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On (α, m, h)-Convexity

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Abstract: The main objective of this article is to introduce the class of (α, m, h)-convex functions. It is shown that this class unifies several new and known classes of convex functions. Some new integral inequalities of Hermite-Hadamard type are obtained. Some special cases are also discussed.

Keywords: Convex functions, (α, m)-convex functions, Hermite-Hadamard inequality.
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1 Introduction

Theory of convexity has experienced rapid development in recent years. Consequently several generalizations for classical convex sets and convex functions have been proposed by many researchers, see [4]. An important generalization in this regard is h-convex functions, which was introduced and studied by Varosanec [17]. It has been observed that the class of h-convex function unifies several other known classes of convex functions, such as, s-convex functions [3], Godunova-Levin functions [10], P-functions [9] and s-Godunova-Levin functions [6] respectively. For some recent investigations on h-convex functions, see [2,5].

The relationship between theory of convexity and theory of inequalities has attracted many researchers. Many inequalities have been obtained via convex functions. One of the most intensively studied inequality is Hermite-Hadamard's inequality. This inequality provides us a necessary and sufficient condition for a function to be convex. It also provides the mean value of convex function function over an interval. This classical result of Hermite and Hadamard reads as:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. It is known that f is a convex function if and only if, the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

holds. For some recent studies on Hermite-Hadamard type inequalities, see [1,2,7,8,11,13,14,15].

In this paper, we introduce the class of (α, m, h)-convex functions. It is shown that this class unifies some known and new classes of convex functions. We also establish some Hermite-Hadamard type inequalities for (α, m, h)-convex functions. Some special cases are also discussed. This is the main motivation of this paper.

2 Preliminaries

In this section, we recall some previously known concepts.

Definition 1([12]). *A function $f : I \to (0, \infty)$ is said to be (α, m)-convex function , if*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y), \quad \forall x, y \in I, t \in [0,1], (\alpha, m) \in [0,1]^2. \quad (1)$$

Note that for $\alpha = 1$, Definition 2.4 reduces to the definition of m-convex functions introduced by [16].

Definition 2([16]). *A function $f : I \to (0, \infty)$ is said to be m-convex function, if*

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y), \quad \forall x, y \in I, t \in [0,1], m \in [0,1]. \quad (2)$$

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Definition 3([17]). *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ a nonnegative function. We say that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is h-convex function $(f\in SX(h,I))$ if f is nonnegative and*

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y) \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$. If (3) holds in the reversed sense, then f is h-concave, $(f\in SV(h,I))$.

Now, we define the new concept of (α,m,h)-convex functions.

Definition 4. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ a real function. We say that $f:I\subseteq\to(0,\infty)$ is (α,m,h)-convex function $(f\in SX(h,I))$, if*

$$f(tx+m(1-t)y)\leq h(t^{\alpha})f(x)+h(1-t^{\alpha})f(y), \tag{4}$$

for all $x,y\in I$ and $t\in(0,1)$.

Now we discuss some special cases of our proposed definition of (α,m,h)-convex functions.
I. If $h(t)=t$, then, we have definition of (α,m)-convex functions.
II. If $\alpha=1=m$ and $h(t)=t$, then, we have definition of classical convex functions.
III. If $\alpha=1=m$, then, we have definition of h-convex functions.
IV. If $\alpha=1$ and $h(t)=t^s$, then, we have definition of (s,m)-convex functions.
V. If $\alpha=1$ and $h(t)=t^{-s}$, then, we have definition of (s,m)-Godunova-Levin functions.
VI. If $\alpha=1$ and $h(t)=t^{-1}$, then, we have definition of m-Godunova-Levin functions.
VII. If $m=1$ and $h(t)=t^s$, then, we have definition of (α,s)-convex functions.
VIII. If $m=1$ and $h(t)=t^{-s}$, then, we have definition of (α,s)-Godunova-Levin functions.
IX. If $m=1$ and $h(t)=t^{-1}$, then, we have definition of α-Godunova-Levin functions.
One can see that the class of (α,m,h)-convex functions is quite unifying one as it contains several other classes of convex functions as sepcial cases.
We need an auxiliary result, which will be used in obtaining our main results.

