|\int_0^t[K_i(t,s,x)-K_i(t,s,\overline{x})]ds\right\|_\eta \le N_{K_i}\|x-\overline{x}\|_\eta,$$
$$(t,s)\in J\times J,\quad (x,\overline{x})\in\mathbb{X}_\eta^2, i=1,2,3$$

and $\max_{t\in J}\|\int_0^t K_i(t,s,0)ds\|\le N_{K_i}^*$.

(H7)The function $g:\mathbb{X}_\eta\to\mathbb{X}_\eta$ is continuous function and we can find constants $N_g>0$ and $N_g^*>0$ ensure that

$$\|g(x)-g(\overline{x})\|_\eta\le N_g\|x-\overline{x}\|_\eta \quad\text{and}$$
$$\max_{t\in J}\|g(0)\|_\eta\le N_g^*,\quad (x,\overline{x})\in\mathbb{X}_\eta^2.$$

(H8)The function $I_k : \mathbb{X}_\eta\to\mathbb{X}_\eta, k=1,2,\cdots,m$ are continuous functions and we can observe constants $N_I>0$ and $N_I^*>0$ ensure that

$$\|I_k(x)-I_k(\overline{x})\|_\eta\le N_I\|x-\overline{x}\|_{PC} \quad\text{and}$$
$$\max_{t\in J}\|I_k(0)\|_\eta\le N_I^*,\quad (x,\overline{x})\in\mathbb{X}_\eta^2.$$

Presently, we are in position to show the principle result of this section.

Theorem 2. *Assume that hypotheses (H1)-(H8) are fulfilled and*

$$\begin{aligned}L^* &= M_S[1+M_SN_I]+(M_S+1)\|(-A)^{-\beta}\|N_1(1+N_{K_1})\\ &+\frac{T^{\alpha\beta}}{\alpha\beta}N_1(1+N_{K_1})(M_R+M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\\ &+M_R[N_2(1+N_{K_2})+N_3(1+N_{K_3})]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}<1,\end{aligned}\tag{3.1}$$

$$\begin{aligned}\widetilde{L}^* =& M_S[r+M_S(N_Ir+N_I^*)]+(M_S+1)\|(-A)^{-\beta}\\ &\|[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*]\\ &+\frac{T^{\alpha\beta}}{\alpha\beta}[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*]\\ &(M_R+M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\\ &+M_R[N_2(1+N_{K_2})r+N_2N_{K_2}^*+N_2^*\\ &+N_3(1+N_{K_3})r+N_3N_{K_3}^*+N_3^*]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}\le r.\end{aligned}\tag{3.2}$$

At that point, the framework (1.1)- (1.3) concedes a mild solution in J.

Proof. Assume that the space $\mathscr{B}=\{x:[0,T]\to\mathbb{X} : x|_{[0,T]}\in PC\}$ with the uniform convergence topology. To demonstrate the outcome, we characterize the operator $\widetilde{\Psi}:\mathscr{B}\to\mathscr{B}$

$$\widetilde{\Psi}x(t)=\begin{cases}S_\alpha(t)[x_0+g(x)-A_1(0,x(h_1(0)),0)]\\ +A_1\Big(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\Big)\\ +\int_0^t AR_\alpha(t-s)\\ A_1\Big(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\Big)ds\\ +\int_0^t\int_0^s B(s-\tau)R_\alpha(t-s)\\ A_1\Big(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\Big)d\tau ds\end{cases}$$

$$\widetilde{\Psi}x(t)=\begin{cases}+\int_0^t R_\alpha(t-s)\\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds\\ +\int_0^t R_\alpha(t-s)\\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds,\\ t\in[0,t_1],\\ S_\alpha(t-t_1)\Big[x(t_1^-)+I_1(x(t_1^-))\\ -A_1\left(t_1,x(h_1(t_1^-)),\int_0^{t_1}K_1(t_1,s,x(h_2(s)))ds\right)\Big]\\ +A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\\ +\int_{t_1}^t AR_\alpha(t-s)\\ A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds\\ +\int_{t_1}^t\int_0^s B(s-\tau)R_\alpha(t-s)\\ A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds\\ +\int_{t_1}^t R_\alpha(t-s)\\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds\\ +\int_{t_1}^t R_\alpha(t-s)\\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds,\\ t\in(t_1,t_2],\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ S_\alpha(t-t_m)\Big[x(t_m^-)+I_m(x(t_m^-))\\ -A_1\left(t_m,x(h_1(t_m^-)),\int_0^{t_m}K_1(t_m,s,x(h_2(s)))ds\right)\Big]\\ +A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\\ +\int_{t_m}^t AR_\alpha(t-s)\\ A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds\\ +\int_{t_m}^t\int_0^s B(s-\tau)R_\alpha(t-s)\\ A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds\\ +\int_{t_m}^t R_\alpha(t-s)\\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds\\ +\int_{t_m}^t R_\alpha(t-s)\\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds,\\ t\in(t_m,T].\end{cases}$$

The map $\widetilde{\Psi}:\mathscr{B}\to\mathscr{B}$ is a well defined. With a specific end goal to show that there exists a mild solution for the issue $(1.1)-(1.3)$, it is adequate to demonstrate that $\widetilde{\Psi}$ admits a fixed point.

