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Characterization and Estimation of Weighted Maxwell-Boltzmann Distribution

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Abstract: In this paper, we have introduced weighted Maxwell-Boltzmann distribution and abbrivated as WMD. Different characteristic properties of the introduced distribution have been studied in detail. Although the estimators are not derived in a closed form but parametrs are estimated through the fitting of WMD to a particular data sets using the technique of MLE. In order to show the validilty, potentiality and flexibility of WMD in statistical modelling, we have fitted it to four different types of data sets. After the fitting of WMD to the considered data sets, comparison has been made between the special cases of WMD in terms of having least values of BIC, AIC & $AICC$. Random numbers from WMD are generated by using the Inverse Cdf method. Simulation has been carried out with the help of programming language R.

Keywords: Length-biased Maxwell-Boltzmann distribution (LBMD), Area-biased Maxwell-Boltzmann distribution (ABMD), Moment generating function, Characteristic function, Reliability, Entropy, Bonferroni curve, Lorenz curve, Order Statistics, AIC, $AICC$, BIC.

1 Introduction

The concept of weighted distributions can be traced from the work of Fisher [25], in connection with his studies, on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was developed and formulated in general terms by C.R. Rao [5] in connection with modelling statistical data, where the usual practice of using standard distributions for the purpose was not found to be appropriate. It is quite obvious that while studying the real world random phenomena, the observations may be recorded with an amount of inherent bias. As a result of which these recorded observations will not have the original distribution unless every observation is given an equal chance of being recorded. C.R. Rao [5] introduced a new class of distributions known as weighted distributions after analyzing the situations where observations are recorded with varying probabilities.

The concept of weighted distribution is very important, because of the fact that weighted distributions take into consideration the method of ascertainment, by adjusting the probabilities of actual occurrence of events. We may arrive at wrong conclusions, while failing to make such adjustment. Thus, it is very imperative to use the concept of weighted distribution while dealing with a stochastic process in which the observations are being generated or recorded with varying probability. In order to increase the accuracy and to draw sound results, our main motive becomes to give importance to model specification. One of the unifying approaches for this purpose is to use the concept of weighted distributions. The importance of weighted distributions can be understood from L.L. Macdonald [20] discussing the need for teaching weighted distribution theory.

There are some traditional theories and practices which have been occupied with replication and randomisation like environmentric theory. Observations also fall in the non-experimental, non-replicated and non-randomised categories. Thus our main interest lies in drawing the inference about random phenomena with higher degree of accuracy. We can't guarantee the degree of accuracy of results unless the suitable and flexible model are used for statistical modelling. G. P. Patil and C. R. Rao [9] quoted "Although the situations that involve weighted distributions seem to occur frequently in various fields, the underlying concept of weighted distributions as

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a major stochastic concept does not seem to have been widely recognized". Thus it is very essential to identify the stochastic processes where observations are recorded with varying probabilities so that the validity and importance of weighted distributions in statistical modelling can be understood.

The concept of weighted distributions attracted a lot of researchers to contemplate on and to carry out research on the same. G.P.Patil and C.R.Rao [9] studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Van Deusen [23] arrived at size biased distribution theory independently and applied it in fitting assumed distributions to data arising from horizontal point sampling. Subsequently, Lappi and Bailey [15] used weighted distributions to analyze HPS diameter increment data. Rao [6] studied the weighted distributions arising out of method of ascertainment. In fisheries, Taillie et. al [7] modelled populations of fish stocks using weights. In ecology, Dennis and Patil [4] used stochastic differential equations to arrive at weighted properties of size-biased Gamma distribution. K.G. Janardan [18] characterized the weighted Lagrange distributions. K.G. Janardan and B.R. Rao [17] studied the Lagrange distributions of the second kind and weighted distributions. G.S. Lingappaiah [11] discussed Lagrange-negative binomial distribution in its simple and weighted forms. R.S. Ambagaspitiya [24] defined weighted generalized Negative Binomial distribution. Asgharian, M. *et al.* [19] worked on the length-biased sampling with right censoring. Gove [16] studied the estimation and application of size-biased distributions in forestry. Kvam [22] discussed about the Length bias in the measurements of Carbon Nanotubes. Dar *et al.*[1] characterized the transmuted weighted Exponential distribution and discussed some of it's application. Reshi *et al.* [14] worked on new moment method of estimation of parameters of size-biased classical gamma distribution.

Definition: Let us suppose that X be a continuous random variable of interest such that $X \sim f(x;\theta)$. However if the sample observations are selected with probability proportional to weighted function $w(x) = x^{\omega}$, where $\omega > 0$ is the weight parameter. Then the distribution, with Pdf given by:

$$f\left(x;\theta,\omega\right) = \frac{w\left(x\right) f\left(x;\theta\right)}{E\left[w\left(x\right)\right]} \quad (1)$$

Is called the weighted distribution of random variable X.

