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A Parametric Type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi Polynomials

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Abstract: By defining six specific generating functions, we introduce a kind of parametric Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, we study their basic properties in a systematic manner. As an application of the introduced polynomials, we use them in computing some new series of the Taylor type.

Keywords: Appell polynomials, Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Apostol-Genocchi polynomials, hurwitz-Lerch zeta function, Stirling numbers, Cauchy product and binomial convolution; parametric generalization, generating functions, computation of Taylor type series

1 Introduction

Let $f(t)$ be a formal power series in t. The Appell polynomials $A_n(x)$ defined by

$$f(t)\, e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \tag{1}$$

have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [2,24]. Three special cases of these polynomials are the Bernoulli polynomials $B_n(x)$, the Euler polynomials $E_n(x)$ and the Genocchi polynomials [19] $G_n(x)$ (see [19]) that are generated by choosing in (1) the following values of $f(t)$:

$$f(t) = \frac{t}{e^t - 1}, \quad f(t) = \frac{2}{e^t + 1} \quad \text{and} \quad f(t) = \frac{2t}{e^t + 1},$$

respectively, so that we have

$$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi),$$

$$\frac{2 e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \qquad (|t| < \pi)$$

and

$$\frac{2t e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \qquad (|t| < \pi).$$

In this sense, the Bernoulli numbers $B_n := B_n(0)$, the Euler numbers $E_n := 2^n E_n\left(\frac{1}{2}\right)$ and the Genocchi numbers $G_n := G_n(0)$ have found considerable applications in Number Theory, Special Functions, Combinatorics and Numerical Analysis. It is clear that

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi),$$

$$\frac{2e^t}{e^{2t} + 1} = \frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \left(|t| < \frac{\pi}{2}\right)$$

and

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \qquad (|t| < \pi).$$

The Apostol-Bernoulli polynomials defined by

$$\frac{t e^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathscr{B}_n(x;\lambda) \frac{t^n}{n!}$$

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$\left(\lambda \in \mathbb{C};\ |t| < 2\pi \quad \text{when} \quad \lambda = 1;\ |t| < |\log \lambda| \quad \text{when} \quad \lambda \neq 1\right)$, where $\mathscr{B}_{n,\lambda} := \mathscr{B}_n(0;\lambda)$ denotes the Apostol-Bernoulli numbers, were introduced by Apostol [1] (see also [22]) in order to evaluate the Hurwitz-Lerch zeta function $\Phi(z,s,a)$:

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right)$ for negative integer values of s, Z_0^- being the set of non-positive integers. Apostol [1] gave several elementary properties of $\mathscr{B}_n(x;\lambda)$ including (for example) the following interesting recursion formula for the numbers $\mathscr{B}_{n,\lambda}$:

$$\mathscr{B}_{n,\lambda} = n\sum_{k=0}^{n-1} \frac{k!(-\lambda)^k}{(\lambda-1)^{k+1}}\, S(n-1,k) \qquad (\lambda \neq 1),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind defined by

$$S(n,k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n.$$

The Apostol-Euler polynomials $\mathscr{E}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathscr{G}_n(x;\lambda)$ are defined, respectively, by

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathscr{E}_n(x;\lambda)\, \frac{t^n}{n!} \qquad \left(|t| < \frac{1}{2}\ |\log(-\lambda)|\right)$$

and

$$\frac{2te^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathscr{G}_n(x;\lambda)\, \frac{t^n}{n!} \qquad \left(|t| < |\log(-\lambda)|\right),$$

where

$$\mathscr{E}_{n,\lambda} := \mathscr{E}_n(0;\lambda) \qquad \text{and} \qquad \mathscr{G}_{n,\lambda} := \mathscr{G}_n(0;\lambda)$$

denote the corresponding Apostol-Euler number and the Apostol-Genocchi numbers, respectively.

Recently, many authors studied these Apostol type Bernoulli, Euler and Genocchi polynomials and the corresponding numbers. In particular, Cenkci and Can [3] considered a q-analogue of the Apostol-Bernoulli polynomials $\mathscr{B}_n(x;\lambda)$. Luo (see [13] [14]) computed the Fourier expansions and integral representations of the Apostol-Bernoulli polynomials $\mathscr{B}_n(x;\lambda)$ and the Apostol-Euler polynomials $\mathscr{E}_n(x;\lambda)$. Prévost [21], on the other hand, investigated the Padé approximation for these polynomials. Also, in [8] and [11], a q-extension of Apostol-Euler polynomials $\mathscr{E}_n(x;\lambda)$ was studied. Other notable developments involving these Apostol type polynnomials, including also the Genocchi and Apostol-Genocchi polynomials, and their various extensions and generalizations, see (for example) [4], [5], [6], [7], [9], [10], [12], [15], [16], [17] and [23].

Our present paper is organized as follows. In Section 2, we introduce a parametric type of the Apostol-Bernoulli polynomials $\mathscr{B}_n(x;\lambda)$, the Apostol-Euler polynomials $\mathscr{E}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathscr{G}_n(x;\lambda)$ by means of three separate generating functions. In Section 3, we obtain several basic properties of the introduced parametric Apostol-Bernoulli polynomials and, in Sections 4 and 5, we simply record without proof the corresponding basic properties of the introduced parametric Apostol-Euler polynomials and the Apostol-Genocchi polynomials. Finally, in Section 6, an application of the introduced polynomials is presented by computing some new series of the Taylor type involving the Apostol-Bernoulli numbers $\mathscr{B}_{n,\lambda}$, the Apostol-Euler numbers $\mathscr{E}_{n,\lambda}$ and the Apostol-Genocchi numbers $\mathscr{G}_{n,\lambda}$.

2 Parametric Type of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

We begin by introducing the binomial convolution of two sequences a_n and b_n given by

$$A(t) = \sum_{n=0}^{\infty} a_n\, \frac{t^n}{n!} \qquad \text{and} \qquad B(t) = \sum_{n=0}^{\infty} b_n\, \frac{t^n}{n!},$$

so that, by the Cauchy product, we have

$$C(t) := A(t)B(t) = \sum_{n=0}^{\infty} c_n\, \frac{t^n}{n!},$$

where c_n denotes the binomial convolution of the sequences a_n and b_n defined as follows (see [18]):

$$c_n = a_n * b_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} b_k. \tag{2}$$

For $p,q \in \mathbb{R}$, it was proved in [20] that the Taylor expansions of the two functions $e^{pt}\cos(qt)$ and $e^{pt}\sin(qt)$ are given by

$$e^{pt}\cos(qt) = \sum_{k=0}^{\infty} C_k(p,q)\, \frac{t^k}{k!}$$

and

$$e^{pt}\sin(qt) = \sum_{k=0}^{\infty} S_k(p,q)\, \frac{t^k}{k!},$$

where

$$C_k(p,q) = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k}{2j} p^{k-2j}\, q^{2j} \tag{3}$$