Let $\{\zeta_n : n\in\mathbb{N}\}$ be a decreasing sequence in $(0,T)\supset(0,t_1)$ ensure that $\lim_{n\to\infty}\zeta_n=0$. For setting the above hypothesis, we consider the issue represented as

$$\begin{aligned}{}^C D_t^\alpha\Big[x(t)-A_1\Big(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\Big)\Big]&\\ =Ax(t)+\int_0^t B(t-s)x(s)ds&\\ +A_2\Big(t,x(h_3(t)),\int_0^t K_2(t,s,x(h_4(s)))ds\Big)&\\ +A_3\Big(t,x(h_5(t)),\int_0^t K_3(t,s,x(h_6(s)))ds\Big),&\\ t\in J=[0,T], t\neq t_k, 0<T<\infty,& \qquad (3.3)\\ \Delta x(t_k)=S_\alpha(\zeta_n)I_k(x(t_k^-)),\ k=1,2,\cdots,m,& \qquad (3.4)\\ x(0)=x_0+g(x)\in\mathbb{X},\quad x'(0)=0.& \qquad (3.5)\end{aligned}$$

Presently, we need to demonstrate that there exists at least one mild solution $x_n\in\mathscr{B}, n\in\mathbb{N}$ for the framework (3.3)-(3.5). To demonstrate the outcome, we characterize the operator $\Psi:\mathscr{B}\to\mathscr{B}$ by the arrangement of $x\in\mathscr{B}$ ensure that

$$\Psi x(t)=\begin{cases}S_\alpha(t)[x_0+g(x)-A_1(0,x(h_1(0)),0)]\\ +A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\\ +\int_0^t AR_\alpha(t-s)\\ A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds\\ +\int_0^t\int_0^s B(s-\tau)R_\alpha(t-s)\\ A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds\\ +\int_0^t R_\alpha(t-s)\\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds\\ +\int_0^t R_\alpha(t-s)\\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds,\\ t\in[0,t_1],\\ S_\alpha(t-t_1)\Big[x(t_1^-)+S_\alpha(\zeta_n)I_1(x(t_1^-))\\ -A_1\left(t_1,x(h_1(t_1^-)),\int_0^{t_1}K_1(t_1,s,x(h_2(s)))ds\right)\Big]\\ +A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\end{cases}$$

$$\Psi x(t)=\begin{cases} +\int_{t_1}^{t} AR_\alpha(t-s) \\ A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds \\ +\int_{t_1}^{t}\int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds \\ +\int_{t_1}^{t} R_\alpha(t-s) \\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \\ +\int_{t_1}^{t} R_\alpha(t-s) \\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds, \\ t\in(t_1,t_2], \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ S_\alpha(t-t_m)\left[x(t_m^-)+S_\alpha(\zeta_n)I_m(x(t_m^-))\right. \\ \left.-A_1\left(t_m,x(h_1(t_m^-)),\int_0^{t_m} K_1(t_m,s,x(h_2(s)))ds\right)\right] \\ +A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right) \\ +\int_{t_m}^{t} AR_\alpha(t-s) \\ A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds \\ +\int_{t_m}^{t}\int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds \\ +\int_{t_m}^{t} R_\alpha(t-s) \\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \\ +\int_{t_m}^{t} R_\alpha(t-s) \\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds, \\ t\in(t_m,T], \end{cases}$$

For better readability, we break the verification into following steps:

Step 1: We demonstrate that Ψ has bounded values in $\mathscr{B}$. For $t\in[0,t_1]$ and $x\in B_r(0,\mathscr{B})=\{x\in\mathscr{B}:\|x\|_\eta\le r\}$, we have

$$\|(-A)^\eta(\Psi x)(t)\| \le \sum_{j=1}^{7} I_j. \tag{3.6}$$

$$\begin{aligned} I_1 &= \|S_\alpha(t)(-A)^\eta[x_0+g(x)]\| \\ &\le M_S[\|x_0\|_\eta+N_gr+N_g^*] \\ I_2 &= \|S_\alpha(t)(-A)^\eta A_1(0,x(h_1(0)),0)\| \\ &\le M_S\|(-A)^{-\beta}\|[|t|+N_1r+N_1^*] \\ I_3 &= \left\|(-A)^\eta A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)\right\| \\ &\le \|(-A)^{-\beta}\|[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*] \\ I_4 &= \left\|(-A)^\eta\int_0^t AR_\alpha(t-s)\right. \\ &\quad \left. A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds\right\| \\ &\le M_R\frac{T^{\alpha\beta}}{\alpha\beta}[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*] \\ I_5 &= \left\|(-A)^\eta\int_0^t\int_0^s B(s-\tau)R_\alpha(t-s)\right. \\ &\quad \left. A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds\right\| \\ &\le M_B\|(-A)^{-\beta}\|[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*]\|\mu\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta} \\ I_6 &= \left\|(-A)^\eta\int_0^t R_\alpha(t-s)\right. \\ &\quad \left. A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds\right\| \\ &\le M_R[N_2(1+N_{K_2})r+N_2N_{K_2}^*+N_2^*]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)} \\ I_7 &= \left\|(-A)^\eta\int_0^t R_\alpha(t-s)\right. \\ &\quad \left. A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds\right\| \\ &\le M_R[N_3(1+N_{K_3})r+N_3N_{K_3}^*+N_3^*]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}. \end{aligned}$$

Using $(I_1)-(I_7)$ in equation (3.6), we obtain

$$\begin{aligned} &\|(-A)^\eta(\Psi x)(t)\| \\ &\le M_S[\|x_0\|_\eta+N_gr+N_g^*]+M_S\|(-A)^{-\beta}\|[|t|+N_1r+N_1^*] \\ &\quad +\|(-A)^{-\beta}\|[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*] \\ &\quad +\frac{T^{\alpha\beta}}{\alpha\beta}[N_1(1+N_{K_1})r+N_1N_{K_1}^*+N_1^*] \\ &\quad (M_R+M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1}) \\ &\quad +M_R[N_2(1+N_{K_2})r+N_2N_{K_2}^*+N_2^* \\ &\quad +N_3(1+N_{K_3})r+N_3N_{K_3}^*+N_3^*]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}. \end{aligned}$$