2 Derivation of Weighted Maxwell Distribution (WMD)

In Physics and Chemistry there is a lot of applications of Maxwell (or Maxwell-Boltzmann) distribution. The Maxwell distribution forms the basis of the kinetic energy of gases, which explains many fundamental properties of gases, including pressure and diffusion. This distribution is sometimes referred to as the distribution of velocities, energy and magnitude of momenta of molecules. It was Tyagi and Bhattacharya [26,27] who considered the Maxwell distribution as a lifetime model for the first time and discussed the Baye's and minimum variance unbiased estimation procedures for it's parameter and reliability function. Chaturvedi and Rani [2] obtained classical and Baye's estimators for the Maxwell distribution, after generalizing it by introducing one more parameter. Empirical Baye's estimation for the Maxwell distribution was studied by Bekker and Roux [3]. Kazmi *et al.* [28] carried out the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data. Herein, we considered Maxwell distribution and constructed its weighted version. The *Pdf* of a random variable X following Maxwell distribution with rate parameter θ is given by (2).

$$f\left(x;\theta\right) = \sqrt{2/\pi}\,\theta^{\frac{3}{2}} x^2 exp\left(-\theta x^2/2\right); x,\theta > 0 \quad (2)$$

Weight function: The weight function considered is $w(x) = x^{\omega}$, where $\omega > 0$ is the weight parameter. Therefore,

$$E\left[w(x)\right] = \frac{2^{\omega/2+1}\Gamma\left((\omega+3)/2\right)}{\sqrt{\pi\theta^{\omega}}} \quad (3)$$

Now, from the definition(1), we will have the *Pdf* of WMD as given by (4):

$$f_{\omega}\left(x;\theta,\omega\right) = \frac{\theta^{(\omega+3)/2} x^{\omega+2} exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma\left((\omega+3)/2\right)} \quad (4)$$

Cdf, Reliability function and hazard rate of WMD are respectively given by (5), (6) and (7)

$$F_{\omega}\left(x;\theta,\omega\right) = 1 - \frac{\Gamma\left((\omega+3)/2, \theta x^2/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (5)$$

$$R_{\omega}\left(x;\theta,\omega\right) = \frac{\Gamma\left((\omega+3)/2, \theta x^2/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (6)$$

$$h_{\omega}\left(x;\theta,\omega\right) = \frac{\theta^{(\omega+3)/2} x^{\omega+2} exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}, \theta x^2/2\right)} \quad (7)$$

Table-1: Special cases of WMD at different values of ω

weight (ω)	$\omega = 0$	$\omega = 1$	$\omega = 2$
Distribution	**MD**	**LBMD**	**ABMD**
$f\omega(x;\theta)$	$\sqrt{\frac{2}{\pi}}\theta^{3/2}x^2e^{-\theta x^2/2}$	$(1/2)\theta^2x^3e^{-\theta x^2/2}$	$\left\{2^{-3/2}/\Gamma(5/2)\right\}\theta^{5/2}x^4e^{-\theta x^2/2}$
$F\omega(x;\theta)$	$1-\frac{\Gamma(3/2,\theta x^2/2)}{\Gamma(3/2)}$	$1-\frac{\Gamma(2,\theta x^2/2)}{\Gamma(2)}$	$1-\frac{\Gamma(5/2,\theta x^2/2)}{\Gamma(5/2)}$

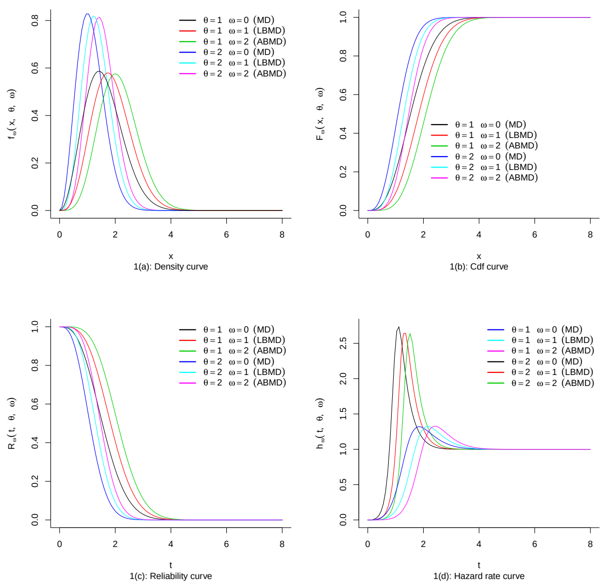

Fig. 1: Curves of density, distribution, reliability and hazard rate function at different values of rate and weight parameter.

3 Structural properties of WMD

In this section, various structural properties of WMD has been discussed.

Theorem 3.1. The r^{th} moment about origin of a random variable X following WMD is given by:

$$\mu_r' = \left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma\left((\omega+r+3)/2\right)}{\Gamma\left((\omega+3)/2\right)}; r = 1,2,3,\ldots \quad (8)$$

Proof.

$$\mu_r' = \int_0^\infty x^r f_\omega\left(x;\theta,\omega\right) dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\theta^{(\omega+3)/2} x^{\omega+2}}{2^{(\omega+1)/2}\Gamma\left((\omega+3)/2\right)} exp\left(-\theta x^2/2\right) dx$$

$$\mu_r' = \frac{\theta^{(\omega+3)/2}}{2^{(\omega+1)/2}\Gamma\left((\omega+3)/2\right)} \int_0^\infty x^{\omega+r+2} exp\left(-\theta x^2/2\right) dx$$

$$\mu_r' = \left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma\left((\omega+r+3)/2\right)}{\Gamma\left((\omega+3)/2\right)}$$

Moments: First four moments about origin are given as follows:

$$\mu_1' = \sqrt{\frac{2}{\theta}} \frac{\Gamma\left((\omega+4)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (9)$$

$$\mu_2' = \left(\frac{2}{\theta}\right) \frac{\Gamma\left((\omega+5)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (10)$$

$$\mu_3' = \left(\frac{2}{\theta}\right)^{3/2} \frac{\Gamma\left((\omega+6)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (11)$$

$$\mu_4' = \left(\frac{2}{\theta}\right)^{2} \frac{\Gamma\left((\omega+7)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (12)$$