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and

$$S_k(p,q)=\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}\,q^{2j+1}. \tag{4}$$

By noting the definitions of $C_n(p,q)$, $S_n(p,q)$ and the Apostol type numbers $\mathscr{B}_{n,\lambda}$, $\mathscr{E}_{n,\lambda}$ and $\mathscr{G}_{n,\lambda}$, we can introduce two parametric kinds of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials as follows:

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\mathscr{B}_{n,\lambda}*C_n(p,q) \quad \text{and} \quad \mathscr{B}_n^{(s)}(p,q;\lambda)=\mathscr{B}_{n,\lambda}*S_n(p,q), \tag{5}$$

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\mathscr{E}_{n,\lambda}*C_n(p,q) \quad \text{and} \quad \mathscr{E}_n^{(s)}(p,q;\lambda)=\mathscr{E}_{n,\lambda}*S_n(p,q) \tag{6}$$

and

$$\mathscr{G}_n^{(c)}(p,q;\lambda)=\mathscr{G}_{n,\lambda}*C_n(p,q) \quad \text{and} \quad \mathscr{G}_n^{(s)}(p,q;\lambda)=\mathscr{G}_{n,\lambda}*S_n(p,q), \tag{7}$$

whose exponential generating functions are given, respectively, by

$$\frac{te^{pt}}{\lambda e^t-1}\cos(qt)=\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!} \quad \text{and} \quad \frac{te^{pt}}{\lambda e^t-1}\sin(qt)=\sum_{n=0}^{\infty}\mathscr{B}_n^{(s)}(p,q;\lambda)\frac{t^n}{n!}, \tag{8}$$

$$\frac{2e^{pt}}{\lambda e^t+1}\cos(qt)=\sum_{n=0}^{\infty}\mathscr{E}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{pt}}{\lambda e^t+1}\sin(qt)=\sum_{n=0}^{\infty}\mathscr{E}_n^{(s)}(p,q;\lambda)\frac{t^n}{n!} \tag{9}$$

$$\frac{2te^{pt}}{\lambda e^t+1}\cos(qt)=\sum_{n=0}^{\infty}\mathscr{G}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!} \quad \text{and} \quad \frac{2te^{pt}}{\lambda e^t+1}\sin(qt)=\sum_{n=0}^{\infty}\mathscr{G}_n^{(s)}(p,q;\lambda)\frac{t^n}{n!}. \tag{10}$$

Hence, according to the relation (2), we can represent these polynomials as follows:

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k,\lambda}C_k(p,q) \quad \text{and} \quad \mathscr{B}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k,\lambda}S_k(p,q), \tag{11}$$

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{n-k,\lambda}C_k(p,q) \quad \text{and} \quad \mathscr{E}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{n-k,\lambda}S_k(p,q) \tag{12}$$

and

$$\mathscr{G}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{G}_{n-k,\lambda}C_k(p,q) \quad \text{and} \quad \mathscr{G}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{G}_{n-k,\lambda}S_k(p,q). \tag{13}$$

We note from the above equations that

$$\mathscr{B}_n^{(c)}(p,0;\lambda)=\mathscr{B}_n(p;\lambda),\quad \mathscr{E}_n^{(c)}(p,0;\lambda)=\mathscr{E}_n(p;\lambda) \quad \text{and} \quad \mathscr{G}_n^{(c)}(p,0;\lambda)=\mathscr{G}_n(p;\lambda).$$

Thus, for example, we have

$$\mathscr{B}_0^{(c)}(p,q;\lambda)=0,\qquad \mathscr{B}_1^{(c)}(p,q;\lambda)=\frac{1}{\lambda-1},\qquad \mathscr{B}_2^{(c)}(p,q;\lambda)=\frac{2\lambda-2}{(\lambda-1)^2}\,p-\frac{2\lambda}{(\lambda-1)^2},$$

$$\mathscr{B}_0^{(s)}(p,q;\lambda)=0,\qquad \mathscr{B}_1^{(s)}(p,q;\lambda)=0,\qquad \mathscr{B}_2^{(s)}(p,q;\lambda)=\frac{2q}{\lambda-1},$$

$$\mathscr{E}_0^{(c)}(p,q;\lambda)=\frac{2}{\lambda+1},\qquad \mathscr{E}_1^{(c)}(p,q;\lambda)=\frac{2\lambda+2}{(\lambda+1)^2}\,p-\frac{2\lambda}{(\lambda+1)^2},$$

$$\mathscr{E}_2^{(c)}(p,q;\lambda)=\frac{2\lambda^2+4\lambda+2}{(\lambda+1)^3}\,p^2-\frac{4\lambda^2+4\lambda}{(\lambda+1)^3}\,p-\frac{2\lambda^2+4\lambda+2}{(\lambda+1)^3}\,q^2+\frac{2\lambda^2-2\lambda}{(\lambda+1)^3},$$

$$\mathscr{E}_0^{(s)}(p,q;\lambda)=0,\qquad \mathscr{E}_1^{(s)}(p,q;\lambda)=\frac{2q}{\lambda+1},\qquad \mathscr{E}_2^{(s)}(p,q;\lambda)=\frac{4q\lambda+4q}{(\lambda+1)^2}\,p-\frac{4q\lambda}{(\lambda+1)^2},$$

$$\mathscr{G}_0^{(c)}(p,q;\lambda)=0,\qquad \mathscr{G}_1^{(c)}(p,q;\lambda)=\frac{2}{\lambda+1},\qquad \mathscr{G}_2^{(c)}(p,q;\lambda)=\frac{4\lambda+4}{(\lambda+1)^2}\,p-\frac{4\lambda}{(\lambda+1)^2},$$

$$\mathscr{G}_0^{(s)}(p,q;\lambda)=0,\qquad \mathscr{G}_1^{(s)}(p,q;\lambda)=0,\qquad \mathscr{G}_2^{(s)}(p,q;\lambda)=\frac{4q}{\lambda+1}.$$

3 Basic Properties of $\mathscr{B}_n^{(c)}(p,q;\lambda)$ and $\mathscr{B}_n^{(s)}(p,q;\lambda)$

Proposition 1 *For every $n\in\mathbb{N}$, the following identities hold true*:

$$\lambda\mathscr{B}_n^{(c)}(1+p,q;\lambda)-\mathscr{B}_n^{(c)}(p,q;\lambda)=nC_{n-1}(p,q) \tag{14}$$

and

$$\lambda\mathscr{B}_n^{(s)}(1+p,q;\lambda)-\mathscr{B}_n^{(s)}(p,q;\lambda)=nS_{n-1}(p,q), \tag{15}$$

$\mathbb{N}$ being the set of positive integers and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

Proof. We have

$$\begin{aligned}\lambda\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(1+p,q;\lambda)\frac{t^n}{n!}&=\frac{te^{pt}(\lambda e^t-1+1)}{\lambda e^t-1}\cos(qt)=te^{pt}\cos(qt)+\frac{te^{pt}}{\lambda e^t-1}\cos(qt)\\&=\sum_{n=0}^{\infty}C_n(p,q)\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}nC_{n-1}(p,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!},\end{aligned}$$

which proves the first assertion (14). The proof of the second assertion (15) is similar.

