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A Note on the New Extended Beta and Gauss Hypergeometric Functions

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Abstract: A variety of extensions of the classical beta function and the Gauss hypergeometric function ${}_2F_1$ have been presented and investigated. In this sequel, we aim to give a further extension of the extended beta function, which is used to extend the ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$. Then we investigate to present certain properties and formulas associated with these three extended functions. The results presented here, being very general, are pointed out to be specialized to yield numerous known and new representations and formulas.

Keywords: Gamma function, Beta function, Extended Beta functions, extended hypergeometric functions, extended confluent hypergeometric functions

1 Introduction

Special functions and their extensions have proved powerful and far-reaching tools for development of many branches of pure and applied mathematics (see, e.g., [9], [13], [15]). During last two decades or so, several interesting and useful extensions of many special functions have been investigated (see, e.g., [1], [2], [5], [8], [3], [4], [10], [12], [14]).

In 1997, Chaudhry et al. [3] presented the following interesting extension of Euler's beta function:

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[-\frac{p}{t(1-t)}\right]\mathrm{d}t \tag{1}$$

$$\left(\Re(p)>0;\ p=0, \min\{\Re(x),\Re(y)\}>0\right).$$

Obviously $B_0(x,y)$ reduces to the familiar beta function $B(x,y)$ (see, e.g., [16, Section 1.1]). They [3] showed that the extension (1) has certain connections with Macdonald function, error function and Whittaker function. Chaudhry et al. [4] used the $B_p(x,y)$ to extend the hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$, respectively, as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{2}$$

$$\left(p\in\mathbb{R}_0^+;\ |z|<1;\ \Re(c)>\Re(b)>0\right)$$

and

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{3}$$

$$\left(p\in\mathbb{R}_0^+;\ \Re(c)>\Re(b)>0\right),$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by (see, e.g., [16, p. 2 and pp. 4-6]):

$$\begin{aligned}(\lambda)_n &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)\\ &= \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}).\end{cases}\end{aligned} \tag{4}$$

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Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\Gamma(\lambda)$ is the familiar Gamma function.

They [3] called the functions (1), (2), and (3) as extended beta function (EBF), extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. It is clear that $F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$ and $\Phi_0(b;c;z) = {}_1F_1(b;c;z)$.

They [3] investigated the extended functions to present their diverse properties such as differentiation formulas, Mellin transforms, recurrence relations, summation and asymptotic formulas including the following integral representations

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \times \exp\left[-\frac{p}{t(1-t)}\right] \mathrm{d}t \tag{5}$$

$$\left(p \in \mathbb{R}^+;\ p=0,\ |\arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0\right)$$

and

$$\Phi_p(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times \exp\left[zt - \frac{p}{t(1-t)}\right] \mathrm{d}t \tag{6}$$

$$\left(p \in \mathbb{R}^+;\ p=0,\ \Re(c) > \Re(b) > 0\right).$$

Choi et al. [6] presented a further extension of the extended beta function (1) as follows:

$$B_{p,q}(x,y) = B(x,y;p,q) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{t} - \frac{q}{1-t}\right] \mathrm{d}t \tag{7}$$

$$(\min\{\Re(p),\Re(q)\} > 0;\ p=q=0,\ \min\{\Re(x),\Re(y)\} > 0).$$

Clearly $B_{p,p}(x,y) = B_p(x,y)$ and $B_{0,0}(x,y) = B(x,y)$.

They [6] used the $B_{p,q}(x,y)$ to extend the hypergeometric function and the confluent hypergeometric function, respectively, as follows:

$$F_{p,q}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{8}$$

$$\left(p,q \in \mathbb{R}_0^+;\ |z| < 1;\ \Re(c) > \Re(b) > 0\right)$$

and

$$\Phi_{p,q}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_{p,q}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{9}$$

$$\left(p,q \in \mathbb{R}_0^+;\ \Re(c) > \Re(b) > 0\right).$$

Obviously $F_{p,p}(a,b;c;z) = F_p(a,b;c;z)$, $F_{0,0}(a,b;c;z) = {}_2F_1(a,b;c;z)$, $\Phi_{p,p}(b;c;z) = \Phi_p(b;c;z)$, and $\Phi_{0,0}(b;c;z) = \Phi(b;c;z)$.