Variance:

$$\sigma^2 = \frac{2\left[\Gamma\left((\omega+3)/2\right)\Gamma\left((\omega+5)/2\right) - \left\{\Gamma\left((\omega+4)/2\right)\right\}^2\right]}{\theta\left[\Gamma\left((\omega+3)/2\right)\right]^2} \quad (13)$$

Variation, Skewness and Kurtosis: Coefficient of variation, skewness and kurtosis are respectively given by (14), (15) and (16)

$$c.v. = \frac{\sqrt{\Gamma\left((\omega+3)/2\right)\Gamma\left((\omega+5)/2\right) - \left\{\Gamma\left((\omega+4)/2\right)\right\}^2}}{\Gamma\left((\omega+4)/2\right)} \quad (14)$$

$$\gamma_1 = \frac{\left\{\Gamma\left((\omega+3)/2\right)\right\}^2}{\left[\Gamma\left((\omega+3)/2\right)\Gamma\left((\omega+5)/2\right) - \left\{\Gamma\left((\omega+4)/2\right)\right\}^2\right]^{3/2}} \times \left[\Gamma\left(\frac{\omega+6}{2}\right) - 3\frac{\Gamma\left(\frac{\omega+4}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} + 2\frac{\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^3}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2}\right] \quad (15)$$

$$\gamma_2 = \frac{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2}{\left[\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right) - \left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2\right]^2} \times \left[\Gamma\left(\frac{\omega+7}{2}\right) - \frac{4\Gamma\left(\frac{\omega+6}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} + \frac{6\Gamma\left(\frac{\omega+5}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^2} - 3\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^4 / \left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^3\right] \quad (16)$$

Theorem 3.2. The moment generating function and characteristic function of a random variable X following WMD are respectively given by (17) and (18).

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma\left((\omega+r+3)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (17)$$

$$\Psi_x(t) = \sum_{r=0}^{\infty} \frac{(\iota t)^r}{r!}\left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma\left((\omega+r+3)/2\right)}{\Gamma\left((\omega+3)/2\right)} \quad (18)$$

Proof. From the definition of mgf we have:

$$M_x(t) = E\left[e^{tx}\right]$$
$$M_x(t) = \int_0^\infty e^{tx} f_\omega(x;\theta,\omega)\,dx$$
$$M_x(t) = \int_0^\infty e^{tx} \frac{\theta^{(\omega+3)/2} x^{\omega+2} exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)} dx$$
$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\theta^{(\omega+3)/2} \int_0^\infty x^{\omega+r+2} exp\left(-\theta x^2/2\right) dx}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)}$$
$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)}$$

Also, we have

$$\Phi_x(t) = M_x(\iota t)$$

Therefore,

$$\Phi_x(t) = \sum_{r=0}^{\infty} \frac{(\iota t)^r}{r!} \left(\frac{2}{\theta}\right)^{r/2} \frac{\Gamma((\omega+r+3)/2)}{\Gamma((\omega+3)/2)}$$

Fisher's Information matrix: Fisher's Information matrix of WMD is given by:

$$I(\theta,\omega) = -nE\begin{bmatrix} \frac{\partial^2 \log\{f_\omega(x;\theta,\omega)\}}{\partial\theta^2} & \frac{\partial^2 \log\{f_\omega(x;\theta,\omega)\}}{\partial\theta\partial\omega} \\ \frac{\partial^2 \log\{f_\omega(x;\theta,\omega)\}}{\partial\omega\partial\theta} & \frac{\partial^2 \log\{f_\omega(x;\theta,\omega)\}}{\partial\omega^2} \end{bmatrix}$$
$$I(\theta,\omega) = \begin{bmatrix} \frac{n(\omega+3)}{2\theta^2} & \frac{n}{2\theta} \\ \frac{n}{2\theta} & \frac{n}{4}\Psi'\left(\frac{\omega+3}{2}\right) \end{bmatrix} \quad (19)$$

where $\Psi(x) = \frac{\partial \log\Gamma(z)}{\partial z} = \frac{\Gamma'(z)}{\Gamma(z)}$ is known as digamma or Psi function.

4 Information measures of WMD

Theory which deals with the study of transmission, processing, utilization, and extraction of information is called Information theory. Abstractly, information can be viewed as the resolution of uncertainty. It was Claude E. Shannon [8], who originally proposed the information theory in a landmark article. In this article, "information" is thought of as a set of possible messages, where the goal is to send these messages over a noisy channel, and then to have the receiver reconstruct the message with low probability of error, in spite of the channel noise. The quantification, storage, and communication of information is having a key measure known as entropy. The amount of uncertainty in the value of a random variable or the outcome of a random process is measured in terms of entropy measure. A number of Information measures have been proposed by various authors. Shannon and Renyi entropy are two of them.