Corollary 1 *The relations* (14) *and* (15) *imply that*

$$\lambda\mathscr{B}_{2n+1}^{(c)}(1,q;\lambda)-\mathscr{B}_{2n+1}^{(c)}(0,q;\lambda)=(2n+1)(-1)^n\,q^{2n}$$

and

$$\lambda\mathscr{B}_{2n}^{(s)}(1,q;\lambda)-\mathscr{B}_{2n}^{(s)}(0,q;\lambda)=2n(-1)^{n+1}\,q^{2n-1}.$$

Proposition 2 *For every $n\in\mathbb{N}$, the following identities hold true*:

$$\mathscr{B}_n^{(c)}(p+r,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)\,r^{n-k} \tag{16}$$

and

$$\mathscr{B}_n^{(s)}(p+r,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k^{(s)}(p,q;\lambda)\,r^{n-k}. \tag{17}$$

Proof. We apply the relation (8) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p+r,q;\lambda)\frac{t^n}{n!}&=\left(\frac{te^{pt}}{\lambda e^t-1}\cos(qt)\right)e^{rt}\\&=\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p,q;\lambda)\frac{t^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty}r^n\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)r^{n-k}\right)\frac{t^n}{n!},\end{aligned}$$

which proves the result (16). The other result (17) can be proved similarly.

Corollary 2 *It is asserted that*

$$\mathscr{B}_n^{(c)}(p+1,q;\lambda)-\mathscr{B}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)$$

and

$$\mathscr{B}_n^{(s)}(p+1,q;\lambda)-\mathscr{B}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}_k^{(s)}(p,q;\lambda).$$

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Now, by combining these results and Proposition 1, we find the following recurrence relations:

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\frac{1}{\lambda-1}\left[nC_{n-1}(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)\right] \tag{18}$$

and

$$\mathscr{B}_n^{(s)}(p,q;\lambda)=\frac{1}{\lambda-1}\left[nS_{n-1}(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{B}_k^{(s)}(p,q;\lambda)\right], \tag{19}$$

where

$$\mathscr{B}_0^{(c)}(p,q;\lambda)=\mathscr{B}_0^{(s)}(p,q;\lambda)=0.$$

Proposition 3 *For every $n\in\mathbb{N}$, the following identities hold true*:

$$\frac{\partial}{\partial p}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}=n\mathscr{B}_{n-1}^{(c)}(p,q;\lambda), \tag{20}$$

$$\frac{\partial}{\partial q}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}=-n\mathscr{B}_{n-1}^{(s)}(p,q;\lambda), \tag{21}$$

$$\frac{\partial}{\partial p}\left\{\mathscr{B}_n^{(s)}(p,q;\lambda)\right\}=n\mathscr{B}_{n-1}^{(s)}(p,q;\lambda) \tag{22}$$

and

$$\frac{\partial}{\partial q}\left\{\mathscr{B}_n^{(s)}(p,q;\lambda)\right\}=n\mathscr{B}_{n-1}^{(c)}(p,q;\lambda). \tag{23}$$

Proof. In view of the equation (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial p}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}\frac{t^n}{n!}&=\frac{t^2e^{pt}}{\lambda e^t-1}\cos(qt)=\sum_{n=0}^{\infty}\mathscr{B}_n^{(c)}(p,q;\lambda)\frac{t^{n+1}}{n!}\\&=\sum_{n=1}^{\infty}\mathscr{B}_{n-1}^{(c)}(p,q;\lambda)\frac{t^n}{(n-1)!}=\sum_{n=1}^{\infty}n\mathscr{B}_{n-1}^{(c)}(p,q;\lambda)\frac{t^n}{n!},\end{aligned}$$

which proves the result (20). The other results (21), (22) and (23) can be proved similarly.

Proposition 4 *The polynomials $\mathscr{B}_n^{(c)}(p,q;\lambda)$ and $\mathscr{B}_n^{(s)}(p,q;\lambda)$ are, respectively, of degrees $n-1$ and $n-2$ in the variable p It is also asserted that*

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\frac{n}{\lambda-1}p^{n-1}-\frac{n(n-1)\lambda}{(\lambda-1)^2}p^{n-2}+\cdots \tag{24}$$

and

$$\mathscr{B}_n^{(s)}(p,q;\lambda)=\frac{n(n-1)q}{\lambda-1}p^{n-2}-\frac{n(n-1)(n-2)q\lambda}{(\lambda-1)^2}p^{n-3}+\cdots. \tag{25}$$

Furthermore, if they are considered as polynomials in the variable q, then

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\begin{cases}\dfrac{(-1)^{\frac{n+2}{2}}}{\lambda-1}n(n-1)\left(p-\dfrac{\lambda}{\lambda-1}\right)q^{n-2}+\dfrac{(n-3)(-1)^{\frac{n}{2}}}{\lambda-1}\dbinom{n}{3}\\ \quad\cdot\left[p^3-\dfrac{3\lambda p^2}{\lambda-1}+\dfrac{3\lambda(\lambda+1)p}{(\lambda-1)^2}-\dfrac{\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^3}\right]q^{n-4}+\cdots\\ \qquad (n\ \text{even})\\[2mm] \dfrac{n(-1)^{\frac{n-1}{2}}}{\lambda-1}q^{n-1}+\dfrac{3(-1)^{\frac{n+1}{2}}}{\lambda-1}\dbinom{n}{3}\left[p^2-\dfrac{2\lambda p}{\lambda-1}+\dfrac{\lambda(\lambda+1)}{(\lambda-1)^2}\right]q^{n-3}+\cdots\\ \qquad (n\ \text{odd})\end{cases} \tag{26}$$

and

$$\mathscr{B}_n^{(s)}(p,q;\lambda)=\begin{cases}\dfrac{n(-1)^{\frac{n+2}{2}}}{\lambda-1}q^{n-1}+\dfrac{3(-1)^{\frac{n}{2}}}{\lambda-1}\dbinom{n}{3}\left(p^2-\dfrac{2\lambda p}{\lambda-1}+\dfrac{\lambda(\lambda+1)}{(\lambda-1)^2}\right)q^{n-3}+\cdots\\ \qquad (n\ \text{even})\\[2mm] \dfrac{(-1)^{\frac{n+1}{2}}}{\lambda-1}n(n-1)\left(p-\dfrac{\lambda}{\lambda-1}\right)q^{n-2}+\dfrac{(n-3)(-1)^{\frac{n-1}{2}}}{\lambda-1}\dbinom{n}{3}\\ \quad\cdot\left[p^3-\dfrac{3\lambda p^2}{\lambda-1}+\dfrac{3\lambda(\lambda+1)p}{(\lambda-1)^2}-\dfrac{\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^3}\right]q^{n-4}+\cdots\\ \qquad (n\ \text{odd}).\end{cases} \tag{27}$$