For each $t \in (t_i, t_{i+1}], i = 1, 2, \cdots, m,$

$$\begin{aligned}
&\|(-A)^{\eta}(\Psi x)(t)\| \\
&\leq M_S[r + M_S(N_I r + N_I^*)] + (M_S+1)\|(-A)^{-\beta}\| \\
&\quad [N_1(1+N_{K_1})r + N_1 N_{K_1}^* + N_1^*] \\
&\quad + \frac{T^{\alpha\beta}}{\alpha\beta}[N_1(1+N_{K_1})r + N_1 N_{K_1}^* + N_1^*] \\
&\quad (M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1}) \\
&\quad + M_R[N_2(1+N_{K_2})r + N_2 N_{K_2}^* + N_2^* \\
&\quad + N_3(1+N_{K_3})r + N_3 N_{K_3}^* + N_3^*]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}
\end{aligned}$$

Therefore, Ψ is bounded.

Step 2: We show that Ψ is a contraction in $\mathscr{B}$. For $t \in [0, t_1]$ and $x, \overline{x} \in \mathscr{B}$, we have

$$\|(-A)^{\eta}\Psi x(t) - (-A)^{\eta}\Psi\overline{x}(t)\| \leq \sum_{i=8}^{14} I_i. \quad (3.7)$$

$$\begin{aligned}
I_8 &= \|S_\alpha(t)[(-A)^{\eta}g(x) - (-A)^{\eta}g(\overline{x})]\| \\
&\leq M_S N_g \sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_9 &\leq \|(-A)^{-\beta}\|M_S N_1 \sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_{10} &\leq \|(-A)^{-\beta}\|N_1[1+N_{K_1}]\sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_{11} &\leq M_R\frac{T^{\alpha\beta}}{\alpha\beta}N_1[1+N_{K_1}]\sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_{12} &\leq M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta}N_1[1+N_{K_1}] \\
&\quad \sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_{13} &\leq M_R N_2\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}[1+N_{K_2}]\sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta} \\
I_{14} &\leq M_R N_3\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}[1+N_{K_3}]\sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta}.
\end{aligned}$$

Using $I_8 - I_{14}$ in equation (3.7), we get

$$\begin{aligned}
&\|(-A)^{\eta}\Psi x(t) - (-A)^{\eta}\Psi\overline{x}(t)\| \\
&\leq \Big\{M_S[N_g + \|(-A)^{-\beta}\|N_1] + \|(-A)^{-\beta}\|N_1[1+N_{K_1}] \\
&\quad + \left(M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1}\right)\frac{T^{\alpha\beta}}{\alpha\beta}N_1[1+N_{K_1}] \\
&\quad + M_R\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}\left(N_2[1+N_{K_2}] + N_3[1+N_{K_3}]\right)\Big\} \\
&\quad \sup_{s\in J}\|x(s) - \overline{x}(s)\|_{\eta}.
\end{aligned}$$

Similarly, for $t \in [t_i, t_{i+1}]$, $i = 1, \cdots, m$, we have

$$\begin{aligned}
&\|(-A)^{\eta}(\Psi x)(t) - (-A)^{\eta}(\Psi\overline{x})(t)\| \\
&\leq \Big\{M_S[1 + M_S N_I] + (M_S+1)\|(-A)^{-\beta}\|N_1(1+N_{K_1}) \\
&\quad + \frac{T^{\alpha\beta}}{\alpha\beta}N_1(1+N_{K_1})(M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1}) \\
&\quad + M_R[N_2(1+N_{K_2}) + N_3(1+N_{K_3})]\frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)}\Big\}\|x - \overline{x}\|_{PC}.
\end{aligned}$$

In this manner, we finish up

$$\|(-A)^{\eta}(\Psi x)(t) - (-A)^{\eta}(\Psi\overline{x})(t)\| \leq L^*\|x - \overline{x}\|_{PC}, \forall t \in J.$$

Taking supremum over t, we get

$$\|(-A)^{\eta}(\Psi x) - (-A)^{\eta}(\Psi\overline{x})\|_{PC} \leq L^*\|x - \overline{x}\|_{PC},$$

By the inequality (3.1), we have $L^* < 1$, it shows that the map Ψ is contraction on $\mathscr{B}$. Hence, by Banach contraction principle, we understand that Ψ includes a unique fixed point $x \in \mathbb{X}_{\eta}$ which is a mild solution of the issue (3.3) – (3.5) in $(-\infty, T]$. The proof is now completed.

Our next outcome depends on the following Krasnoselskii-Schaefer's type fixed point hypothesis.

In order to use this theorem, we have to expect another arrangement of presumptions on A_2 and A_3.

(H9) (i)$A_i : \mathbb{X}_{\eta} \times \mathbb{X}_{\eta} \to \mathbb{X}_{\eta}, i = 2, 3$ are uniformly strongly continuous for every $t \in J$, and $x \in \mathbb{X}_{\eta}$, the function $A_i(\cdot, x, y) : J \to \mathbb{X}_{\eta}$ are strongly measurable to t;
(ii)we can find functions $\phi_i(t) > 0$ and a continuous increasing function $\psi : [0, \infty) \to (0, \infty)$, ensure that for any $(t, x, y) \in J \times \mathbb{X}_{\eta} \times \mathbb{X}_{\eta}$, we have

$$\|A_i(t, x, y)\| \leq \phi_i(t)\psi(\|x\|_{\eta} + \|y\|_{\eta}), \; i = 2, 3$$

.