They [6] presented various properties and formulas for the extended functions (7), (8), and (9), for example, the following Euler type integral representations:

$$F_{p,q}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t} - \frac{q}{1-t}\right] \mathrm{d}t \tag{10}$$

$$\left(p,q \in \mathbb{R}^+;\ p=q=0,\ |\arg(1-z)| < \pi,\ \Re(c) > \Re(b) > 0\right)$$

and

$$\Phi_{p,q}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{p}{t} - \frac{q}{1-t}\right] \mathrm{d}t \tag{11}$$

$$\left(p,q \in \mathbb{R}^+;\ p=q=0,\ \Re(c) > \Re(b) > 0\right).$$

Here, we further generalize the extended beta function (7) as follows:

$$B_{p,q}^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left(\alpha;\beta;-\frac{p}{t} - \frac{q}{1-t}\right) \mathrm{d}t \tag{12}$$

$$(\min\{\Re(p),\Re(q)\} > 0;\ p=q=0,\ \min\{\Re(x),\Re(y)\} > 0;\ \alpha \in \mathbb{C},\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Clearly $B_{p,q}^{(\alpha,\alpha)}(x,y) = B_{p,q}(x,y)$, $B_{p,p}^{(\alpha,\beta)}(x,y) = B_p^{(\alpha,\beta)}(x,y)$, $B_{p,p}^{(\alpha,\alpha)}(x,y) = B_p(x,y)$, and $B_{0,0}^{(\alpha,\beta)}(x,y) = B(x,y)$.

We use the extended beta function (12) to extend the hypergeometric and confluent hypergeometric functions, respectively, as follows:

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{13}$$

$$\left(p,q \in \mathbb{R}_0^+;\ |z| < 1;\ \Re(c) > \Re(b) > 0;\ \alpha \in \mathbb{C},\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$$

and

$$\Phi_{p,q}^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{14}$$

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$\left(p,q\in\mathbb{R}_0^+;\ \Re(c)>\Re(b)>0;\ \alpha\in\mathbb{C},\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$

We call $F_{p,q}^{(\alpha,\beta)}(a,b;c;z)$ and $\Phi_{(p,q)}^{(\alpha,\beta)}(b;c;z)$ as further generalized Gauss hypergeometric function (FGGHF) and further generalized confluent hypergeometric function (FGCHF), respectively.

Obviously $F_{p,q}^{(\alpha,\alpha)}(a,b;c;z) = F_{p,q}(a,b;c;z)$, $F_{p,p}^{(\alpha,\beta)}(a,b;c;z)=F_p^{(\alpha,\beta)}(a,b;c;z)$, $F_{p,p}^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z)$, $F_{0,0}^{(\alpha,\beta)}(a,b;c;z)={}_2F_1(a,b;c;z)$, $\Phi_{p,q}^{(\alpha,\alpha)}(b;c;z) = \Phi_{p,q}(b;c;z)$, $\Phi_{p,p}^{(\alpha,\beta)}(b;c;z)=\Phi_p^{(\alpha,\beta)}(b;c;z)$, $\Phi_{p,p}^{(\alpha,\alpha)}(b;c;z) = \Phi_p(b;c;z)$, and $\Phi_{0,0}^{(\alpha,\beta)}(b;c;z)=\Phi(b;c;z)$.

In this paper, we aim to present various properties and formulas associated with the extended functions (12), (13), and (14).

2 Properties and formulas

Theorem 1. *Let $n\in\mathbb{N}_0$. Then the following summation formula holds.*

$$B_{p,q}^{(\alpha,\beta)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}B_{p,q}^{(\alpha,\beta)}(x+k,y+n-k) \tag{15}$$

$$\left(\min\{\Re(p),\Re(q)\}>0;\ p=q=0,\quad \min\{\Re(x),\Re(y)\}>0;\ \alpha\in\mathbb{C},\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Proof. The case $n=0$ of (15) holds trivially. The case $n=1$ of (15) holds as follows:

$$\begin{aligned}&B_{p,q}^{(\alpha,\beta)}(x+1,y)+B_{p,q}^{(\alpha,\beta)}(x,y+1)\\&=\int_0^1\left[t^x(1-t)^{y-1}+t^{x-1}(1-t)^y\right]{}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right)\mathrm{d}t\\&=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right)\mathrm{d}t=B_{p,q}^{(\alpha,\beta)}(x,y).\end{aligned}$$

Continuing this process, by mathematical induction on n, we can prove (15) for all $n\in\mathbb{N}_0$. We omit the details.