Proof. We first prove (24) by applying the principle of mathematical induction on n. Indeed, it is known from (18) that

$$\mathscr{B}_1^{(c)}(p,q;\lambda)=\frac{1}{\lambda-1},$$

$$\mathscr{B}_2^{(c)}(p,q;\lambda)=\frac{2}{\lambda-1}p-\frac{2\lambda}{(\lambda-1)^2}$$

and

$$\mathscr{B}_3^{(c)}(p,q;\lambda)=\frac{3}{\lambda-1}p^2-\frac{6\lambda}{(\lambda-1)^2}p-\frac{3}{\lambda-1}q^2+\frac{3\lambda(\lambda+1)}{(\lambda-1)^3}.$$

Therefore, the assertion (24) holds true for $n=1,2,3$. We now assume that it is valid for $n-1$. By referring to (20), we have

$$\frac{\partial}{\partial p}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}=\frac{n(n-1)}{\lambda-1}p^{n-2}-\frac{n(n-1)(n-2)\lambda}{(\lambda-1)^2}p^{n-3}+\cdots.$$

In order to complete the proof, it is sufficient to integrate the above equation with respect to the variable p to get the result (24). By virtue of the relation (23), the result (25) can be derived similarly.

To prove (26), we suppose that it holds true for $1,2,3,\cdots,n-1$. If $n=2m$, then from (18) we have

$$\begin{aligned}\mathscr{B}_{2m}^{(c)}(p,q;\lambda)=\frac{1}{\lambda-1}\Bigg[&2m\sum_{k=0}^{m-1}(-1)^k\binom{2m-1}{2k}p^{2m-1-2k}q^{2k}\\&-\lambda\sum_{k=0}^{2m-1}\binom{2m}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)\Bigg].\end{aligned} \tag{28}$$

Hence, clearly, the coefficient of q^{2m-2} in the right-hand side of (28) is equal to

$$\begin{aligned}&\frac{1}{\lambda-1}\left[2m(-1)^{m-1}\binom{2m-1}{2m-2}p^{2m-1-2m+2}-\lambda\binom{2m}{2m-1}\frac{(2m-1)(-1)^{m-1}}{\lambda-1}\right]\\&\quad=\frac{(-1)^{m+1}}{\lambda-1}2m(2m-1)\left(p-\frac{\lambda}{\lambda-1}\right)\end{aligned}$$

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and the coefficient of q^{2m-4} is equal to

$$\begin{aligned}
&\frac{1}{\lambda-1}\left[2m(-1)^{m-2}\binom{2m-1}{2m-4}p^3-\lambda\left\{\binom{2m}{2m-1}\frac{3(-1)^m}{\lambda-1}\binom{2m-1}{3}\right.\right.\\
&\qquad\cdot\left(p^2-\frac{2\lambda}{\lambda-1}p+\frac{\lambda(\lambda+1)}{(\lambda-1)^2}\right)\\
&\qquad+\binom{2m}{2m-2}\frac{(-1)^m}{\lambda-1}(2m-2)(2m-3)\left(p-\frac{\lambda}{\lambda-1}\right)\\
&\qquad\left.\left.+\binom{2m}{2m-3}\frac{(2m-3)(-1)^{m-2}}{\lambda-1}\right\}\right]\\
&=\frac{(2m-3)(-1)^m}{\lambda-1}\binom{2m}{3}\left(p^3-\frac{3\lambda}{\lambda-1}p^2+\frac{3\lambda(\lambda+1)}{(\lambda-1)^2}p-\frac{\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^3}\right).
\end{aligned}$$

So, the assertion (26) is true for $n=2m$. In the second case, taking $n=2m+1$ in (18), we get

$$\begin{aligned}
\mathscr{B}_{2m+1}^{(c)}(p,q;\lambda)=\frac{1}{\lambda-1}\Bigg[&(2m+1)\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}p^{2m-2k}q^{2k}\\
&-\lambda\sum_{k=0}^{2m}\binom{2m+1}{k}\mathscr{B}_k^{(c)}(p,q;\lambda)\Bigg].
\end{aligned}\tag{29}$$

Hence, clearly, the coefficient of q^{2m} in the right-hand side of (29) is equal to

$$\frac{1}{\lambda-1}\left[(2m+1)(-1)^m\binom{2m}{2m}\right]=\frac{(2m+1)(-1)^m}{\lambda-1}$$

and the coefficient of q^{2m-2} is equal to

$$\begin{aligned}
&\frac{1}{\lambda-1}\left[(2m+1)(-1)^{m-1}\binom{2m}{2m-2}p^2-\lambda\left\{\binom{2m+1}{2m}\frac{(-1)^{m+1}}{\lambda-1}2m(2m-1)\left(p-\frac{\lambda}{\lambda-1}\right)\right.\right.\\
&\qquad\left.\left.+\binom{2m+1}{2m-1}\frac{(2m-1)(-1)^{m-1}}{\lambda-1}\right\}\right]\\
&=\frac{3(-1)^{m+1}}{\lambda-1}\binom{2m+1}{3}\left(p^2-\frac{2\lambda}{\lambda-1}p+\frac{\lambda(\lambda+1)}{(\lambda-1)^2}\right),
\end{aligned}$$

which completes the proof of (26). By combining (23) and (26), we can also obtain the result (27).

Proposition 5 *The following identities hold true*:

$$\mathscr{B}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k}\mathscr{B}_{n-2k}^{(c)}(p,0;\lambda)q^{2k}\tag{30}$$

and

$$\mathscr{B}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{\left[\frac{n-2}{2}\right]}(-1)^k\binom{n}{2k+1}\mathscr{B}_{n-2k-1}^{(c)}(p,0;\lambda)q^{2k+1},\tag{31}$$

in which

$$\mathscr{B}_{n-2k}^{(c)}(p,0;\lambda)=\mathscr{B}_{n-2k}(p;\lambda)\qquad\text{and}\qquad\mathscr{B}_{n-2k-1}^{(c)}(p,0;\lambda)=\mathscr{B}_{n-2k-1}(p;\lambda)$$

are the Apostol-Bernoulli polynomials.