(H10)The functions $K_i : J \times J \times \mathbb{X}_{\eta} \to \mathbb{X}_{\eta}, i = 2, 3;$ are continuous and there exists constants $M_{K_i} > 0, i = 2, 3;$ ensure that

$$\|K_i(t, s, x)\|_{\eta} \leq M_{K_i}(t, s)\|x\|_{\eta} \quad \textit{and}$$

$$M_{K_i}^* = \max_{t\in J}\int_0^t M_{K_i}(t, s)ds.$$

Theorem 3.*Suppose $(H1) - (H3), (H7) - (H10)$ holds. If $\mu = 1 - [M_S + M_S^2 N_I + (M_S+1)\|(-A)^{-\beta}\|N_1(1+N_{K_1}) + (M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\frac{T^{\alpha\beta}}{\alpha\beta}N_1(1+N_{K_1})]$ and*

$$\int_0^T m(s)ds \leq \int_{C^*}^{\infty}\frac{ds}{2\psi(s)},$$

where $m(t) = \max\{\omega\phi_2(t)(1+M_{K_2}^*), \omega\phi_3(t)(1+M_{K_3}^*)\}$ and $C^* = M_S^2 N_I^* + (M_S+1)\|(-A)^{-\beta}\|[N_1 N_{K_1}^* + N_1^*] + (M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\frac{T^{\alpha\beta}}{\alpha\beta}[N_1 N_{K_1}^* + N_1^*]$, then there exist at least one mild solution of the system (3.3)-(3.5) on $[0, T]$.

Proof. Consider an operator $\Psi : \mathscr{B} \to \mathscr{B}$ as in Theorem 2. From the Theorem 2, we observe that the map Ψ is well-defined on $\mathscr{B}$. It is enough to prove our result for $t \in (t_m, T]$. We show Ψ has at least one fixed point.

We introduce the decomposition $\Psi = \Psi_1 + \Psi_2$ such that

$$(\Psi_1 x)(t) = \begin{cases} -S_\alpha(t)A_1(0, x(h_1(0)), 0)] \\ +A_1\left(t, x(h_1(t)), \int_0^t K_1(t,s,x(h_2(s)))ds\right) \\ +\int_0^t AR_\alpha(t-s) \\ A_1\left(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds \\ +\int_0^t \int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\left(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds, \\ t \in [0,t_1], \\ \vdots \qquad \vdots \qquad \vdots \\ -S_\alpha(t-t_m) \\ A_1\left(t_m, x(h_1(t_m^-)), \int_0^{t_m} K_1(t_m,s,x(h_2(s)))ds\right) \\ +A_1\left(t, x(h_1(t)), \int_0^t K_1(t,s,x(h_2(s)))ds\right) \\ +\int_{t_m}^t AR_\alpha(t-s) \\ A_1\left(s, x(h_1(s)), \int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds \\ +\int_{t_m}^t \int_0^s B(s-\tau)R_\alpha(t-s) \\ A_1\left(\tau, x(h_1(\tau)), \int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds, \\ t \in (t_m, T] \end{cases}$$

and

$$(\Psi_2 x)(t) = \begin{cases} S_\alpha(t)[x_0 + g(x)] \\ +\int_0^t R_\alpha(t-s) \\ A_2\left(s, x(h_3(s)), \int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \\ +\int_0^t R_\alpha(t-s) \\ A_3\left(s, x(h_5(s)), \int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds, \\ t \in [0,t_1], \\ \vdots \qquad \vdots \qquad \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + S_\alpha(\zeta_n)I_m(x(t_m^-))] \\ +\int_{t_m}^t R_\alpha(t-s) \\ A_2\left(s, x(h_3(s)), \int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \end{cases}$$

$$(\Psi_2 x)(t) = \begin{cases} +\int_{t_m}^t R_\alpha(t-s) \\ A_3\left(s, x(h_5(s)), \int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds, \\ t \in (t_m, T]. \end{cases}$$

Now, we divide the proof into four steps.

Step 1: We will prove that the operator Ψ_1 is continuous. Firstly, we shall show that Ψ_1 is a contraction in $\mathscr{B}$. For $t \in [t_i, t_{i+1}]$, $i = 1, \cdots, m$, and $x, \overline{x} \in \mathscr{B}$, we get

$$\begin{aligned} &\|(-A)^\eta(\Psi_1 x)(t) - (-A)^\eta(\Psi_1 \overline{x})(t)\| \\ &\leq \Big\{(M_S+1)\|(-A)^{-\beta}\|N_1(1+N_{K_1}) \\ &\quad + \frac{T^{\alpha\beta}}{\alpha\beta}N_1(1+N_{K_1})(M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\Big\}\|x-\overline{x}\|_{PC}. \end{aligned}$$

Since $\left\{(M_S+1)\|(-A)^{-\beta}\|N_1(1+N_{K_1}) + \frac{T^{\alpha\beta}}{\alpha\beta}N_1(1+N_{K_1})(M_R + M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})\right\} < 1$. This shows that Ψ_1 is contraction in $\mathscr{B}$.