Theorem 2. *The following formula holds.*

$$B_{p,q}^{(\alpha,\beta)}(x,1-y)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_{p,q}^{(\alpha,\beta)}(x+n,1) \tag{16}$$

$$\left(\min\{\Re(p),\Re(q)\}>0;\ \alpha\in\mathbb{C},\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Proof. We have

$$B_{p,q}^{(\alpha,\beta)}(x,1-y)=\int_0^1 t^{x-1}(1-t)^{-y}{}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right)\mathrm{d}t. \tag{17}$$

By using the generalized binomial theorem

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!}\quad(|t|<1;\ y\in\mathbb{C}) \tag{18}$$

in the factor $(1-t)^{-y}$ in the right side of (17), and interchanging the order of integral and summation in the right side resulting expression, which is verified under the given conditions, and using (12), we obtain the desired identity (16).

Theorem 3. *The following formula holds.*

$$B_{p,q}^{(\alpha,\beta)}(x,y)=\sum_{n=0}^{\infty}B_{p,q}^{(\alpha,\beta)}(x+n,y+1) \tag{19}$$

$$\left(\min\{\Re(p),\Re(q)\}>0;\ \alpha\in\mathbb{C},\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Proof. We have

$$(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n\quad(|\,t\,|<1). \tag{20}$$

Substituting the series expression in (20) for the factor $(1-t)^{y-1}$ in (12), similarly as in the proof of Theorem 2, we can get the desired result (19). We omit the details.

Theorem 4. *Let $\Re(c)>\Re(b)>0$, $\alpha\in\mathbb{C}$, and $\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then the following integral formula holds.*

$$\begin{aligned}&F_{p,q}^{(\alpha,\beta)}(a,b;c;z)\\&=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}{}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right)\mathrm{d}t\end{aligned} \tag{21}$$

$$\left(p,q\in\mathbb{R}_0^+;\ p=q=0,\ |\arg(1-z)|<\pi\right).$$

Proof. Applying (12) to (13) and interchanging the order of summation and integral, which is verified under the conditions here, we obtain

$$\begin{aligned}F_{p,q}^{(\alpha,\beta)}(a,b;c;z)&=\frac{1}{B(b,c-b)}\\&\times\int_0^1 t^{b-1}(1-t)^{c-b-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right)\\&\times\left(\sum_{n=0}^{\infty}(a)_n\frac{(tz)^n}{n!}\right)\mathrm{d}t.\end{aligned} \tag{22}$$

Using the generalized binomial theorem (18) for the summation in the parentheses in (22), we obtain the desired result (21).

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Theorem 5. *Let $\Re(c) > \Re(b) > 0$, $\alpha \in \mathbb{C}$, and $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Also, let $p, q \in \mathbb{R}_0^+$. Then the following integral formula holds.*

$$\Phi_{p,q}^{(\alpha,\beta)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} \times {}_1F_1\left(\alpha;\beta;-\frac{p}{t}-\frac{q}{1-t}\right) \mathrm{d}t \tag{23}$$

Proof. A similar argument as in the proof of Theorem 4 will establish the result (23). We omit the details.

Theorem 6. *The following differential formula holds: For $n \in \mathbb{N}_0$,*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{F_{p,q}^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(b)_n (a)_n}{(c)_n} F_{p,q}^{(\alpha,\beta)}(a+n,b+n;c+n;z) \tag{24}$$

$$\left(p, q \in \mathbb{R}_0^+; \mid z \mid < 1; \Re(c) > \Re(b) > 0; \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. Differentiating each side of (13) with respect to the variable z and taking termwise differentiation on the right side of (13), which is verified under the conditions here, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left\{F_{p,q}^{(\alpha,\beta)}(a,b;c;z)\right\} &= \sum_{n=1}^{\infty} (a)_n \frac{B_{p,q}^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}^{(\alpha,\beta)}(b+n+1,c-b)}{B(b,c-b)} \frac{z^n}{n!}.\end{aligned} \tag{25}$$

