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Multiple Complex Soliton Solutions for the Integrable Sinh-Gordon and the Modified KdV-Sinh-Gordon Equation

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Abstract: In this work, we study the integrable sinh-Gordon (ShG) and the modified KdV-sinh-Gordon (MKdV-ShG) equations. We show that these two equations pass the Painlevé test to confirm its integrabilities. We establish new complex forms of the simplified Hirota's method, to formally derive multiple complex soliton solutions for each equation. Our results show that the complex simplified Hirota's method explicitly constructs new multiple complex soliton solutions in addition to the multiple real soliton solutions that each equation generates.

Keywords: Sinh-Gordon equation; modified KdV-sinh-Gordon equation; complex forms of the simplified Hirota's method; multiple complex soliton solutions.

1 Introduction

The sinh-Gordon equation

$$u_{xt} + \sinh u = 0, \tag{1}$$

is an elliptic PDE which appears naturally in surface theory, and is widely used in physics and other sciences. The sinh-Gordon equation (1) appears in integrable quantum field theory, kink dynamics, fluid dynamics, and in many other scientific applications [1]- [16]. It involves the d'Alembert operator u_{xt} and the sinh of the unknown function $u(x,t)$.

The modified KdV-sinh-Gordon (MKdV-ShG) equation

$$u_{xt} - \alpha\left(\frac{3}{2}u_x^2 u_{xx} - u_{xxxx}\right) - \beta \sinh u = 0, \tag{2}$$

where α and β are nonlinear constants, is a nonlinear partial differential equation (PDE) involving the d'Alembert operator u_{xt}, three distinct derivatives of $u(x,t)$ with respect to x, and the sinh of the unknown function $u(x,t)$ [1]- [16]. For $\alpha = 1$ and $\beta = 0$, the (MKdV-ShG) equation (2) passes into the defocusing modified KdV equation for the function u_x [9, 16]

$$(u_x)_t - \frac{3}{2}(u_x)^2(u_x)_x + (u_x)_{xxx} = 0. \tag{3}$$

For $\alpha = 0$ and $\beta = 1$, we obtain the sinh-Gordon equation (ShG) equation. For any α and β, the equation is integrable as will be proved later. The constants α and $\frac{3}{2}\alpha$ are related to the dispersion and nonlinear effects of the medium. The MKdV, ShG, and MKdV-ShG equations are completely integrable equations.

The sinh-Gordon equation (1) and the MKdV-ShG equation (2) arise in models of interacting charged particles in plasma physics [17]- [31]. The sinh-Gordon equation (1) and the MKdV-ShG equation (2) are used to model the interaction of neighboring particles of equal mass in a lattice formation with a crystal. More applications of these two equations are in the field of thermodynamics, where partition and correlation functions can be precisely computed and on effect of weak dislocation potential on nonlinear wave propagation in anharmonic crystal.

It is well known that completely integrable equations possess remarkable properties, such as infinitely many symmetries, infinitely many conservation laws, the Painlevé property, Bäcklund transformations, Darboux

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transformations, bilinear forms, Lax pair, Hamiltonian and bi-Hamiltonian structures, etc. A Painlevé integrable model indicates that this model possesses the Painlevé property.

The extensive research work was focused on obtaining real soliton solutions, and mostly multiple real soliton solutions. To the best of author's knowledge, much research has been done, for the last decades, on the traditional real soliton solutions. However, the complex solitons and the multiple complex soliton solutions have not been investigated frequently in the literature. The primary purpose of the present paper is to develop a new reliable method which will effectively construct multiple complex soliton solutions for integrable equations. Recall that solitons can take on complex features, such as dipole solitons, multi hump solitons, solitons organized as necklaces, and even complex beams carrying angular momentum, like rotating propellers [7,8].

We mainly aim to show that the sinh-Gordon (ShG) equation and the modified KdV-sinh-Gordon (MKdV-ShG) equation give multiple real soliton solutions and multiple complex soliton solutions as well. For this reason, complex forms of the simplified Hirota's method will be developed. For comparison reasons, we will briefly report the multiple soliton solution for these two equations. We then will introduce the complex forms of the simplified Hirota's method. We finally will close our work by employing the newly developed complex simplified Hirota's forms to formally derive multiple complex soliton solutions for each of the sinh-Gordon equation (1) and the modified KdV-sinh-Gordon (MKdV-ShG) equation (2).