4.1 Renyi Entropy

Information measure propose by Alferd Renyi known as Renyi entropy of order δ for a random variable X is given by:

$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty \{f(x)\}^\delta dx\right] \quad (20)$$

where $\delta \geq 0, \delta \neq 1$

4.2 Shannon Entropy

Shannon's measure of information for the outcome of a random process is given by:

$$\gamma_S = E[-\log f(x)] \quad (21)$$

Shannon entropy is one of the special case of Renyi entropy as the order (δ) of Renyi entropy tends to 1. i.e.

$$\gamma_S = \lim_{\delta\to 1}\gamma_R(\delta) \quad (22)$$

Theorem 4.1. Renyi and Shannon entropy of WMD is respectively given by:

$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\frac{(2\theta)^{(\delta-1)/2}\Gamma((2\omega+2\delta+1)/2)}{\delta^{(\delta\omega+2\delta+1)/2}\{\Gamma((\omega+3)/2)\}^\delta}\right] \quad (23)$$

$$\gamma_S = \frac{1}{2}[3+\omega-\log(2\theta)+2\log\{\Gamma((\omega+3)/2)\} -(\omega+2)\Psi((\omega+3)/2)] \quad (24)$$

Proof. From (20), we have,

$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty \{f_\omega(x;\theta,\omega)\}^\delta dx\right]$$
$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty \left\{\frac{\theta^{\frac{\omega+3}{2}} x^{\omega+2} exp(\frac{-\delta\theta x^2}{2})}{2^{\frac{\omega+1}{2}}\Gamma((\omega+3)/2)}\right\}^\delta dx\right]$$
$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\frac{\theta^{\frac{\delta(\omega+3)}{2}} \int_0^\infty x^{\delta(\omega+2)} exp(\frac{-\delta\theta x^2}{2})dx}{2^{\delta(\omega+1)/2}\{\Gamma((\omega+3)/2)\}^\delta}\right]$$
$$\gamma_R(\delta) = \frac{1}{1-\delta}\log\left[\frac{(2\theta)^{(\delta-1)/2}\Gamma((2\omega+2\delta+1)/2)}{\delta^{(\delta\omega+2\delta+1)/2}\{\Gamma((\omega+3)/2)\}^\delta}\right]$$

From (22), we have,

$$\gamma_S = \lim_{\delta\to 1}\frac{1}{1-\delta}\log\left[\frac{(2\theta)^{(\delta-1)/2}\Gamma((2\omega+2\delta+1)/2)}{\delta^{(\delta\omega+2\delta+1)/2}\{\Gamma((\omega+3)/2)\}^\delta}\right]$$
$$\gamma_S = \lim_{\delta\to 1}\frac{1}{1-\delta}\log\left[\frac{(2\theta)^{(\delta-1)/2}\Gamma((2\omega+2\delta+1)/2)}{\delta^{(\delta\omega+2\delta+1)/2}\{\Gamma((\omega+3)/2)\}^\delta}\right]$$
$$\gamma_S = \frac{1}{2}[3+\omega-\log(2\theta)+2\log\{\Gamma((\omega+3)/2)\} -(\omega+2)\Psi((\omega+3)/2)]$$

hence the Theorem.

Table-2: Characteristics of WMD at diff. values of θ & ω

θ	ω	Mean	variance	c.v.	γ_1	γ_2	Renyi Entropy δ 0.2	0.5	0.7	0.9999	Shannon Entropy
1	0	1.59577	0.45352	0.42202	0.48569	3.10816	1.41796	1.16050	1.07737	0.9961759	0.996154
	1	1.87997	0.46571	0.36299	0.40569	3.05929	1.45294	1.18975	1.10356	1.0192720	1.019250
	2	2.12769	0.47293	0.32321	0.35424	3.03698	1.47584	1.20662	1.11816	1.0318190	1.031796
2	0	1.12838	0.22676	0.42202	0.48569	3.10816	1.07139	0.81393	0.73080	0.6496023	0.649580
	1	1.32934	0.23285	0.36299	0.40569	3.05929	1.10637	0.84318	0.75699	0.6726989	0.672676
	2	1.50451	0.23646	0.32321	0.35424	3.03698	1.12927	0.86005	0.77159	0.6852455	0.685222
5	0	0.71365	0.09070	0.42202	0.48569	3.10816	0.61324	0.35578	0.27270	0.1914570	0.191435
	1	0.84075	0.09314	0.36299	0.40569	3.05929	0.64822	0.38503	0.29884	0.2145535	0.214531
	2	0.95153	0.09459	0.32321	0.35424	3.03698	0.67112	0.40190	0.31344	0.2271001	0.227077

From Table-2, it is quite evident that on increasing the value of weight parameter (ω) for the fixed value of rate parameter (θ), mean, variance and entropy increase whereas the coefficient of variation, skewness and kurtosis start decreasing. While on increasing the value of rate parameter for the fixed value of weight parameter, mean, variance and entropy decreases whereas the other three characteristics i.e. coefficient of variation, skewness and kurtosis remains unaffected due to their independence from θ . It can also be seen from the last two columns of the table-2 that Renyi entropy approaches to Shannon entropy as the order (δ) of Renyi entropy tends to 1.

5 Statistical properties of WMD

5.1 Bonferroni and Lorenz curve

In economics, Lorenz curve is a graphical representation of the distribution of income or of wealth. It was American economist Max O. Lorenz [21] who developed Lorenz curve in 1905 for representing inequality of the wealth distribution. The Lorenz curve is usually represented by a function $L(p)$, where p is the cumulative portion of the population, represented by the horizontal axis, and L denotes the cumulative portion of the total wealth or income, represented by the vertical axis. Lorenz curve and the associated Gini index are off course the most popular indices of income inequality. However, there are some measures which despite possessing interesting characteristics are not used often for measuring income inequality. Bonferroni curve (BC) is one such measure, which have the advantage of being represented graphically in the unit square and can also be related to the Lorenz curve (Giorgi and Mondani[13] , Giorgi [12]). The Bonferroni and Lorenz curves are not only used in economics in order to study the relation between income and poverty, it is also being used in reliability, medicine, insurance and demography. The Bonferroni and the Lorenz curves for a non-negative random variable $X \sim f(x)$ are respectively given by (25) and (26).