Next, we present that Ψ_2 is completely continuous in $\mathscr{B}$

Step 2: We demonstrate that $\Psi_2 : \mathscr{B} \to \mathscr{B}$ maps bounded. It is sufficient to show that there exists constant $l^* > 0$ ensure that for every $x \in B_r = \{x \in \mathscr{B} : \|x\|_{\mathscr{B}} \leq r\}$ and $\|\Psi_2 x\| \leq l^*$. Let $x \in B_r$ then for every $t \in (t_m, T]$, we get

$$\|(-A)^\eta(\Psi_2 x)(t)\| \leq \sum_{i=15}^{17} I_i. \tag{3.8}$$

By the hypothesis $(H7)-(H10)$, we get

$$\begin{aligned} I_{15} &= \|S_\alpha(t-t_m)(-A)^\eta[x(t_m^-) + S_\alpha(\zeta_n)I_m(x(t_m^-))]\| \\ &\leq M_S[r + M_S(N_I r + N_I^*)] \\ I_{16} &\leq M_R \frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)} \int_{t_m}^t \phi_2(s)\psi(r + M_{K_2}^* r)ds \\ I_{17} &\leq M_R \frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)} \int_{t_m}^t \phi_3(s)\psi(r + M_{K_3}^* r)ds \end{aligned}$$

From $I_{15} - I_{17}$, we get the equation (3.8)

$$\begin{aligned} &\|(-A)^\eta(\Psi_2 x)(t)\| \\ &\leq M_S[r + M_S(N_I r + N_I^*)] \\ &\quad + M_R \frac{T^{\alpha(1-\eta)}}{\alpha(1-\eta)} \int_{t_m}^t [\phi_2(s)\psi(r + M_{K_2}^* r) \\ &\quad + \phi_3(s)\psi(r + M_{K_3}^* r)]ds \\ &= l^*. \end{aligned}$$

This implies that the set is bounded in $\mathscr{B}$.

Step 3: We demonstrate that maps bounded sets into Ψ_2 is equicontinuous on B_r. Let $t_m < q_1 < q_2 < T$, for each

$x \in B_r$,

$$\begin{aligned}
&\|(-A)^\eta(\Psi_2 x)(q_2)-(-A)^\eta(\Psi_2 x)(q_1)\| \\
&\le \|S_\alpha(q_2-t_m)-S_\alpha(q_1-t_m)\|[r+M_S(N_I r+N_I^*)] \\
&\quad +\int_{t_m}^{q_1}\|(-A)^\eta[R_\alpha(q_2-s)-R_\alpha(q_1-s)]\| \\
&\quad \phi_2(s)\psi(r+M_{K_2}^* r)ds \\
&\quad +M_R\phi_2(t)\psi(r+M_{K_2}^* r)\frac{(q_2-q_1)^{\alpha(1-\eta)}}{\alpha(1-\eta)} \\
&\quad +\int_{t_m}^{q_1}\|(-A)^\eta[R_\alpha(q_2-s)-R_\alpha(q_1-s)]\| \\
&\quad \phi_3(s)\psi(r+M_{K_3}^* r)ds \\
&\quad +M_R\phi_3(t)\psi(r+M_{K_3}^* r)\frac{(q_2-q_1)^{\alpha(1-\eta)}}{\alpha(1-\eta)}.
\end{aligned}$$

Since $S_\alpha(t), t>0$ and $R_\alpha(t), t>0$ are compact, therefore $\|(-A)^\eta\Psi_2 x(q_2)-(-A)^\eta\Psi_2 x(q_1)\|\to 0$ as $q_2\to q_1$. The compactness of the operator suggests the continuity in the uniform operator topology and the set $\{S_\alpha(\zeta_n)(-A)^\eta I_k(x(t_k)) : x\in B_r(\mathscr{B})\}$ is relatively compact in $\mathbb{X}$. It is easy to show that the equicontinuity ar $t=0$ ($S_\alpha(t)$ is compact). We can easily verify that equicontinuity for the case $q_1<q_2\le 0$ or $q_1\le 0\le q_2\le T$. Hence, Ψ_2 maps B_r into an equicontinuous family of the functions.

Step 4: We show that Ψ_2 is continuous. Let x_n be sequence in $\mathscr{B}$ such that $\lim_{n\to\infty} x_n(t)=x(t)$, i.e., $x_n\to x$ as $n\to\infty$ in $\mathscr{B}$. Since A_2, A_3 and I_k are continuous. Therefore, by the continuity of A_2, A_3 and $I_k, k=1,2,\dots,m$, we deduce that

$$\begin{aligned}
&A_2\left(s,x_n(h_3(s)),\int_0^s K_2(s,\tau,x_n(h_4(\tau)))d\tau\right) \\
&\quad \to A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right) \quad \text{as}
\end{aligned}$$

$n\to\infty$,

$$\begin{aligned}
&A_3\left(s,x_n(h_5(s)),\int_0^s K_3(s,\tau,x_n(h_6(\tau)))d\tau\right) \\
&\quad \to A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right) \quad \text{as}
\end{aligned}$$

$n\to\infty$,

$g(x_n)\to g(x)$ as $n\to\infty$,

$I_k(x_n(t_i))\to I_k(x(t_i))$ as $n\to\infty$.

Now for every $t\in(t_m,T]$, we receive

$$\begin{aligned}
&\|(-A)^\eta(\Psi_2 x_n)(t)-(-A)^\eta(\Psi_2 x)(t)\| \\
&\le M_S\|(-A)^\eta[x_n(t_m^-)-x(t_m^-)]\| \\
&\quad +M_S^2\|(-A)^\eta[I_m(x_n(t_m^-))-I_m(x(t_m^-))]\| \\
&\quad +M_R\int_{t_m}^t (t-s)^{\alpha(1-\eta)-1} \\
&\quad \Bigg\|A_2\left(s,x_n(h_3(s)),\int_0^s K_2(s,\tau,x_n(h_4(\tau)))d\tau\right) \\
&\quad -A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)\Bigg\|ds \\
&\quad +M_R\int_{t_m}^t (t-s)^{\alpha(1-\eta)-1} \\
&\quad \Bigg\|A_3\left(s,x_n(h_5(s)),\int_0^s K_3(s,\tau,x_n(h_6(\tau)))d\tau\right) \\
&\quad -A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)\Bigg\|ds.
\end{aligned}$$

Then by the continuity of A_2, A_3 and $I_k, k=1,2,\dots,m$, and dominated convergence theorem, we get that $\Psi_2 x_n(t)$ converges to $\Psi_2 x(t)$ in $\mathbb{X}_\eta$, i.e., $\lim_{n\to\infty}(-A)^\eta\Psi_2 x_n(t)=(-A)^\eta\Psi_2 x(t)$ in $\mathbb{X}_\eta$ for each $t\in(t_m,T]$. Hence this proves the continuity of the map Ψ_2.