2 Painlevé test

In this section, we will use the Painlevé test to confirm the integrability of the sinh-Gordon (ShG) equation (1) and the modified KdV-sinh-Gordon (MKdV-ShG) equation (2).

2.1 The sinh-Gordon equation

To test the ShG equation (1) for complete integrability, we introduce the transformation

$$v = e^{u}, \tag{4}$$

which leads to

$$u(x,t) = \ln v(x,t), \tag{5}$$

and

$$\sinh u = \frac{1}{2}(v - \frac{1}{v}). \tag{6}$$

Substituting (4)–(6) into (1) changes the sinh-Gordon equation to an equivalent partial differential equation given as

$$2vv_{xt} - 2v_x v_t + v^3 - v = 0, \tag{7}$$

Assuming (7) has a solution as a Laurent expansion about a singular manifold $\psi = \psi(x,t)$ as

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)\psi^{k-\gamma}, \tag{8}$$

where $v_k(x,t)'s$ $(k = 0,1,2,...)$ are the functions of x and t. On substitution of (8) in equation (7), then we can show that the characteristic equation for resonances has one branch with two resonances at $k = -1$ and 2. However, as usual, the resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x,t) = 0$. After detailed computations, we observed explicit expressions for u_1, and u_2 turns out to be arbitrary function and, hence compatibility condition, for $k = 2$, is satisfied identically which implies that equation (1) passes the Painlevé test for complete integrability.

2.2 The modified KdV-sinh-Gordon equation

To confirm the integrability of the MKdV-ShG equation (2), we proceed as before, and introduce the transformation

$$v = e^{u}, \tag{9}$$

which leads to

$$u(x,t) = \ln v(x,t), \tag{10}$$

and this in turn gives

$$\sinh u = \frac{1}{2}(v - \frac{1}{v}). \tag{11}$$

Substituting (9)–(11) into (2) changes the MKdV-ShG equation (2) to an equivalent partial differential equation given as

$$2\alpha v^3 v_{xxxx} - 8\alpha v^2 v_x v_{xxx} - 6\alpha v^2 (v_{xx})^2 + 21\alpha v (v_x)^2 v_{xx}$$

$$-9\alpha(v_x)^4 + 2v^3 v_{xt} - 2v^2 v_x v_t - \beta v^5 + \beta v^3 = 0. \tag{12}$$

Using a Laurent expansion as

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)\psi^{k-\gamma}, \tag{13}$$

where $v_k(x,t)'s$ $(k = 0,1,2,...)$ are the functions of x and t. Substituting (13) in equation (12), then we can show that the characteristic equation for resonances has one branch with four resonances at $k = -3,-1,4,6$. The resonance at $k = -3$ is ignored because it is < -1. However, as usual, the resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x,t) = 0$. After detailed computations, we observed explicit expressions for u_1,u_2,u_3,u_5. However, it turns out that u_4 and u_6 are arbitrary function and, hence compatibility condition, for $k = 4$ and $k = 6$, are satisfied identically which implies that equation (2) passes the Painlevé test for complete integrability.

In what follows, we briefly present the multiple soliton solutions for the sinh-Gordon equation and the modified KdV-sinh-Gordon equation obtained before by distinct methods in [1]- [20]. For more details about the formal construction of these solutions, we refer to the works in [1]- [20] and some of the references therein.

3 Brief summary of multiple soliton solutions

For comparison reasons with the newly developed complex derivations, we briefly summarize the obtained results in [1]- [31].

3.1 The sinh-Gordon equation

We use sinh-Gordon equation

$$u_{xt} + \sinh u = 0, \tag{14}$$

where we approximate $\sinh u$ by the linear term u, to find the dispersion relation as

$$\alpha_i = \frac{1}{k_i}, i = 1, 2, \cdots N, \tag{15}$$

and hence θ_i becomes

$$\theta_i = k_i x - \frac{1}{k_i} t. \tag{16}$$

In [1-10], it was shown that the multi-soliton solutions of the sinh-Gordon equation are expressed by

$$u(x,t) = 4\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right). \tag{17}$$