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)\, dx \tag{25}$$

$$L(p) = pB(p) = \frac{1}{\mu}\int_0^q x f(x)\, dx \tag{26}$$

where, $q = F^{-1}(p)$ and $\mu = E[x]$

Theorem 5.1. The Bonferroni and Lorenz curve for a random variable X following WMD is respectively given by (27) and (28)

$$B(p) = \frac{\Gamma((\omega+4)/2) - \Gamma\left((\omega+4)/2, q^2\theta/2\right)}{p\Gamma((\omega+4)/2)} \tag{27}$$

$$L(p) = \frac{\Gamma((\omega+4)/2) - \Gamma\left((\omega+4)/2, q^2\theta/2\right)}{\Gamma((\omega+4)/2)} \tag{28}$$

Proof. From (25), we can write,

$$B(p) = \frac{1}{p\mu}\int_0^q x f_\omega(x;\theta,\omega)\, dx$$

$$B(p) = \frac{\Gamma((\omega+3)/2)}{p\sqrt{2/\theta}\,\Gamma((\omega+4)/2)} \times \int_0^q x \frac{\theta^{(\omega+3)/2} x^{\omega+2} exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\Gamma((\omega+3)/2)} dx$$

$$B(p) = \frac{\theta^{(\omega+4)/2}\int_0^q x^{\omega+3} exp\left(-\theta x^2/2\right) dx}{p\, 2^{(\omega+2)/2}\Gamma((\omega+4)/2)}$$

$$B(p) = \frac{\theta^{\frac{\omega+4}{2}}}{p\, 2^{\frac{\omega+2}{2}}\Gamma\left(\frac{\omega+4}{2}\right)} \frac{2^{\frac{\omega+2}{2}}\left[\Gamma\left(\frac{\omega+4}{2}\right)) - \Gamma\left(\frac{\omega+4}{2}, \frac{q^2\theta}{2}\right)\right]}{\theta^{(\omega+4)/2}}$$

$$B(p) = \frac{\Gamma((\omega+4)/2) - \Gamma\left((\omega+4)/2, q^2\theta/2\right)}{p\Gamma((\omega+4)/2)}$$

Also from (26), we have, $L(p) = pB(p)$. Therefore,

$$L(p) = \frac{\Gamma((\omega+4)/2) - \Gamma\left((\omega+4)/2, q^2\theta/2\right)}{\Gamma((\omega+4)/2)}$$

hence the Theorem.

5.2 Order Statistics

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be an ordered random sample of odd size (i.e. $n = 2m+1,\ m = 0,1,2,...$) from WMD. Then the *Pdf* of $X_{(1)}$ (minimum order statistics), $X_{(n)}$ (maximum order statistics), $X_{(r)}$ (r^{th} order statistics) and $X_{(m+1)}$ (sample median) are respectively given by (29), (30), (31) and (32)

Pdf of $X_{(1)}$:

$$f_{X_{1:n}}(x) = \frac{n\left\{\Gamma\left((\omega+3)/2, \theta x^2/2\right)\right\}^{n-1}}{2^{(\omega+1)/2}\left\{\Gamma((\omega+3)/2)\right\}^n} \times \left[\theta^{(\omega+3)/2} x^{\omega+2} exp\left(-\theta x^2/2\right)\right] \tag{29}$$

Pdf of $X_{(n)}$:

$$f_{X_{n:n}}(x)=\frac{n\left\{\Gamma((\omega+3)/2)-\Gamma\left((\omega+3)/2,\theta x^2/2\right)\right\}^{n-1}}{2^{(\omega+1)/2}\left\{\Gamma((\omega+3)/2)\right\}^{n}}\times\left[\theta^{(\omega+3)/2}x^{\omega+2}exp\left(-\theta x^2/2\right)\right] \quad (30)$$

Pdf of $X_{(r)}$:

$$f_{X_{r:n}}(x)=\frac{n!\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^{n-r}\left\{\Gamma\left(\frac{\omega+3}{2}\right)-\Gamma\left(\frac{\omega+3}{2},\frac{\theta x^2}{2}\right)\right\}^{r-1}}{(r-1)!(n-r)!2^{\frac{\omega+1}{2}}\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^{n}}\times\left[\theta^{\frac{\omega+3}{2}}x^{\omega+2}exp\left(-\theta x^2/2\right)\right] \quad (31)$$

Pdf of $X_{(m+1)}$:

$$f_{X_{r:n}}(x)=\frac{(2m+1)!}{m!\,m!}\frac{\theta^{(\omega+3)/2}x^{\omega+2}exp\left(-\theta x^2/2\right)}{2^{(\omega+1)/2}\left[\Gamma\left(\frac{\omega+3}{2}\right)\right]^{2m+1}} \quad (32)$$
$$\times\left[\Gamma\left(\frac{\omega+3}{2},\frac{\theta x^2}{2}\right)\right]^{2m}\left[\frac{\Gamma\left(\frac{\omega+3}{2}\right)}{\Gamma\left(\frac{\omega+3}{2},\frac{\theta x^2}{2}\right)}-1\right]^{m}$$

6 Characterization of WMD

Theorem 6.1. Let $X_1,X_2,..,X_n$, be a random sample of size n drawn from WMD. Then, the square of sample coefficient of variation is asymptotically unbiased estimator of the square of population coefficient of variation.
Mathematically, $\lim\limits_{n\to\infty}E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\lim\limits_{n\to\infty}E\left[\frac{S_n}{\overline{X_n}}\right]^2\to\left(\frac{\sigma}{\mu}\right)^2$, where $\overline{X_n}$ and S_n^2 are respectively the mean and variance of the sample.