Step 5: We demonstrate Ψ_2 maps $B_r(\mathscr{B})$ into a relatively compact in $\mathbb{X}_\eta$. To demonstrate it, we present the disintegration of the map $\Psi_2:\Gamma_1\to\Gamma_2$. Right here the map $\Psi_2: B_r\to B_r$ is given by means of $\Psi_2 x$, the set $\Gamma_1\in\mathscr{B}$ such that

$$\Gamma_1(t)=\begin{cases}\int_{t_m}^t R_\alpha(t-s) \\ A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \\ +\int_{t_m}^t R_\alpha(t-s) \\ A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds, \\ t\in(t_m,T],\end{cases}$$

and set $\Gamma_2\in\mathscr{B}$, we have

$$\Gamma_2(t)=S_\alpha(t-t_m)[x(t_m^-)+S_\alpha(\zeta_n)I_m(x(t_m^-))], \qquad t\in(t_m,T].$$

For $t\in(t_m,t_{m+1}]$. Let $t_m<t\le s\le t_{m+1}$ and ε be a real number ensure that $0<\varepsilon<t$. For $x\in B_r(\mathscr{B})$, we consider

$$\begin{aligned}
\Gamma_1^{1,\varepsilon}(t)&=\int_{t_m}^{t-\varepsilon}R_\alpha(t-s) \\
&\quad A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds \\
&\quad +\int_{t_m}^{t-\varepsilon}R_\alpha(t-s) \\
&\quad A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds \\
&\le R_\alpha(\varepsilon)\int_{t_m}^{t-\varepsilon}R_\alpha(t-s-\varepsilon)
\end{aligned}$$

$$A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds$$
$$+R_\alpha(\varepsilon)\int_{t_m}^{t-\varepsilon} R_\alpha(t-s-\varepsilon)$$
$$A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds.$$

Therefore, we deduce that the set $V_\varepsilon(t)=\{\Gamma_1^{1,\varepsilon}(t): x\in B_r(\mathscr{B})\}$ is relatively compact in $\mathbb{X}_\eta$ for each $\varepsilon, 0<\varepsilon<t$ by utilizing compactness of $R_\alpha(t), 0<t$. For $x\in B_r(\mathscr{B})$, we get

$$\|(-A)^\eta\Gamma_1 x(t)-(-A)^\eta\Gamma_1^{1,\varepsilon}x(t)\|$$
$$\le M_R\int_{t-\varepsilon}^t (t-s)^{\alpha(1-\eta)-1}\phi_2(s)\psi(r+M_{K_2}^* r)ds$$
$$+M_R\int_{t-\varepsilon}^t (t-s)^{\alpha(1-\eta)-1}\phi_3(s)\psi(r+M_{K_3}^* r)ds.$$

Absolutely, the right-hand facet of the above inequality tends to 0 as $\varepsilon\to 0$. Since there are relatively compact sets arbitrarily close to the set $V(t)=\{\Gamma_1(t): x\in B_r(\mathscr{B})\}$. Hence, Γ_1 is a compact by Arzela-Ascoli lemma. Subsequent, we need to expose that $V^1(t)=\{\Gamma_2(t): x\in B_r(\mathscr{B})\|$ is relatively compact for each $t\in[0,T]$. For $x\in B_r(\mathscr{B})$ and $t\in(t_i,t_{i+1}], i=1,\cdots,m$, we obtain that there exists $r'>0$ such that

$$[\overline{\Gamma}_2]_{t_i}(t)=\begin{cases} S_\alpha(t-t_i)[x(t_i^-)+S_\alpha(\zeta_n)I_i(x(t_i^-))], \\ \quad t\in(t_i,t_{i+1}], x\in B_{r'}(\mathscr{B}), \\ S_\alpha(t_{i+1}-t_i)[x(t_i^-)+S_\alpha(\zeta_n)I_i(x(t_i^-))], \\ \quad t=t_{i+1}, x\in B_{r'}(\mathscr{B}), \\ [x(t_i^-)+S_\alpha(\zeta_n)I_i(x(t_i^-))], t=t_k, x\in B_{r'}(\mathscr{B}), \end{cases}$$

where $B_{r'}$ indicates an open ball of range r'. From the compactness of operator $S_\alpha(t), 0<t$, it is anything but difficult to confirm that $\{S_\alpha(\zeta_n)I_i(x(t_i^-)): x\in B_{r'}(\mathscr{B}), i=1,\cdots,m\}$ are relatively compact in $\mathbb{X}$. Thus, we get that $[\overline{\Gamma}_2]_{t_i}(t)$ is relatively compact in $\mathbb{X}$ for each $t\in(t_i,t_{i+1}], i=1,\cdots,m$. In this manner, we reason that Γ_2 is relatively compact by Lemma 2.5. Subsequently, by Arzela-Ascoli lemma, we understand that Ψ_2 is compact. Therefore, Ψ_2 is completely continuous.