Based on this result, we can rewrite (17) as

$$\tanh\left(\frac{u(x,t)}{4}\right) = \frac{f(x,t)}{g(x,t)}. \tag{18}$$

Using the hyperbolic identity for $\sinh u$, we can show that

$$\sinh u = \frac{4gf(g^2+f^2)}{(g^2-f^2)^2}. \tag{19}$$

The auxiliary functions $f(x,t)$ and $g(x,t)$ for the single soliton solution are defined by

$$\begin{aligned} f(x,t) &= e^{\theta_1} = e^{k_1 x - \alpha_1 t}, \\ g(x,t) &= 1. \end{aligned} \tag{20}$$

Consequently, the single soliton solution reads

$$u(x,t) = 4\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right) = 4\text{arctanh}\left(e^{k_1 x - \alpha_1 t}\right). \tag{21}$$

For the two-soliton solutions the two auxiliary functions $f(x,t)$ and $g(x,t)$ are defined by

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} = e^{k_1 x - \alpha_1 t} + e^{k_2 x - \alpha_2 t}, \\ g(x,t) &= 1 + a_{12} e^{\theta_1 + \theta_2} = 1 + a_{12} e^{(k_1+k_2)x - (\alpha_1+\alpha_2)t}. \end{aligned} \tag{22}$$

Using (22) in (17) and proceeding as before, we find the phase shift a_{12} is given by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{23}$$

and hence we set

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{24}$$

Consequently, the two-soliton solutions are obtained by substituting (23) and (34) into (17), where we obtain

$$u(x,t) = 4\text{arctanh}\left(\frac{e^{k_1 x - \alpha_1 t} + e^{k_2 x - \alpha_2 t}}{1 + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x - (\alpha_1+\alpha_2)t}}\right). \tag{25}$$

For the three-soliton solutions, the auxiliary functions take the forms

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + b_{123} e^{\theta_1+\theta_2+\theta_3}, \\ g(x,t) &= 1 + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3}. \end{aligned} \tag{26}$$

where

$$b_{123} = a_{12} a_{23} a_{13}. \tag{27}$$

Based on this result, the three–soliton solutions for the sinh-Gordon equation (14) are obtained immediately.

3.2 The modified KdV-sinh-Gordon equation

We give here a brief summary of the results obtained in [1]- [14] for the MKdV-ShG equation

$$u_{xt} - \alpha\left(\frac{3}{2} u_x^2 u_{xx} - u_{xxxx}\right) - \beta \sinh u = 0. \tag{28}$$

The dispersion relation is given as

$$c_i = \alpha k_i^3 - \frac{\beta}{k_i}, i = 1, 2, \cdots N, \tag{29}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x - \left(\alpha k_i^3 - \frac{\beta}{k_i}\right) t. \tag{30}$$

In [16]- [31], it was shown that the multi-soliton solutions of the MKdV-ShG equation are expressed by

$$u(x,t) = 4\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right), \tag{31}$$

where we can show that

$$\sinh u = \frac{4gf(g^2+f^2)}{(g^2-f^2)^2}. \quad (32)$$

The single soliton solution reads

$$u(x,t) = 4\text{arctanh}\left(e^{k_1 x - \left(\alpha k_1^3 - \frac{\beta}{k_1}\right)t}\right). \quad (33)$$

For the two-soliton solutions, we use the two auxiliary functions $f(x,t)$ and $g(x,t)$ as

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2}, \\ g(x,t) &= 1 + a_{12} e^{\theta_1+\theta_2}, \end{aligned} \quad (34)$$

to find the phase shift a_{12} by the form

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \quad (35)$$

For the three-soliton solutions, the auxiliary functions take the forms

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + b_{123} e^{\theta_1+\theta_2+\theta_3}, \\ g(x,t) &= 1 + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3}. \end{aligned} \quad (36)$$

where

$$b_{123} = a_{12} a_{23} a_{13}. \quad (37)$$

Based on this result, the three–soliton solutions for the MKdV-ShG equation (28) are obtained immediately upon using (31).

As stated earlier, we plan to show that the integrable ShG equation (1) and the integrable MKdV-ShG equation (28) give multiple complex soliton solutions in addition to the traditional multiple soliton solutions, we therefor present the complex forms of the simplified Hirota's method. The complex forms will shed light on the new development of multiple complex soliton solutions for integrable equations.