Proof. According to (21) and (23), we first observe that

$$\frac{\partial^{2k}}{\partial q^{2k}}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}=(-1)^k\frac{n!}{(n-2k)!}\mathscr{B}_{n-2k}^{(c)}(p,q;\lambda)\qquad\left(k=0,1,2,\cdots,\left[\frac{n-1}{2}\right]\right)$$

and

$$\frac{\partial^{2k+1}}{\partial q^{2k+1}}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}=(-1)^{k+1}\frac{n!}{(n-2k-1)!}\mathscr{B}_{n-2k-1}^{(s)}(p,q;\lambda)\quad\left(k=0,1,2,\cdots,\left[\frac{n-3}{2}\right]\right),$$

because $\mathscr{B}_n^{(c)}(p,q;\lambda)$ is a polynomial in the variable q of degree n for even n and of degree $n-1$ for odd n according to Proposition 4. The Taylor expansion of $\mathscr{B}_n^{(c)}(p,q;\lambda)$ gives

$$\mathscr{B}_n^{(c)}(p,q+h;\lambda)=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial^k}{\partial q^k}\left\{\mathscr{B}_n^{(c)}(p,q;\lambda)\right\}h^k,$$

in which $h\in\mathbb{R}$. Since

$$\mathscr{B}_n^{(s)}(p,0;\lambda)=0$$

for every $n\in\mathbb{Z}^+$, by setting $q=0$ and $h=q$, we obtain the assertion (30). In a similar way, the result (31) can be derived.

Proposition 6 *If $m\in\mathbb{N}$, $n\in\mathbb{Z}^+$ and $\lambda>0$, then*

$$\mathscr{B}_n^{(c)}(mp,q;\lambda^{\frac{1}{m}})=m^{n-1}\sum_{k=0}^{m-1}\lambda^{\frac{k}{m}}\mathscr{B}_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right)\tag{32}$$

and

$$\mathscr{B}_n^{(s)}(mp,q;\lambda^{\frac{1}{m}})=m^{n-1}\sum_{k=0}^{m-1}\lambda^{\frac{k}{m}}\mathscr{B}_n^{(s)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right).\tag{33}$$

Proof. To prove (32), it suffices to consider the following relation:

$$\sum_{n=0}^{\infty}\lambda^{\frac{k}{m}}\mathscr{B}_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right)\frac{t^n}{n!}=\lambda^{\frac{k}{m}}\frac{t\mathrm{e}^{(p+\frac{k}{m})t}}{\lambda\mathrm{e}^t-1}\cos\left(\frac{qt}{m}\right)$$

and then take a sum from both sides of the above equation to obtain

$$\begin{aligned}
&\sum_{k=0}^{m-1}\left[\sum_{n=0}^{\infty}\lambda^{\frac{k}{m}}\mathscr{B}_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right)\frac{t^n}{n!}\right]\\
&=\frac{t\mathrm{e}^{pt}}{\lambda\mathrm{e}^t-1}\cos\left(\frac{qt}{m}\right)\sum_{k=0}^{m-1}\left(\lambda^{\frac{1}{m}}\mathrm{e}^{\frac{t}{m}}\right)^k\\
&=m\frac{\frac{t}{m}\mathrm{e}^{mp\left(\frac{t}{m}\right)}}{\lambda^{\frac{1}{m}}\mathrm{e}^{\frac{t}{m}}-1}\cos\left(\frac{qt}{m}\right)=\sum_{n=0}^{\infty}m^{1-n}\mathscr{B}_n^{(c)}\left(mp,q;\lambda^{\frac{1}{m}}\right)\frac{t^n}{n!}.
\end{aligned}$$

In a similar way, we can prove (33).

In the next two sections, we just present the corresponding basic properties of $\mathscr{E}_n^{(c)}(p,q;\lambda)$, $\mathscr{E}_n^{(s)}(p,q;\lambda)$, $\mathscr{G}_n^{(c)}(p,q;\lambda)$ and $\mathscr{G}_n^{(s)}(p,q;\lambda)$. The proofs are similar and will, therefore, be omitted.

4 Basic Properties of $\mathscr{E}_n^{(c)}(p,q;\lambda)$ and $\mathscr{E}_n^{(s)}(p,q;\lambda)$

Proposition 7 *For every $n\in\mathbb{N}$, the following identities hold true*:

$$\lambda\mathscr{E}_n^{(c)}(1+p,q;\lambda)+\mathscr{E}_n^{(c)}(p,q;\lambda)=2C_n(p,q)\tag{34}$$

and

$$\lambda\mathscr{E}_n^{(s)}(1+p,q;\lambda)+\mathscr{E}_n^{(s)}(p,q;\lambda)=2S_n(p,q).\tag{35}$$

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Corollary 3 *The relations (34) and (35) imply that*

$$\lambda \mathscr{E}_{2n}^{(c)}(1,q;\lambda)+\mathscr{E}_{2n}^{(c)}(0,q;\lambda)=2(-1)^n\, q^{2n}$$

and

$$\lambda \mathscr{E}_{2n+1}^{(s)}(1,q;\lambda)+\mathscr{E}_{2n+1}^{(s)}(0,q;\lambda)=2(-1)^n\, q^{2n+1}.$$

Proposition 8 *For every $n \in \mathbb{Z}^+$, the following identities hold true*:

$$\mathscr{E}_n^{(c)}(p+r,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_k^{(c)}(p,q;\lambda)\, r^{n-k}$$

and

$$\mathscr{E}_n^{(s)}(p+r,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_k^{(s)}(p,q;\lambda)\, r^{n-k}.$$

Corollary 4 *It is asserted that*

$$\mathscr{E}_n^{(c)}(p+1,q;\lambda)-\mathscr{E}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{E}_k^{(c)}(p,q;\lambda)$$

and

$$\mathscr{E}_n^{(s)}(p+1,q;\lambda)-\mathscr{E}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{E}_k^{(s)}(p,q;\lambda).$$

Now, by combining these results and Proposition 7, we have the following recurrence relations:

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\frac{1}{\lambda+1}\left[2C_n(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{E}_k^{(c)}(p,q;\lambda)\right]$$

and

$$\mathscr{E}_n^{(s)}(p,q;\lambda)=\frac{1}{\lambda+1}\left[2S_{n-1}(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{E}_k^{(s)}(p,q;\lambda)\right],$$

where

$$\mathscr{E}_0^{(c)}(p,q;\lambda)=\frac{2}{\lambda+1} \qquad \text{and} \qquad \mathscr{E}_0^{(s)}(p,q;\lambda)=0.$$

Proposition 9 *For every $n \in \mathbb{N}$, the following identities hold true*:

$$\frac{\partial}{\partial p}\left\{\mathscr{E}_n^{(c)}(p,q;\lambda)\right\}=n\mathscr{E}_{n-1}^{(c)}(p,q;\lambda),$$

$$\frac{\partial}{\partial q}\left\{\mathscr{E}_n^{(c)}(p,q;\lambda)\right\}=-n\mathscr{E}_{n-1}^{(s)}(p,q;\lambda),$$

$$\frac{\partial}{\partial p}\left\{\mathscr{E}_n^{(s)}(p,q;\lambda)\right\}=n\mathscr{E}_{n-1}^{(s)}(p,q;\lambda)$$

and

$$\frac{\partial}{\partial q}\left\{\mathscr{E}_n^{(s)}(p,q;\lambda)\right\}=n\mathscr{E}_{n-1}^{(c)}(p,q;\lambda).$$