Using the identities $B(b,c-b) = \frac{c}{b} B(b+1,c-b)$ and $(a)_{n+1} = a(a+1)_n$ in the second summation in (25), in terms of (13), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\left\{F_{p,q}^{(\alpha,\beta)}(a,b;c;z)\right\} \\ &= \frac{ba}{c} \sum_{n=0}^{\infty} (a+1)_n \frac{B_{p,q}^{(\alpha,\beta)}(b+n+1,c-b)}{B(b+1,c-b)} \frac{z^n}{n!} \\ &= \frac{ba}{c} F_{p,q}^{(\alpha,\beta)}(a+1,b+1;c+1;z).\end{aligned} \tag{26}$$

Repeating this argument in the last expression, as in (26), in each step, $n-1$ times, we get the desired result (24).

Theorem 7. *The following differential formula holds: For $n \in \mathbb{N}_0$,*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\Phi_{p,q}^{(\alpha,\beta)}(b;c;z)\right\} = \frac{(b)_n}{(c)_n} \quad \Phi_{p,q}^{(\alpha,\beta)}(b+n;c+n;z) \tag{27}$$

$$\left(p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0; \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. The proof would run parallel to that of Theorem 6. We omit the details.

Theorem 8. *The following transformation formula holds: For $\mid \arg(1-z) \mid < \pi$,*

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;z) = (1-z)^{-a} F_{q,p}^{(\alpha,\beta)}\left(a,c-b;c;\frac{z}{z-1}\right) \tag{28}$$

$$\left(p, q \in \mathbb{R}_0^+; \mid z \mid < 1; \Re(c) > \Re(b) > 0; \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. Replacing t by $1-t$ in (21) and writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a},$$

we have

$$\begin{aligned}F_{p,q}^{(\alpha,\beta)}(a,b;c;z) &= \frac{(1-z)^{-a}}{B(b,c-b)} \\ &\times \int_0^1 (1-t)^{b-1} t^{c-b-1}\left(1-\frac{z}{z-1}t\right)^{-a} \\ &\times {}_1F_1\left(\alpha,\beta;-\frac{p}{1-t}-\frac{q}{t}\right) \mathrm{d}t \\ &= (1-z)^{-a} F_{q,p}^{(\alpha,\beta)}\left(a,c-b;c;\frac{z}{z-1}\right).\end{aligned}$$

Theorem 9. *The following transformation formula holds.*

$$\Phi_{p,q}^{(\alpha,\beta)}(b;c;z) = e^z \, \Phi_{q,p}^{(\alpha,\beta)}(c-b;c;-z) \tag{29}$$

$$\left(p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0; \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. A similar argument as in the proof of Theorem 8 will establish the result here. We omit the details.

Theorem 10. *The following summation formula holds.*

$$F_{p,q}^{(\alpha,\beta)}(a,b;c;1) = \frac{B_{p,q}^{(\alpha,\beta)}(b,c-a-b)}{B(b,c-b)} \tag{30}$$

$$\left(p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0; \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$$

Proof. Setting $z = 1$ in (21) and using (12), we obtain the result (30).

3 Special cases and remarks

The results presented in Section 2, being very general, can be specialized to yield numerous known and new identities. Setting $t = u/(1+u)$ and $t = \sin^2\theta$ in (21), we get integral representations of $F_{p,q}^{(\alpha,\beta)}(a,b;c;z)$ with the range of integral from 0 to ∞ and the integrand of trigonometric functions, respectively. The special cases of the results in Section 2 when $p = 0 = q$ yield the corresponding identities for the Gauss hypergeometric series ${}_2F_1$ and the confluent hypergeometric series ${}_1F_1$.

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For example, setting $p = 0 = q$ in (30) gives the well-known Gauss summation theorem for ${}_2F_1(1)$:

$$ {}_2F_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{31} $$

$$ \left(\Re(c-a-b) > 0;\ c \in \mathbb{C}\setminus\mathbb{Z}_0^-\right). $$

4 Conclusion

In the paper, we have presented a variety of extensions of the classical beta function and the Gauss hypergeometric function ${}_2F_1$. We gave a further extension of the extended beta function, which is used to extend the ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$. After that, we have investigated to present certain properties and formulas associated with these three extended functions. The results obtained here, which seems to be very general, are a corresponding generalizations of some known beta function and Gauss hypergeometric function ${}_2F_1$.

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