Step:6 Now, we will show that $Q(\Psi)=\{x(\cdot):\lambda\Psi_1\left(\frac{x}{\lambda}\right)+\lambda\Psi_2 x=x, \text{for a few} 0<\lambda<1\}$ is bounded. Let $x(\cdot)\in Q(\Psi)$. Then $\lambda\Psi_1\left(\frac{x}{\lambda}\right)+\lambda\Psi_2 x=x$ for some $1>\lambda>0$. Thus for each $t\in(t_m,T]$.

$$x(t)=S_\alpha(t-t_m)\Big[x(t_m^-)+S_\alpha(\zeta_n)I_m(x(t_m^-))$$
$$-A_1\left(t_m,x(h_1(t_m^-)),\int_0^{t_m}K_1(t_m,s,x(h_2(s)))ds\right)\Big]$$
$$+A_1\left(t,x(h_1(t)),\int_0^t K_1(t,s,x(h_2(s)))ds\right)$$
$$+\int_{t_m}^t AR_\alpha(t-s)$$
$$A_1\left(s,x(h_1(s)),\int_0^s K_1(s,\tau,x(h_2(\tau)))d\tau\right)ds$$
$$+\int_{t_m}^t\int_0^s B(s-\tau)R_\alpha(t-s)$$
$$A_1\left(\tau,x(h_1(\tau)),\int_0^\tau K_1(\tau,\xi,x(h_2(\xi)))d\xi\right)d\tau ds$$
$$+\int_{t_m}^t R_\alpha(t-s)$$
$$A_2\left(s,x(h_3(s)),\int_0^s K_2(s,\tau,x(h_4(\tau)))d\tau\right)ds$$
$$+\int_{t_m}^t R_\alpha(t-s)$$
$$A_3\left(s,x(h_5(s)),\int_0^s K_3(s,\tau,x(h_6(\tau)))d\tau\right)ds$$

From the hypotheses $(H1)-(H3)$ and $(H8)-(H10)$ Then for each $t\in(t_m,T]$, we sustain

$$\|x\|_\eta\le M_S[\|x\|_\eta+M_S(N_I\|x\|_\eta+N_I^*)+\|(-A)^{-\beta}$$
$$\|[N_1(1+N_{K_1})\|x\|_\eta+N_1N_{K_1}^*+N_1^*]]$$
$$+\|(-A)^{-\beta}\|[N_1(1+N_{K_1})\|x\|_\eta+N_1N_{K_1}^*+N_1^*]$$
$$+\frac{T^{\alpha\beta}}{\alpha\beta}[N_1(1+N_{K_1})\|x\|_\eta+N_1N_{K_1}^*+N_1^*]$$
$$(M_R+M_B\|(-A)^{-\beta}\|\|\mu\|_{L^1})$$
$$+\omega\int_{t_m}^t[\phi_2(s)\psi(\|x\|_\eta+M_{K_2}^*\|x\|_\eta)+$$
$$\phi_3(s)\psi(\|x\|_\eta+M_{K_3}^*\|x\|_\eta)]ds,$$

where $\omega=\dfrac{M_RT^{\alpha(1-\eta)}}{\alpha(1-\eta)\mu}$. Then for all $t\in(t_m,T]$, we have

$$\|x\|_\eta\le\beta_\lambda(t)=\frac{C^*}{\mu}$$
$$+\omega\int_{t_m}^t\left[\phi_2(s)(1+M_{K_2}^*)+\phi_3(s)(1+M_{K_3}^*)\right]\psi(\|x\|_\eta)ds$$

and $\beta_\lambda(0)=\frac{C^*}{\mu}$. Thus

$$\beta'_\lambda(t)\le\omega\{(\phi_2(t)(1+M_{K_2}^*)+\phi_3(t)(1+M_{K_3}^*))\psi(\|x\|_\eta)\}$$
$$\le\omega\{(\phi_2(t)(1+M_{K_2}^*)+\phi_3(t)(1+M_{K_3}^*))\psi(\beta_\lambda(t))\}$$
$$\le 2m(t)\psi(\beta_\lambda(t))$$

From (H9), we note that ψ is positive and non-decreasing function. Now, we integrating the above estimation on both sides, we obtain

$$\int_{\beta_\lambda(0)}^{\beta_\lambda(t)} \frac{ds}{2\psi(s)} \le \int_0^t m(s)ds < \int_{C^*}^{\infty} \frac{ds}{2\psi(s)}, t \in J.$$

From this inequality and mean value theorem, we observe that there exists constant r, independent of $\lambda \in (0,1)$ such that $x(t) \le r$ for $t \in J$ and hence $\|x(t)\| \le r$ for $t \in J$ and consequently, we have

$$\|x\|_\eta = \sup\{\|x(t)\| : t \in J\} \le r, \forall x \in \zeta(\Psi).$$

This shows that the set Q is bounded in $\mathscr{B}$. Consequently, by Theorem 2 the operator Ψ has a fixed a point in $\mathscr{B}$. Thus the problem $(3.3)-(3.5)$ has a solution on $\mathscr{B}$. The proof is now completed.