Proposition 10 *If $\mathscr{E}_n^{(c)}(p,q)$ and $\mathscr{E}_n^{(s)}(p,q)$ are considered as polynomials in the variable p, then they are of degree n and $n-1$, respectively, and it is asserted that*

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\frac{2}{\lambda+1}\,p^n-\frac{2n\lambda}{(\lambda+1)^2}\,p^{n-1}+\cdots$$

and

$$\mathscr{E}_n^{(s)}(p,q;\lambda)=\frac{2nq}{\lambda+1}p^{n-1}-\frac{2n(n-1)q\lambda}{(\lambda+1)^2}\,p^{n-2}+\cdots.$$

Furthermore, if $\mathscr{E}_n^{(c)}(p,q)$ and $\mathscr{E}_n^{(s)}(p,q)$ are considered as polynomials in the variable q, then

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\begin{cases}\dfrac{(-1)^{\frac{n-1}{2}}}{\lambda+1}2n\left(p-\dfrac{\lambda}{\lambda+1}\right)q^{n-1}+\dfrac{2(-1)^{\frac{n+1}{2}}}{\lambda+1}\\ \qquad\cdot\dbinom{n}{3}\left(p^3-\dfrac{3\lambda p^2}{\lambda+1}+\dfrac{3\lambda(\lambda-1)p}{(\lambda+1)^2}-\dfrac{\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^3}\right)q^{n-3}+\cdots & (n\ \text{odd})\\ \dfrac{2(-1)^{\frac{n}{2}}}{\lambda+1}\,q^n+\dfrac{(-1)^{\frac{n+2}{2}}n(n-1)}{\lambda+1}\left(p^2-\dfrac{2\lambda p}{\lambda+1}+\dfrac{\lambda(\lambda-1)}{(\lambda+1)^2}\right)q^{n-2}+\cdots & (n\ \text{even})\end{cases}$$

and

$$\mathscr{E}_n^{(s)}(p,q;\lambda)=\begin{cases}\dfrac{2(-1)^{\frac{n-1}{2}}}{\lambda+1}\,q^n+\dfrac{n(n-1)(-1)^{\frac{n+1}{2}}}{\lambda+1}\left(p^2-\dfrac{2\lambda p}{\lambda+1}+\dfrac{\lambda(\lambda-1)}{(\lambda+1)^2}\right)q^{n-2}+\cdots & (n\ \text{odd})\\ \dfrac{2n(-1)^{\frac{n+2}{2}}}{\lambda+1}\left(p-\dfrac{\lambda}{\lambda+1}\right)q^{n-1}+\dfrac{2(-1)^{\frac{n}{2}}}{\lambda+1}\\ \qquad\cdot\dbinom{n}{3}\left(p^3-\dfrac{3\lambda p^2}{\lambda+1}+\dfrac{3\lambda(\lambda-1)p}{(\lambda+1)^2}-\dfrac{\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^3}\right)q^{n-3}+\cdots & (n\ \text{even}).\end{cases}$$

Proposition 11 *The following identities hold true*:

$$\mathscr{E}_n^{(c)}(p,q;\lambda)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\mathscr{E}_{n-2k}^{(c)}(p,0;\lambda)\,q^{2k}$$

and

$$\mathscr{E}_n^{(s)}(p,q;\lambda)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\mathscr{E}_{n-2k-1}^{(c)}(p,0;\lambda)\,q^{2k+1},$$

in which

$$\mathscr{E}_{n-2k}^{(c)}(p,0;\lambda)=\mathscr{E}_{n-2k}(p;\lambda)$$

and

$$\mathscr{E}_{n-2k-1}^{(c)}(p,0;\lambda)=\mathscr{E}_{n-2k-1}(p;\lambda)$$

are the Apostol-Euler polynomials.

Proposition 12 *If $n \in \mathbb{N}$, $\lambda > 0$ and m is an odd positive integer, then*

$$\mathscr{E}_n^{(c)}\left(mp,q;\lambda^{\frac{1}{m}}\right)=m^n\sum_{k=0}^{m-1}(-1)^k\lambda^{\frac{k}{m}}\,\mathscr{E}_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right)$$

and

$$\mathscr{E}_n^{(s)}\left(mp,q;\lambda^{\frac{1}{m}}\right)=m^n\sum_{k=0}^{m-1}(-1)^k\lambda^{\frac{k}{m}}\,\mathscr{E}_n^{(s)}\left(p+\frac{k}{m},\frac{q}{m};\lambda\right).$$

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5 Basic Properties of $\mathscr{G}_n^{(c)}(p,q;\lambda)$ and $\mathscr{G}_n^{(s)}(p,q;\lambda)$

Proposition 13 *For every $n \in \mathbb{N}$, the following identities hold*

$$\lambda \mathscr{G}_n^{(c)}(1+p,q;\lambda) + \mathscr{G}_n^{(c)}(p,q;\lambda) = 2nC_{n-1}(p,q) \tag{36}$$

and

$$\lambda \mathscr{G}_n^{(s)}(1+p,q;\lambda) + \mathscr{G}_n^{(s)}(p,q;\lambda) = 2nS_{n-1}(p,q). \tag{37}$$

Corollary 5 *The relations* (36) *and* (37) *imply that*

$$\lambda \mathscr{G}_{2n+1}^{(c)}(1,q;\lambda) + \mathscr{G}_{2n+1}^{(c)}(0,q;\lambda) = 2(2n+1)(-1)^n\, q^{2n}$$

and

$$\lambda \mathscr{G}_{2n}^{(s)}(1,q;\lambda) + \mathscr{G}_{2n}^{(s)}(0,q;\lambda) = 4n(-1)^{n+1}\, q^{2n-1}.$$

Proposition 14 *For every $n \in \mathbb{N}$, the following identities hold true*:

$$\mathscr{G}_n^{(c)}(p+r,q;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{G}_k^{(c)}(p,q;\lambda)\, r^{n-k}$$

and

$$\mathscr{G}_n^{(s)}(p+r,q;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{G}_k^{(s)}(p,q;\lambda)\, r^{n-k}.$$

Corollary 6 *It is asserted that*

$$\mathscr{G}_n^{(c)}(p+1,q;\lambda) - \mathscr{G}_n^{(c)}(p,q;\lambda) = \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{G}_k^{(c)}(p,q;\lambda)$$

and

$$\mathscr{G}_n^{(s)}(p+1,q;\lambda) - \mathscr{G}_n^{(s)}(p,q;\lambda) = \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{G}_k^{(s)}(p,q;\lambda).$$