Lemma 1([14]). *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. If $f''\in L[a,b]$, then*

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\mathrm{d}x$$
$$=\frac{(mb-a)^2}{2}\int_0^1(t-t^2)f''(ta+m(1-t)b)\mathrm{d}t.$$

The well known gamma function is defined as:

$$\Gamma(x)=\int_0^{\infty}e^{-x}t^{x-1}\mathrm{d}t,$$

3 Main Results

In this section, we derive our main results.

Theorem 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0 and $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. Suppose $|f''|$ is (α,m,h)-convex functions, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\mathrm{d}x\right|$$
$$=\frac{(mb-a)^2}{2}[M(\alpha,m,h;t)|f''(a)|+N(\alpha,m,h;t)|f''(b)|],$$

where

$$M(\alpha,m,h;t)=\int_0^1(t-t^2)h(t^{\alpha})\mathrm{d}t$$
$$N(\alpha,m,h;t)=\int_0^1 m(t-t^2)h(1-t^{\alpha})\mathrm{d}t.$$

Proof. Using Lemma 2.5 and the fact that $|f''|$ is (α,m,h)-convex function, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\mathrm{d}x\right|$$
$$=\left|\frac{(mb-a)^2}{2}\int_0^1(t-t^2)f''(ta+m(1-t)b)\mathrm{d}t\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1(t-t^2)|f''(ta+m(1-t)b)|\mathrm{d}t$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1(t-t^2)[h(t^{\alpha})|f''(a)|$$
$$+mh(1-t^{\alpha})|f''(b)|]\mathrm{d}t$$
$$=\frac{(mb-a)^2}{2}[M(\alpha,m,h;t)|f''(a)|+N(\alpha,m,h;t)|f''(b)|].$$

This completes the proof. □

We now discuss some special cases.

I. If $h(t)=t^s$, then we have

Corollary 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0 and $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. Suppose $|f''|$ is (α,m,s)-convex functions, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\mathrm{d}x\right|$$
$$=\frac{(mb-a)^2}{2}[M_1(\alpha,m,s;t)|f''(a)|+N_1(\alpha,m,s;t)|f''(b)|],$$

where

$$M_1(\alpha,m,s;t) = \int_0^1 (t-t^2)(t^\alpha)^s \mathrm{d}t = \frac{1}{6+5s\alpha+s^2\alpha^2},$$

and

$$\begin{aligned} &N_1(\alpha,m,s;t) \\ &= \int_0^1 m(t-t^2)(1-t^\alpha)^s \mathrm{d}t \\ &= \frac{m}{6}\Gamma[1+s]\left\{\left[\frac{3\Gamma\left(\frac{2+\alpha}{\alpha}\right)}{\Gamma(1+s+2/\alpha)}\right] - \left[\frac{2\Gamma\left(\frac{3+\alpha}{\alpha}\right)}{\Gamma(1+s+3/\alpha)}\right]\right\}. \end{aligned}$$

II. If $h(t) = t^{-s}$, then we have

Corollary 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable on I^0 and $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$ and $m \in (0,1]$. Suppose $|f''|$ is (α,m,s)-Godunova-Levin functions, then*

$$\begin{aligned} &\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \\ &= \frac{(mb-a)^2}{2}[M_2(\alpha,m,-s;t)|f''(a)| \\ &\quad + N_2(\alpha,m,-s;t)|f''(b)|], \end{aligned}$$

where

$$M_2(\alpha,m,-s;t) = \int_0^1 (t-t^2)(t^\alpha)^{-s} \mathrm{d}t = \frac{1}{6-5s\alpha+s^2\alpha^2},$$

and

$$\begin{aligned} &N_2(\alpha,m,-s;t) \\ &= \int_0^1 m(t-t^2)(1-t^\alpha)^{-s} \mathrm{d}t \\ &= \frac{m}{6}\Gamma[1-s]\left\{\left[\frac{3\Gamma\left(\frac{2+\alpha}{\alpha}\right)}{\Gamma(1-s+2/\alpha)}\right] - \left[\frac{2\Gamma\left(\frac{3+\alpha}{\alpha}\right)}{\Gamma(1-s+3/\alpha)}\right]\right\}. \end{aligned}$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable on I^0 and $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$ and $m \in (0,1]$. Suppose $|f''|^q$ is (α,m,h)-convex functions, then*

$$\begin{aligned} &\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \\ &\leq \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \\ &\quad \times \left(K(\alpha,h;t)|f''(a)|^q + L(\alpha,h;t)|f''(b)|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} K(\alpha,h;t) &= \int_0^1 h(t^\alpha)\mathrm{d}t \\ L(\alpha,m,h;t) &= \int_0^1 mh(1-t^\alpha)\mathrm{d}t. \end{aligned}$$