4 Formulation of the complex forms

To achieve the goal set for this work, we introduce two complex algorithms of the simplified Hirota's method that will be used for the determination of multiple complex soliton solutions for the integrable equations in general. Only the second form will be used for the sine-Gordon equation and the sinh-Gordon equation, however, the first form works effectively for other integrable equations:
1. Type I: The KdV type of dispersive equations:
(i) We assume the dependent variable transformation

$$u(x,t) = R(\ln f(x,t))_{xx}. \quad (38)$$

(ii) For single complex soliton solution we use the auxiliary complex function as

$$f(x,t) = I + e^{k_1 x - c_1 t}, I = \sqrt{-1}. \quad (39)$$

where c_1 is the dispersion relation which will be determined in a regular way.
(iii) For two complex soliton solutions we use the auxiliary complex function as

$$f(x,t) = I + e^{k_1 x - c_1 t} + e^{k_2 x - c_2 t} - I a_{12} e^{(k_1+k_2)x - (c_1+c_2)t}, \quad (40)$$

where a_{12} is the phase shift that results from the interaction of two solitons.
(iv) For three complex soliton solutions we use the auxiliary complex function as

$$f(x,t) = I + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - I a_{12} e^{\theta_1+\theta_2} - I a_{13} e^{\theta_1+\theta_3} - I a_{23} e^{\theta_2+\theta_3} - b_{123} e^{\theta_1+\theta_2+\theta_3}, \quad (41)$$

where the phase variables are given by $\theta_i = k_i x - c_i t, 1 \le i \le 3$. Note that for $b_{123} = a_{12} a_{13} a_{23}$, the equation gives three complex soliton solutions, and hence multiple complex soliton solutions as examined in the standard algorithm.

2. Type II: The modified KdV type of dispersive equations:
(i) We assume the dependent variable transformation

$$u(x,t) = R(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)_x. \quad (42)$$

However, we will use

$$u(x,t) = 4(\arctan\left(\frac{f(x,t)}{g(x,t)}\right), \quad (43)$$

and

$$u(x,t) = 4(\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right), \quad (44)$$

for the sine-Gordon and sinh-Gordon type equations respectively.
(ii) For single complex soliton solution we use the auxiliary complex function as

$$\begin{aligned} f(x,t) &= I e^{k_1 x - c_1 t}, I = \sqrt{-1}, \\ g(x,t) &= 1. \end{aligned} \quad (45)$$

(ii) For the two complex soliton solutions we set auxiliary complex function as

$$\begin{aligned} f(x,t) &= I e^{k_1 x - c_1 t} + I e^{k_2 x - c_2 t)}, \\ g(x,t) &= 1 - a_{12} e^{(k_1+k_2)x - (c_1+c_2)t}, \end{aligned} \quad (46)$$

where a_{12} is the phase shift.

(iii) For the three complex soliton solutions, we use auxiliary complex function as

$$\begin{aligned} f(x,t) &= I e^{\theta_1} + I e^{\theta_2} + I e^{\theta_3} - I a_{12} a_{13} a_{23} e^{\theta_1+\theta_2+\theta_3} \\ g(x,t) &= 1 - a_{12} e^{\theta_1+\theta_2} - a_{13} e^{\theta_1+\theta_3} - a_{23} e^{\theta_2+\theta_3}, \end{aligned} \quad (47)$$

5 Multiple complex soliton solutions

In this section, we employ the aforementioned complex forms to determine multiple complex soliton solutions for each of the two examined equations.

5.1 Multiple complex soliton solutions for the sinh-Gordon equation

The integrable sinh-Gordon equation reads

$$u_{xt} + \sinh u = 0. \tag{48}$$

The dispersion relations for this equation are $c_i = \frac{1}{k_i}$ as derived earlier. We first use the dependent variable transformation

$$u(x,t) = 4(\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right), \tag{49}$$

and by using the hyperbolic identity we find

$$\sinh u = \frac{4gf(g^2+f^2)}{(g^2-f^2)^2}. \tag{50}$$

For single complex soliton solution we use the auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{k_1 x - \frac{1}{k_1}t}, I = \sqrt{-1}, \\ g(x,t) &= 1. \end{aligned} \tag{51}$$

This in turn gives the single complex soliton solution as

$$u(x,t) = 4I(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)$$

$$= 4I(\arctan\left(e^{k_1 x - \frac{1}{k_1}t}\right), I = \sqrt{-1}. \tag{52}$$

For the two complex soliton solutions we set auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{k_1 x - \frac{1}{k_1}t} + Ie^{k_2 x - \frac{1}{k_2}t)}, \\ g(x,t) &= 1 - a_{12}e^{(k_1+k_2)x - (\frac{1}{k_1}+\frac{1}{k_2})t}, \end{aligned} \tag{53}$$

where a_{12} is the phase shift, where we find that the phase shift as

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{54}$$

which can be generalized as

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \leq i < j \leq 3, \tag{55}$$

which is the same as used for deriving the real soliton solutions. Combining all previous results we obtain the two complex soliton solutions.