Now, by combining these results and Proposition 13, we can derive the following recurrence relations:

$$\mathscr{G}_n^{(c)}(p,q;\lambda) = \frac{1}{\lambda+1}\left[2nC_{n-1}(p,q) - \lambda \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{G}_k^{(c)}(p,q;\lambda)\right]$$

and

$$\mathscr{G}_n^{(s)}(p,q;\lambda) = \frac{1}{\lambda+1}\left[2nS_{n-1}(p,q) - \lambda \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{G}_k^{(s)}(p,q;\lambda)\right],$$

where

$$\mathscr{G}_0^{(c)}(p,q;\lambda) = 0 \qquad \text{and} \qquad \mathscr{G}_0^{(s)}(p,q;\lambda) = 0.$$

Proposition 15 *For every $n \in \mathbb{N}$, the following identities hold true*:

$$\frac{\partial}{\partial p}\left\{\mathscr{G}_n^{(c)}(p,q;\lambda)\right\} = n\mathscr{G}_{n-1}^{(c)}(p,q;\lambda),$$

$$\frac{\partial}{\partial q}\left\{\mathscr{G}_n^{(c)}(p,q;\lambda)\right\} = -n\mathscr{G}_{n-1}^{(s)}(p,q;\lambda),$$

$$\frac{\partial}{\partial p}\left\{\mathscr{G}_n^{(s)}(p,q;\lambda)\right\} = n\mathscr{G}_{n-1}^{(s)}(p,q;\lambda)$$

and

$$\frac{\partial}{\partial q}\left\{\mathscr{G}_n^{(s)}(p,q;\lambda)\right\} = n\mathscr{G}_{n-1}^{(c)}(p,q;\lambda).$$

Proposition 16 *If $\mathscr{G}_n^{(c)}(p,q)$ and $\mathscr{G}_n^{(s)}(p,q)$ are considered as polynomials in the variable p, then they are of degrees $n-1$ and $n-2$, respectively, and it is asserted that*

$$\mathscr{G}_n^{(c)}(p,q;\lambda) = \frac{2n}{\lambda+1}\, p^{n-1} - \frac{2n(n-1)\lambda}{(\lambda+1)^2}\, p^{n-2} + \cdots$$

and

$$\mathscr{G}_n^{(s)}(p,q;\lambda) = \frac{2n(n-1)q}{\lambda+1}\, p^{n-2} - 12\binom{n}{3} \frac{q\lambda}{(\lambda+1)^2}\, p^{n-3} + \cdots.$$

Furthermore, if $\mathscr{G}_n^{(c)}(p,q)$ and $\mathscr{G}_n^{(s)}(p,q)$ are considered as polynomials in the variable q, then

$$\mathscr{G}_n^{(c)}(p,q;\lambda) = \begin{cases} \dfrac{(-1)^{\frac{n+2}{2}}}{\lambda+1} 2n(n-1)\left(p - \dfrac{\lambda}{\lambda+1}\right) q^{n-2} + \dfrac{2(n-3)(-1)^{\frac{n}{2}}}{\lambda+1} \\ \quad \cdot \dbinom{n}{3}\left(p^3 - \dfrac{3\lambda p^2}{\lambda+1} + \dfrac{3\lambda(\lambda-1)p}{(\lambda+1)^2} - \dfrac{\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^3}\right) q^{n-4} + \cdots & (n \ \text{even}) \\ \dfrac{2n(-1)^{\frac{n-1}{2}}}{\lambda+1}\, q^{n-1} + \dfrac{6(-1)^{\frac{n+1}{2}}}{\lambda+1} \dbinom{n}{3}\left(p^2 - \dfrac{2\lambda p}{\lambda+1} + \dfrac{\lambda(\lambda-1)}{(\lambda+1)^2}\right) q^{n-3} + \cdots & (n \ \text{odd}) \end{cases}$$

and

$$\mathscr{G}_n^{(s)}(p,q;\lambda) = \begin{cases} \dfrac{2n(-1)^{\frac{n+2}{2}}}{\lambda+1}\, q^{n-1} + \dfrac{6(-1)^{\frac{n}{2}}}{\lambda+1} \dbinom{n}{3}\left(p^2 - \dfrac{2\lambda p}{\lambda+1} + \dfrac{\lambda(\lambda-1)}{(\lambda+1)^2}\right) q^{n-3} + \cdots & (n \ \text{even}) \\ \dfrac{(-1)^{\frac{n+1}{2}}}{\lambda+1} 2n(n-1)\left(p - \dfrac{\lambda}{\lambda+1}\right) q^{n-2} + \dfrac{2(n-3)(-1)^{\frac{n-1}{2}}}{\lambda+1} \\ \quad \cdot \dbinom{n}{3}\left(p^3 - \dfrac{3\lambda p^2}{\lambda+1} + \dfrac{3\lambda(\lambda-1)p}{(\lambda+1)^2} - \dfrac{\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^3}\right) q^{n-4} + \cdots & (n \ \text{odd}). \end{cases}$$

Proposition 17 *The following identities hold true*:

$$\mathscr{G}_n^{(c)}(p,q;\lambda) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k} \mathscr{G}_{n-2k}^{(c)}(p,0;\lambda)\, q^{2k}$$

and

$$\mathscr{G}_n^{(s)}(p,q;\lambda) = \sum_{k=0}^{\left[\frac{n-2}{2}\right]} (-1)^k \binom{n}{2k+1} \mathscr{G}_{n-2k-1}^{(c)}(p,0;\lambda)\, q^{2k+1},$$

in which

$$\mathscr{G}_{n-2k}^{(c)}(p,0;\lambda) = \mathscr{G}_{n-2k}(p;\lambda)$$

and

$$\mathscr{G}_{n-2k-1}^{(c)}(p,0;\lambda) = \mathscr{G}_{n-2k-1}(p;\lambda)$$

are the Apostol-Genocchi polynomials.

Proposition 18 *If $n \in \mathbb{N}$, $\lambda > 0$ and m is an odd positive integer, then*

$$\mathscr{G}_n^{(c)}\left(mp,q;\lambda^{\frac{1}{m}}\right) = m^{n-1} \sum_{k=0}^{m-1} (-1)^k \lambda^{\frac{k}{m}}\, \mathscr{G}_n^{(c)}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda\right)$$

and

$$\mathscr{G}_n^{(s)}\left(mp,q;\lambda^{\frac{1}{m}}\right) = m^{n-1} \sum_{k=0}^{m-1} (-1)^k \lambda^{\frac{k}{m}}\, \mathscr{G}_n^{(s)}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda\right).$$

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6 New Taylor Type Series Involving the Apostol Type Numbers $\mathscr{B}_{n,\lambda}$, $\mathscr{E}_{n,\lambda}$ and $\mathscr{G}_{n,\lambda}$