Proof. Since, the sample mean $(\overline{X_n})$ is an unbiased estimator of population mean (μ) with variance σ^2/n. i.e.

$$E[\overline{X_n}]=\mu,\quad var\left(\overline{X_n}\right)=\sigma^2/n \quad (33)$$

Also, we have,

$$var\left(\overline{X_n}\right)=E[\overline{X_n}^2]-[E\left(\overline{X_n}\right)]^2 \quad (34)$$

$$E[\overline{X_n}^2]=var\left(\overline{X_n}\right)+[E\left(\overline{X_n}\right)]^2$$

Using (9) and (13), we can write

$$E[\overline{X_n}^2]=\frac{2\left[\Gamma\left(\frac{\omega+3}{2}\right))\Gamma\left(\frac{\omega+5}{2}\right)-(1-n)\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2\right]}{n\theta\left[\Gamma\left(\frac{\omega+3}{2}\right)\right]^2} \quad (35)$$

Also, $E[S_n^2]=\sigma^2$, Therefore

$$E[S_n^2]=\frac{2\left[\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2\right]}{\theta\left[\Gamma\left(\frac{\omega+3}{2}\right)\right]^2} \quad (36)$$

Now, we can write

$$E[S_n^2]=E\left[\frac{S_n^2}{\overline{X_n}^2}\overline{X_n}^2\right]$$
$$E[S_n^2]=E\left[\frac{S_n^2}{\overline{X_n}^2}\right]E[\overline{X_n}^2]$$
$$E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\frac{E[S_n^2]}{E[\overline{X_n}^2]}$$

using (35) and (36), we obtain

$$E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\frac{\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2}{\frac{1}{n}\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left(\frac{1}{n}-1\right)\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2}$$

Applying $\lim\limits_{n\to\infty}$ on both sides, we get.

$$\lim_{n\to\infty}E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\frac{\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2}{\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2}$$
$$\lim_{n\to\infty}E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\left[\frac{\sqrt{\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)-\left\{\Gamma\left(\frac{\omega+4}{2}\right)\right\}^2}}{\Gamma\left(\frac{\omega+4}{2}\right)}\right]^2$$
$$\lim_{n\to\infty}E\left[\frac{S_n^2}{\overline{X_n}^2}\right]=\left(\frac{\sigma}{\mu}\right)^2=(c.v.)^2$$

hence the Theorem.

7 Estimation of parameters

7.1 Maximum Likelihood Estimation

Let $x_1,x_2,...,x_n$ be a random sample of size n from WMD. Then it's likelihood function will be given by (37).

$$L(\theta,\omega|x)=\frac{\theta^{n(\omega+3)/2}\prod_{i=1}^{n}x_i^{\omega+2}}{2^{n(\omega+1)/2}\left\{\Gamma((\omega+3)/2)\right\}^{n}}exp\left(-\frac{\theta}{2}\sum_{i=1}^{n}x_i^2\right) \quad (37)$$

Log likelihood function is given by:

$$\log L(\theta,\omega|x)=\left\{\frac{n(\omega+3)}{2}\log(\theta)-\frac{n(\omega+1)}{2}\log(2)-n\log\Gamma((\omega+3)/2)+(\omega+2)\sum_{i=1}^{n}\log x_i-\frac{\theta}{2}\sum_{i=1}^{n}x_i^2\right\} \quad (38)$$

Differentiating Log Likelihood function partially with respect to θ , ω and equating to zero we will have the following system of equations.

$$\frac{\partial \log L(\theta,\omega|x)}{\partial \theta} = \theta - \frac{n(\omega+3)}{\sum_{i=1}^{n} x_i^2} = 0 \quad (39)$$

$$\frac{\partial \log L(\theta,\omega|x)}{\partial \omega} = \sum_{i=1}^{n} [2\log x_i + \log(\frac{\theta}{2}) - \Psi((\omega+3)/2)] \times x_i^{\omega+2} \theta^{(\omega+3)/2} exp\left(-\theta x_i^2/2\right) = 0 \quad (40)$$

The above system of equations is non-linear and can't be solved manually for θ and ω. In order to overcome the said hindrance, programming language R has been used for obtaining the estimates of parameters.