Proof. Using Lemma 2.5 and the fact that $|f''|^q$ is (α,m,h)-convex function, we have

$$\begin{aligned} &\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \\ &= \left|\frac{(mb-a)^2}{2}\int_0^1 (t-t^2)f''(ta+m(1-t)b)\mathrm{d}t\right| \\ &\leq \frac{(mb-a)^2}{2}\left(\int_0^1 (t-t^2)^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^1 |f''(ta+m(1-t)b)|^q \mathrm{d}t\right)^{\frac{1}{q}} \\ &\leq \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^1 \left[h(t^\alpha)|f''(a)|^q + mh(1-t^\alpha)|f''(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\ &= \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \\ &\quad \times \left(K(\alpha,h;t)|f''(a)|^q + L(\alpha,h;t)|f''(b)|^q\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. □

We now discuss some special cases.
I. If $h(t) = t^s$, then we have

Corollary 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable on I^0 and $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$ and $m \in (0,1]$. Suppose $|f''|^q$ is (α,m,s)-convex functions, then*

$$\begin{aligned} &\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \\ &= \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \\ &\quad \times \left(K_1(\alpha,s;t)|f''(a)|^q + L_1(\alpha,s;t)|f''(b)|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where

$$K_1(\alpha,s;t) = \int_0^1 (t^{\alpha s})\mathrm{d}t = \frac{1}{1+s\alpha}$$

$$L_1(\alpha,m,s;t) = \int_0^1 m(1-t^{\alpha})^s\mathrm{d}t = \frac{m\Gamma(1+s)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+s+\frac{1}{\alpha}\right)}.$$

II. If $h(t)=t^{-s}$, then we have

Corollary 4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0 and $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. Suppose $|f''|^q$ is (α,m,s)-Godunova-Levin functions, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| = \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \times\left(K_2(\alpha,-s;t)|f''(a)|^q+L_2(\alpha,-s;t)|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$K_2(\alpha,-s;t) = \int_0^1 (t^{\alpha s})\mathrm{d}t = \frac{1}{1-s\alpha}$$

$$L_2(\alpha,m,-s;t) = \int_0^1 m(1-t^{\alpha})^s\mathrm{d}t = \frac{m\Gamma(1-s)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1-s+\frac{1}{\alpha}\right)}.$$

Theorem 3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0 and $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. Suppose $|f''|^q$ is (α,m,h)-convex functions, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \le \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \times\left(P(\alpha,h;t)|f''(a)|^q+Q(\alpha,m,h;t)|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$P(\alpha,h;t) = \int_0^1 (t-t^2)h(t^{\alpha})\mathrm{d}t$$

$$Q(\alpha,m,h;t) = \int_0^1 m(t-t^2)h(1-t^{\alpha})\mathrm{d}t.$$

Proof. Using Lemma 2.5 and the fact that $|f''|^q$ is (α,m,h)-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \\
&= \left|\frac{(mb-a)^2}{2}\int_0^1 (t-t^2)f''(ta+m(1-t)b)\mathrm{d}t\right| \\
&\le \frac{(mb-a)^2}{2}\left(\int_0^1 (t-t^2)\mathrm{d}t\right)^{1-\frac{1}{q}} \times\left(\int_0^1 (t-t^2)|f''(ta+m(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \times\left(\int_0^1 (t-t^2)\left[h(t^{\alpha})|f''(a)|^q+mh(1-t^{\alpha})|f''(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \times\left(P(\alpha,h;t)|f''(a)|^q+Q(\alpha,m,h;t)|f''(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Some special cases:
I. If $h(t)=t^s$, then we have