One of the applications of the relations (8), (9) and (10) is that they can be considered as the Taylor expansion of some special functions about $t = 0$ involving the Apostol type numbers $\mathscr{B}_{n,\lambda}$, $\mathscr{E}_{n,\lambda}$ and $\mathscr{G}_{n,\lambda}$. In other words, upon substituting the relations (11), (12) and (13) into the relations (8), (9) and (10), we find that

$$f^{(c)}_{\mathscr{B},\lambda}(t;p,q) = \frac{te^{pt}}{\lambda e^t - 1}\cos(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!}, \tag{38}$$

$$f^{(s)}_{\mathscr{B},\lambda}(t;p,q) = \frac{te^{pt}}{\lambda e^t - 1}\sin(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!}, \tag{39}$$

$$f^{(c)}_{\mathscr{E},\lambda}(t;p,q) = \frac{2e^{pt}}{\lambda e^t + 1}\cos(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!}, \tag{40}$$

$$f^{(s)}_{\mathscr{E},\lambda}(t;p,q) = \frac{2e^{pt}}{\lambda e^t + 1}\sin(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!}, \tag{41}$$

$$f^{(c)}_{\mathscr{G},\lambda}(t;p,q) = \frac{2te^{pt}}{\lambda e^t + 1}\cos(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{G}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!} \tag{42}$$

and

$$f^{(s)}_{\mathscr{G},\lambda}(t;p,q) = \frac{2te^{pt}}{\lambda e^t + 1}\sin(qt) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{G}_{n-k,\lambda}C_k(p,q)\right]\frac{t^n}{n!}, \tag{43}$$

where $C_k(p,q)$ and $S_k(p,q)$ are defined in (3) and (4).
In order to evaluate the above functions for some specific parameters, we first prove the following identities:

$$C_k(p,p) = 2^{\frac{k}{2}}\, p^k \cos\left(\frac{k\pi}{4}\right), \tag{44}$$

$$S_k(p,p) = 2^{\frac{k}{2}}\, p^k \sin\left(\frac{k\pi}{4}\right), \tag{45}$$

$$C_k(0,q) = q^k \cos\left(\frac{k\pi}{2}\right), \tag{46}$$

$$S_k(0,q) = q^k \sin\left(\frac{k\pi}{2}\right) \tag{47}$$

and

$$C_k(p,0) = p^k \qquad \text{and} \qquad S_k(p,0) = 0. \tag{48}$$

It is easily observed that

$$\begin{aligned}\cos k\theta + \mathrm{i}\sin(k\theta) &= (\cos\theta + \mathrm{i}\sin\theta)^k \\ &= \sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j}(\sin\theta)^{2j}(\cos\theta)^{k-2j} \\ &\quad + \mathrm{i}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}(\sin\theta)^{2j+1}(\cos\theta)^{k-2j-1}.\end{aligned}$$

Thus, upon setting $\theta = \frac{\pi}{4}$, we obtain

$$\cos\left(\frac{k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{k\pi}{4}\right) = 2^{-\frac{k}{2}}\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j} + \mathrm{i}\,2^{-\frac{k}{2}}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1},$$

which leads to the relations (44) and (45). The relations (46), (47) and (48) are also clear by noting the relations (3) and (4).
We now consider some particular illustrative examples.

Example 1 In (38), we take $p = 0$ and $q = 1$. Then, by noting (46) and (47), we obtain

$$\begin{aligned}f^{(c)}_{\mathscr{B},\lambda}(t;0,1) = \frac{t}{\lambda e^t - 1}\cos t &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k,\lambda}\cos\left(\frac{k\pi}{2}\right)\right]\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\mathscr{B}_{n-2k,\lambda}(-1)^k\right]\frac{t^n}{n!}.\end{aligned}$$

Therefore, we have

$$\frac{t}{\lambda e^t - 1}\cos t = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\mathscr{B}_{n-2k,\lambda}\right)\frac{t^n}{n!}$$

as well as

$$\frac{t}{\lambda e^t - 1}\sin t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\mathscr{B}_{n-2k-1,\lambda}\right]\frac{t^n}{n!},$$

$$\frac{1}{\lambda e^t + 1}\cos t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{2}\binom{n}{2k}\mathscr{E}_{n-2k,\lambda}\right]\frac{t^n}{n!},$$

$$\frac{1}{\lambda e^t + 1}\sin t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{2}\binom{n}{2k+1}\mathscr{E}_{n-2k-1,\lambda}\right]\frac{t^n}{n!},$$

$$\frac{t}{\lambda e^t + 1}\cos t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{2}\binom{n}{2k}\mathscr{G}_{n-2k,\lambda}\right]\frac{t^n}{n!}$$

and

$$\frac{t}{\lambda e^t + 1}\sin t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{2}\binom{n}{2k+1}\mathscr{G}_{n-2k-1,\lambda}\right]\frac{t^n}{n!}.$$

Example 2 Putting $p = q = 1$ in (38), we get

$$f^{(c)}_{\mathscr{B},\lambda}(t;1,1) = \frac{te^t}{\lambda e^t - 1}\cos t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}}\binom{n}{k}\mathscr{B}_{n-k,\lambda}\cos\frac{k\pi}{4}\right]\frac{t^n}{n!}.$$

In a similar way, we have

$$\frac{te^t}{\lambda e^t - 1}\sin t = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}}\binom{n}{k}\mathscr{B}_{n-k,\lambda}\sin\frac{k\pi}{4}\right]\frac{t^n}{n!},$$

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$$\frac{e^t}{\lambda e^t+1}\cos t=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}-1}\binom{n}{k}\mathscr{E}_{n-k,\lambda}\cos\left(\frac{k\pi}{4}\right)\right]\frac{t^n}{n!},$$

$$\frac{e^t}{\lambda e^t+1}\sin t=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}-1}\binom{n}{k}\mathscr{E}_{n-k,\lambda}\sin\left(\frac{k\pi}{4}\right)\right]\frac{t^n}{n!},$$

$$\frac{te^t}{\lambda e^t+1}\cos t=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}-1}\binom{n}{k}\mathscr{G}_{n-k,\lambda}\cos\left(\frac{k\pi}{4}\right)\right]\frac{t^n}{n!}$$

and

$$\frac{te^t}{\lambda e^t+1}\sin t=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{\frac{k}{2}-1}\binom{n}{k}\mathscr{G}_{n-k,\lambda}\sin\left(\frac{k\pi}{4}\right)\right]\frac{t^n}{n!}.$$

7 Perspective

In this paper, we have introduced a new kind of parametric Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials by defining six special generating functions. We have systematically investigated some basic properties of each of these parametric Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials. As an interesting application, we have used such parametric polynomials to explicitly compute some new series of the Taylor type containing the Apostol-Bernoulli numbers $\mathscr{B}_{n,\lambda}$, the Apostol-Euler numbers $\mathscr{E}_{n,\lambda}$ and the Apostol-Genocchi numbers $\mathscr{G}_{n,\lambda}$.

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