7.2 Moment estimator

The Moment estimator for rate parameter θ of WMD can be obtained by equating the first theoretical moment to sample moment $(\overline{x})$ as given below:

$$\sqrt{\frac{2}{\theta}}\frac{\Gamma((\omega+4)/2)}{\Gamma((\omega+3)/2)} = \sum_{i=1}^{n} x_i/n$$

$$\hat{\theta}_{mm} = 2\left(\frac{n\Gamma((\omega+4)/2)}{\Gamma((\omega+3)/2)\sum_{i=1}^{n} x_i}\right)^2 \quad (41)$$

8 Comparison between different special cases of WMD in terms of fitting

In this section, we have fitted WMD to four different types of data sets. The considered data sets include three real life and a simulated one. Comparison is made between the special cases of WMD in terms possessing least values of comparison criteria (AIC, $AICC$, BIC). The R code for generating three real life data sets is as follows.

```
> install.packages("faraway")
> library(faraway)
> lightintensity <- star$light
> lightintensity
[1]  5.23 5.74 4.93 5.74 5.19 5.46 4.65
[8]  5.27 5.57 5.12 5.73 5.45 5.42 4.05
[15] 4.26 4.58 3.94 4.18 4.18 5.89 4.38
[22] 4.22 4.42 4.85 5.02 4.66 4.66 4.90
[29] 4.39 6.05 4.42 5.10 5.22 6.29 4.34
[36] 5.62 5.10 5.22 5.18 5.57 4.62 5.06
[43] 5.34 5.34 5.54 4.98 4.50
```

Dataset named as "lightintensity" is related to the logarithm of light intensity of 47 stars in the star cluster CYG OB1.

```
> currentnoise<-resceram$noise
> currentnoise
[1] 1.11 0.95 0.82 1.70 1.22 0.97 1.60
[8] 1.11 1.52 1.22 1.54 1.18
```

"currentnoise" represents the current noise of four resistors mounted in a combination of 3 on different crematic plates.

```
> wear<-abrasion$wear
> wear
[1]  235 236 218 268 251 241 227 229 234
[10] 273 274 226 195 270 230 225
```

Dataset entitled as "wear" is a vector regarding the amount of wear recorded on feeding four materials into a wear testing machine on using a Latin square design.

8.1 Simulation study

Herein, we have generated a data set of 100 observations from WMD with the help of Inverse transformation method. In Inverse transformation method, random numbers from a particular distribution are generated by solving the equation $F(x) = p$, for x at preassigned values of parameters. Where $F(x)$ is the distribution function characterizing a particular probability distribution and p is any number from the interval [0,1] i.e. $p \sim U(0,1)$. Following the same procedures for generation of random numbers from WMD we will have:

$$F_{\omega}(x;\theta,\omega) = 1 - \frac{\Gamma(\frac{(\omega+3)}{2}, \frac{\theta x^2}{2})}{\Gamma(\frac{(\omega+3)}{2})} = p \quad (42)$$

Now, on solving the equation (42) for x, at 100 different values of p with rate parameter $(\theta) = 3$ and weight parameter $(\omega) = 3$, we will obtain 100 different values of x. Since, Cdf is bijective in nature and is having a unique inverse known as Quantile function. Therefore for each value of p, we will have a unique value of x e.g. if $p = 0.25$, $p = 0.5$ and $p = 0.75$, the resulting solutions will be the first quartile (Q_1) , Median (Q_2) and third quartile (Q_3) respectively. The equation (42) can't be solved manually for x. Hence, the programming language R has been used for obtaining the solution of same equation. The resulting simulated data set along with it's R-code is as follows:

```
> Data<-function(n,s,t,w)
+ {set.seed(s)
+ U=runif(n,0,1)
+ library(zipfR)
+ cdf<-function(x,t,w)
+ {fn<-1-Igamma((w+3)/2,(t*x^2)/2,
+ lower=FALSE)/gamma((w+3)/2)}
+ data=c() #Create an empty vector
+ for(i in 1:length(U)){
+ fn<-function(x){cdf(x,t,w)-U[i]}
+ uni<-uniroot(fn,c(0,100000))
+ data=c(data,uni$root)}
+ return(data)}
> Simulateddata<-Data(100,1,3,3)
> Simulateddata
```

```
[1]  1.09103 1.20576 1.41003 1.90537
[5]  1.01379 1.87957 2.02599 1.50616
[9]  1.47052 0.77178 1.01934 0.98022
[13] 1.53688 1.21804 1.64417 1.33284
[17] 1.57439 2.40454 1.21387 1.65510
[21] 1.98785 1.02713 1.49575 0.90299
[25] 1.09299 1.22010 0.56954 1.21629
[29] 1.81368 1.17271 1.31708 1.43837
[33] 1.32865 0.99339 1.73368 1.51502
[37] 1.68011 0.87210 1.58212 1.24569
[41] 1.72278 1.49052 1.66313 1.38941
[45] 1.36542 1.67270 0.63361 1.31219
[49] 1.59321 1.54373 1.31258 1.79630
[53] 1.27278 1.06696 0.79424 0.85611
[57] 1.14718 1.35410 1.50755 1.24119
[61] 1.91839 1.12252 1.29390 1.16436
[65] 1.49484 1.08242 1.31351 1.63917
[69] 0.82510 1.82569 1.17136 1.75493
[73] 1.17938 1.16580 1.31128 1.86427
[77] 1.80263 1.22406 1.65492 2.10077
[81] 1.26932 1.56798 1.23423 1.15687
[85] 1.62632 1.01510 1.56625 0.89644
[89] 1.06781 0.93158 1.06081 0.76412
[93] 1.48514 1.82775 1.65724 1.68482
[97] 1.29008 1.24449 1.70621 1.44414
```