For the three complex soliton solutions, we use auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{\theta_1} + Ie^{\theta_2} + Ie^{\theta_3} - Ia_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3} \\ g(x,t) &= 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3}, \end{aligned} \tag{56}$$

and substitute these functions in (49).

5.2 Multiple complex soliton solutions for the MKdV-ShG equation

The integrable MKdV-ShG equation reads

$$u_{xt} - \alpha\left(\frac{3}{2}u_x^2 u_{xx} - u_{xxxx}\right) - \beta \sinh u = 0. \tag{57}$$

To determine the dispersion relation, we substitute

$$u(x,t) = e^{\theta_i}, \theta_i = k_i x - c_i t, \tag{58}$$

into the linear terms of (57), where we approximated $\sinh u$ by the linear term u, to find the dispersion relation

$$c_i = \alpha k_i^3 - \frac{\beta}{k_i}, i = 1, 2, \cdots N, \tag{59}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x - \left(\alpha k_i^3 - \frac{\beta}{k_i}\right)t. \tag{60}$$

We first set the dependent variable transformation

$$u(x,t) = 4(\arctan\left(\frac{f(x,t)}{g(x,t)}\right), \tag{61}$$

and by using the hyperbolic identity we find

$$\sinh u = \frac{4gf(g^2+f^2)}{(g^2-f^2)^2}. \tag{62}$$

For single complex soliton solution we use the auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{k_1 x - \left(\alpha k_1^3 - \frac{\beta}{k_1}\right)t}, I = \sqrt{-1}, \\ g(x,t) &= 1. \end{aligned} \tag{63}$$

This in turn gives the single complex soliton solution as

$$u(x,t) = 4\text{arctanh}\left(\frac{f(x,t)}{g(x,t)}\right) = 4I\arctan\left(e^{k_1 x - \left(\alpha k_1^3 - \frac{\beta}{k_1}\right)t}\right) = 4\text{arctanh}\left(Ie^{k_1 x - \left(\alpha k_1^3 - \frac{\beta}{k_1}\right)t}\right). \tag{64}$$

For the two complex soliton solutions we set auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{k_1 x - \left(\alpha k_1^3 - \frac{\beta}{k_1}\right)t} + Ie^{k_2 x - \left(\alpha k_2^3 - \frac{\beta}{k_2}\right)t)}, \\ g(x,t) &= 1 - a_{12}e^{(k_1+k_2)x - \left(\alpha k_1^3 + k_2^3) - (\frac{\beta}{k_1}+\frac{\beta}{k_2}\right)t}, \end{aligned} \tag{65}$$

where a_{12} is the phase shift, where we find that the phase shift as

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{66}$$

which can be generalized as

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \leq i < j \leq 3, \tag{67}$$

which is the same as used for deriving the real soliton solutions. Combining all previous results we obtain the two complex soliton solutions.

For the three complex soliton solutions, we use auxiliary complex function as

$$\begin{aligned} f(x,t) &= Ie^{\theta_1} + Ie^{\theta_2} + Ie^{\theta_3} - Ia_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3} \\ g(x,t) &= 1 - a_{12}e^{\theta_1-\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3}, \end{aligned} \quad (68)$$

and substitute these functions in (61) to obtain the three complex soliton solutions..

6 Conclusion

In summary, we have showed that the integrable sinh-Gordon equation and the integrable modified KdV-sinh-Gordon equation give not only multiple real soliton solutions, but also give multiple complex soliton solutions. To confirm this findings, we established the complex forms of the simplified Hirota's method that will aid in the determination of the complex soliton solutions. To the author's belief, the findings of complex formulas, that will give multiple complex soliton solutions, are presented for the first time. Several well known integrable equations were used to derive multiple complex soliton solutions for each examined model. The obtained results may be helpful to examine other integrable applications for more findings.

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