4 Application

To exemplify our theoretical results, we consider the

$$\begin{aligned} {}^C D_t^\alpha \Big[& z(t,x) - \overline{a}_1(t)z(\sin t,x) - \overline{a}_2(t)\sin z(t,x) \\ & - \frac{1}{1+t^2}\int_0^t \overline{a}_3(s)z(\sin s,x)ds \Big] = \frac{\partial^2}{\partial x^2} z(t,x) \\ & + \int_0^t (t-s)^\xi e^{-\zeta(t-s)} \frac{\partial^2}{\partial x^2} z(s,x)ds \\ & + a_1(t)z(\sin t,x) + a_2(t)\sin z(t,x) \\ & + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s,x)ds \\ & + \widetilde{a}_1(t)z(\sin t,x) + \widetilde{a}_2(t)\sin z(t,x) \\ & + \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s,x)ds, \end{aligned} \quad (5.1)$$

$$\Delta z(t_k,x) = z(t_k^+,x) - z(t_k^-,x) = I_k(z(t_k^-)); \quad (5.2)$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in J = [0,1], 0 \le x \le \pi, \quad (5.3)$$

$$z(0,x) = z_0(x) + \int_0^\pi b(x,v)dv, 0 \le x \le \pi, \quad (5.4)$$

where ${}^C D_t^\alpha$ denotes the Caputo derivative of order α. The functions $a_i, \overline{a}_i, \widetilde{a}_i, i = 1,2,3;$ are continuous on $[0,1]$, $n_i = \sup_{0\le s\le 1} |a_i(s)| < 1, i = 1,2,3;$ $\widetilde{n}_i = \sup_{0\le s\le 1} |\widetilde{a}_i(s)| < 1, i = 1,2,3.$ and $\overline{n}_i = \sup_{0\le s\le 1} |\overline{a}_i(s)| < 1, i = 1,2,3.$ $b : [0,1]\times\mathbb{R}$ are continuous mapping and $z_0(\cdot) \in L^2([0,\pi])$.

Let us consider $X = L^2([0,\pi])$ with the norm $\|\cdot\|$. We now define $A : \mathbb{X} \to \mathbb{X}$ by $Az = z''$. The domain of A is

$$D(A) = \{z \in \mathbb{X} : z, z' \quad \text{are absolutely continuous} \quad z'' \in \mathbb{X}\}.$$

with $\quad z(0) = z(\pi) = 0.$

Then, we have

(i) $Az = \sum_{n=1}^\infty n^2\langle z, z_n\rangle z_n, \quad z \in D(A),$ where $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n = 1,2,\dots,$ is the orthogonal set of eigenvectors of A.

(ii) For every $z \in \mathbb{X}$,

$$(-A)^{-\frac{1}{2}}z = \sum_{n=1}^\infty \frac{1}{n}\langle z, z_n\rangle z_n.$$

(iii)

$$(-A)^{\frac{1}{2}}z = \sum_{n=1}^\infty n\langle z, z_n\rangle z_n$$

on the space $D((-A)^{\frac{1}{2}}) = \{z(\cdot) \in \mathbb{X}; \sum_{n=1}^\infty n\langle z, z_n\rangle z_n \in \mathbb{X}\}$ and $\|(-A)^{-\frac{1}{2}}\| = 1.$

Consequently, A denotes the infinitesimal generator of a strongly continuous, compact, analytic semigroup $T(t)$ and A is sectorial of type and $(P1)$ is satisfied. Also we have $B(t) : \mathbb{X} \subset D(A) \to \mathbb{X}, t \ge 0,$ $B(t)x = t^\xi e^{-\zeta t}x''$ for $x \in D(A)$. In addition, we demonstrate that $(P2)$ and $(P3)$ are satisfied with $te^{-\zeta t}$ and $D(A) = C_0^\infty([0,\pi])$, here $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions ensure that vanish at $x = 0$ and $x = \pi$.

Thus,

$$\begin{aligned} & A_1\left(t, z(h(t)), \int_0^t k_1(t,s,z(h(s)))ds\right)(x) \\ & = \overline{a}_1(t)z(\sin t,x) + \overline{a}_2(t)\sin z(t,x) \\ & \quad + \frac{1}{1+t^2}\int_0^t \overline{a}_3(s)z(\sin s,x)ds, \\ & A_2\left(t, z(\widetilde{h}(t)), \int_0^t k_1(t,s,z(\widetilde{h}(s)))ds\right)(x) \\ & = a_1(t)z(\sin t,x) + a_2(t)\sin z(t,x) \\ & \quad + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s,x)ds, \\ & A_3\left(t, z(\widehat{h}(t)), \int_0^t k_1(t,s,z(\widehat{h}(s)))ds\right)(x) \\ & = \widetilde{a}_1(t)z(\sin t,x) + \widetilde{a}_2(t)\sin z(t,x) \\ & \quad + \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s,x)ds, \\ & g(z)(x) = \int_0^\pi b(\cdot,v)w(v)dv, \quad w \in \mathbb{X}. \end{aligned}$$

Thus, the framework (5.1)-(5.4) can be composed as in the form (1.1)-(1.3). It is easy to verify that with the decisions of the above functions, presumptions (H1)-(H8) of Theorem 2 are fulfilled. From Theorem 2, we reason that nonlocal impulsive Cauchy issue (5.1)-(5.4) has a mild solution in J.

5 Conclusion

In this manuscript, we have examined the existence outcomes for impulsive fractional neutral integro-differential equations with nonlocal conditions in Banach spaces. All the more exactly, by using the semigroup theory, fractional powers of operators, Banach contraction fixed point techniques and Krasnoselskii-Schaefer's fixed point techniques, we build up the existence results with resolvent operator and η-norm. approve the got hypothetical outcomes, an illustration is dissected. The fractional differential equations are exceptionally proficient to portray the genuine wonders; in this manner, it is fundamental to stretch out the present investigation to set up the other subjective and quantitative properties, for example, stability and approximate controllability.

There are two direct issues which require additionally consider. In the first place, we will examine the approximate controllability of fractional neutral stochastic integro-differential frameworks with state-dependent delay both on account of a non-compact operator and an ordinary topological space. Also, we will be dedicated to concentrate the approximate controllability of another class of impulsive fractional stochastic differential equations with state-dependent delay in Hilbert space.

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