Table-3: MLE's and different comparison criteria

Data	*Distn.*	*MLEs* $\hat{\omega}_{mle}$	$\hat{\theta}_{mle}$	$\hat{\theta}_{mm}$	$-2ll$	*AIC*	*BIC*	*AICC*
Light Intensity	WMD	36.543461	1.554255	1.55432	79.65030	83.65030	87.3506	83.9230
	MD	0 *(known)*	0.117949	0.10137	161.8250	163.8250	165.6752	163.914
	LBMD	1 *(known)*	0.157263	0.14069	148.2356	150.2356	152.0858	150.325
	ABMD	2 *(known)*	0.196571	0.18021	138.1773	140.1773	142.0275	140.266
Current Noise	WMD	7.952079	6.746568	6.75085	2.427700	6.427700	7.397513	7.76103
	MD	0 *(known)*	1.847923	1.64286	9.941077	11.94108	12.42598	12.3411
	LBMD	1 *(known)*	2.463956	2.28015	7.294335	9.294335	9.779241	9.69434
	ABMD	2 *(known)*	3.079832	2.92065	5.549149	7.549149	8.034055	7.94915
Amount Wear	WMD	59.566281	0.001085	0.00108	143.6269	147.6269	149.1720	148.549
	MD	0 *(known)*	0.000053	0.00004	178.4599	180.4599	181.2325	180.746
	LBMD	1 *(known)*	0.000070	0.00006	173.6843	175.6843	176.4569	175.970
	ABMD	2 *(known)*	0.000093	0.00008	170.2925	172.2925	173.0651	172.578
Simulated Data	WMD	5.105472	4.050502	4.04862	69.93130	73.93130	79.14164	74.0550
	MD	0 *(known)*	1.499127	1.35267	110.6245	112.6245	115.2297	112.665
	LBMD	1 *(known)*	1.998917	1.87739	92.00237	94.00237	96.60754	94.0431
	ABMD	2 *(known)*	2.498724	2.40475	80.89320	82.89320	85.49837	82.9340

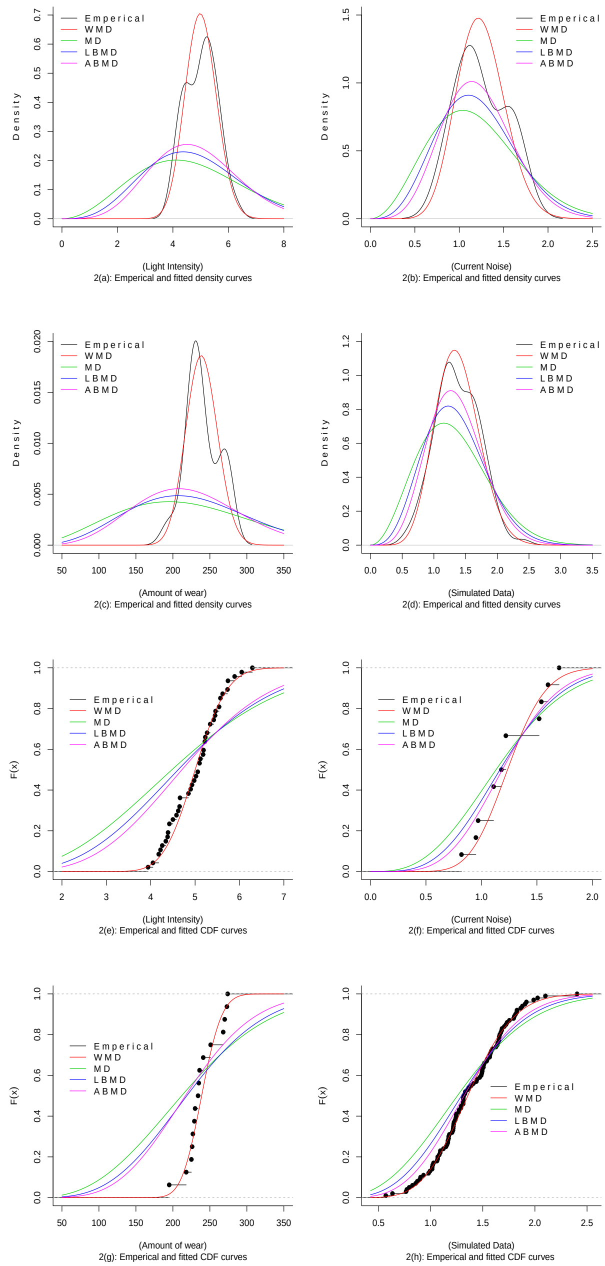

Fig. 2: Density and distribution curves fitted to four different types of data sets.

9 Conclusion

In this paper, various characteristic properties of WMD have been studied and discussed in detail. Three real life data sets and a simulated one is considered for illustrating the validity of WMD in statistical modelling. After the fitting of WMD to the considered data sets, different measures of goodness of fit like *AIC*, *BIC* and *AICC* have been computed for the special cases of WMD and are reported in table-3. The probability model with lowest *AIC*, *BIC* and *AICC* is considered to be the best fitted model. From table-3, it is evident that WMD possesses the least values of *AIC*, *BIC* and *AICC* followed by ABMD, followed by LBMD and then finally followed by Maxwell distribution (MD). Hence, it can be concluded that WMD proves to be more flexible and best fitted distribution in comparison to it's special cases in the current study. Therefore, distributions in the order of best fit for the considered data sets, are given as below: $(Best)\ WMD \rightarrow ABMD \rightarrow LBMD \rightarrow MD\ (Good)$

The main motive behind the construction of WMD and fitting of it's special cases to the considered data sets was to assess its potentiality and flexibility in modelling a particular data set. From the current study, it is concluded that if there is any intuition that the observations in a stochastic process are recorded with probabilities proportional to some weight function $w(x,\omega)$, then it is better to contemplate on the need for studying weighted distributions and their application in modelling.

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