Corollary 5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable on I^0 and $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $m\in(0,1]$. Suppose $|f''|^q$ is (α,m,s)-convex functions, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right| \le \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}} \times\left(M_1(\alpha,s;t)|f''(a)|^q+N_1(\alpha,m,s;t)|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$M_1(\alpha,s;t) = \int_0^1 (t-t^2)(t^{\alpha s})\mathrm{d}t = \frac{1}{6+5s\alpha+s^2\alpha^2},$$

and

$$N_1(\alpha,m,s;t) = \int_0^1 m(t-t^2)(1-t^{\alpha})^s\mathrm{d}t$$

$$= \frac{m}{6}\Gamma[1+s]\left\{\left[\frac{3\Gamma\left(\frac{2+\alpha}{\alpha}\right)}{\Gamma(1+s+2/\alpha)}\right] - \left[\frac{2\Gamma\left(\frac{3+\alpha}{\alpha}\right)}{\Gamma(1+s+3/\alpha)}\right]\right\}.$$

II. If $h(t) = t^{-s}$, then we have

Corollary 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable on I^0 and $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$ and $m \in (0,1]$. Suppose $|f''|^q$ is (α, m, s)-Godunova-Levin functions, then*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\mathrm{d}x\right|$$

$$\leq \frac{(mb-a)^2}{2}\left(\frac{2^{-1-p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}$$

$$\times\left(M_2(\alpha,h;t)|f''(a)|^q + N_2(\alpha,h;t)|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$M_2(\alpha,m,-s;t)$$

$$= \int_0^1 (t-t^2)(t^\alpha)^{-s}\mathrm{d}t = \frac{1}{6-5s\alpha+s^2\alpha^2},$$

and

$$N_2(\alpha,m,-s;t) = \int_0^1 m(t-t^2)(1-t^\alpha)^{-s}\mathrm{d}t$$

$$= \frac{m}{6}\Gamma[1-s]\left\{\left[\frac{3\Gamma\left(\frac{2+\alpha}{\alpha}\right)}{\Gamma(1-s+2/\alpha)}\right] - \left[\frac{2\Gamma\left(\frac{3+\alpha}{\alpha}\right)}{\Gamma(1-s+3/\alpha)}\right]\right\}.$$

4 Conclusion

In this paper, we have introduced and studied a new more generalized class of convex functions. It is noticed that this class includes the known and unknown classes of convex functions as special cases. Several new integral inequalities have been obtained via (α, m, h)-convexity. New refinements of other classical inequalities such as, Ostrowski's inequalities, Simpson's inequalities and Newton's inequalities, can also be obtained via this class of convex functions. This will be an interesting subject of future research. The interested readers are encouraged to consider the applications of these new concepts.

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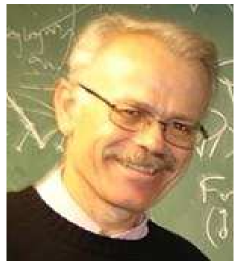
Khalida Inayat Noor is a leading world-known figure in mathematics and is presently employed as HEC Foreign Professor at CIIT, Islamabad. She obtained her PhD from Wales University (UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: Presidents Award for pride of performance on August 14, 2010 for her outstanding contributions in Mathematical Sciences. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/ MS programs at CIIT. Prof. Dr. Khalida Inayat Noor has supervised successfully more than 22 Ph.D students and 40 MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 500 research articles in reputed international journals of mathematical and engineering sciences. She is member of educational boards of several international journals of mathematical and engineering sciences.



Themistocles M. Rassias is a Greek mathematician, and a Professor at the National Technical University of Athens, Greece. Prof Rassias received his Ph.D. in Mathematics from the University of California at Berkeley in June 1976. He has published more than 300 papers, 10 research books and 45 edited volumes in research Mathematics as well as 4 textbooks in Mathematics (in Greek) for university students. His research work has received more than 13,000 citations according to Google Scholar and more than 4,500 citations according to MathSciNet. He serves as a member of the Editorial Board of several international mathematical journals. Dr. Th. M. Rassias has received academic positions at several Universities including Harvard University (Research Associate, 1980), Massachusetts Institute of Technology (Visiting Professor, 1980). He was honored with the titles : Accademico Ordinario of the Accademia Tiberina, Roma in 1987 as well as Fellow of the Royal Astronomical Society of London in 1991. The University of La Verne, California (Athens Campus) honored him with the title Outstanding Faculty Member for the years : 1989 1990, 1990 1991, 191 1992. Three foreign Universities have bestowed upon him the title of Doctor Honoris Causa (DHC). In 2016, Dr. Rassias was honored with the Award for Lifetime Achievements in Mathematics during the International Conference on Ulams Type Stability , Cluj Napoca, Romania.