Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120506

Identities Leading to Distributions of Order Statistics from *innid* Variables

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Received: 14 Mar. 2018, Revised: 22 Jul. 2018, Accepted: 24 Jul. 2018
Published online: 1 Sep. 2018

Abstract: In this study, joint distributions of order statistics of *innid* continuous random variables are expressed in terms of specialized identities based on distributions of *innid* continuous random variables. Then, some results related to *pdf* and *df* order statistics of *innid* continuous random variables are given.

Keywords: Order statistics, permanent, joint distributions, continuous random variable.

1 Introduction

The joint probability density function (*pdf*) and marginal *pdf* of order statistics of independent but not necessarily identically distributed (*innid*) random variables was derived by Vaughan and Venables[1] by means of permanents. In addition, Balakrishnan[2], and Bapat and Beg[3] obtained the joint *pdf* and distribution function (*df*) of order statistics of *innid* random variables by means of permanents. In the first of two papers, Balasubramanian et al.[4] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, \ldots, X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al.[5] generalized their previous results in [4] to the case of the joint distribution function of several order statistics. Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West[6]. Using multinomial arguments, the pdf of $X_{r:n+1}$ $(1 \leq r \leq n+1)$ was obtained by Childs and Balakrishnan[7] by adding another independent random variable to the original n variables $X_1, X_2, \ldots, X_n$. Also, Balasubramanian et al.[8] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg[9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al.[10] derived the expressions for the distribution and density functions by Ryser's method and the distribution of maxima and minima based on permanents.

A multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution was defined by Corley[11]. Guilbaud[12] expressed the probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of independent and identically distributed (*iid*) random vectors and thus also for order statistics of random variables. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller[13]. Several identities and recurrence relations for *pdf* and *df* of order statistics of *iid* random variables were established by numerous authors including Arnold et al.[14], Balasubramanian and Beg[15], David[16], and Reiss[17]. Furthermore, Arnold et al.[14], David[16], Gan and Bain[18], and Khatri[19] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Also, marginal and joint distributions of order statistics from innid / iid continuous and discrete random variables / vectors are obtained in different ways by Güngör and Turan[20,21], Güngör[22,23,24], Güngör et al.[25],

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Yüzbaşı and Güngör[26], Bulut et al.[27], Yüzbaşı et al.[28] and Güngör and Bulut[29].

In this study, joint distributions of order statistics of *innid* continuous random variables are obtained.

Hereafter, subscripts and superscripts are defined in first place in which they are used and these definitions will valid unless they are redefined.

If $a_1, a_2, \ldots$ are defined as column vectors, then matrix obtained by taking m_1 copies of a_1, m_2 copies of $a_2, \ldots$ can be denoted as

$$\underset{m_1}{[a_1} \underset{m_2}{a_2} \ldots]$$

and *per*A denotes permanent of a square matrix A, which is defined as similar to determinants except that all terms in expansion have a positive sign.

Let $X_1, X_2, \ldots, X_n$ be *innid* continuous random variables and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be order statistics obtained by arranging the nX_i's in increasing order of magnitude. A$[s/.)$ is matrix obtained from A by taking rows whose indices are in s.

In this study, *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_d:n}$ $(1 \leq r_1 < r_2 < \ldots < r_d \leq n,\ d = 1, 2, \ldots, n)$ are given. For notational convenience we write $\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,\,m_2}$, $\sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,m_3,m_2}$ and $\sum_{t_d,\ldots,t_2,\,t_1}^{n,\ldots,r_3-1,\,r_2-1}$ instead of $\sum_{m_d=r_d}^{n} \cdots \sum_{m_2=r_2}^{m_3} \sum_{m_1=r_1}^{m_2}$, $\sum_{t_d=m_d}^{n} \cdots \sum_{t_2=m_2}^{m_3} \sum_{t_1=m_1}^{m_2}$ and $\sum_{t_d=r_d}^{n} \cdots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1}$ in the expressions below, respectively.

2 Identities leading to distribution and probability density functions

Identities in the following theorems are used to obtain joint *df* and *pdf* of order statistics of *innid* continuous random variables.

We now express three theorems to establish *df* of order statistics of *innid* continuous random variables.

Theorem 1.

$$per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}] = \sum_P \prod_{w=1}^{d+1} \prod_{l=m_{w-1}+1}^{m_w} \Delta F_{j_l}(x_w), \tag{1}$$

where $x_1 < x_2 < \ldots < x_d$, $\Delta\mathbf{F}(x_w) = (\Delta F_1(x_w), \Delta F_2(x_w), \ldots, \Delta F_n(x_w))'$ is column vector, $x_w \in \mathbb{R}$, $\sum_P$ denotes sum over all $n!$ permutations $(j_1,\ j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$, $m_0 = 0$, $m_{d+1} = n$, $\Delta F_{j_l}(x_w) = F_{j_l}(x_w) - F_{j_l}(x_{w-1})$, $F_{j_l}(x_0) = 0$ and $F_{j_l}(x_{d+1}) = 1$.

Proof. Using expansion of permanent, it can be written

$$\begin{aligned}
&per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]\\
&= \sum_P \Delta F_{j_1}(x_1)\ldots\Delta F_{j_{m_1}}(x_1)\Delta F_{j_{m_1+1}}(x_2)\ldots\Delta F_{j_{m_2}}(x_2)\\
&\cdot \ldots \Delta F_{j_{m_d+1}}(x_{d+1})\ldots\Delta F_{j_n}(x_{d+1})\\
&= \sum_P (\prod_{l=1}^{m_1} \Delta F_{j_l}(x_1))(\prod_{l=m_1+1}^{m_2} \Delta F_{j_l}(x_2)) \ldots \prod_{l=m_d+1}^{n} \Delta F_{j_l}(x_{d+1}).
\end{aligned}$$

Thus, Eq. (1) is obtained.

Eq. (1) in Theorem 1 can be expressed as Eq. (2) using a generalization of binomial expansion.

Theorem 2.

$$\begin{aligned}
&per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]\\
&= \sum_P \left[\prod_{l=1}^{m_1} F_{j_l}(x_1)\right] \prod_{w=2}^{d+1} \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w-t}\\
&\cdot \sum_{n_\tau = t-m_{w-1}} \left[\prod_{l=1}^{t-m_{w-1}} F_{\tau_l}(x_w)\right] \prod_{l=1}^{m_w-t} F_{\tau'_l}(x_{w-1}),
\end{aligned} \tag{2}$$

where $\sum_{n_\tau=t-m_{w-1}}$ denotes sum over all $\binom{m_w-m_{w-1}}{t-m_{w-1}}$ subsets $\tau = \{\tau_1, \tau_2, \ldots, \tau_{t-m_{w-1}}\}$, $\tau' = \{\tau'_1, \tau'_2, \ldots, \tau'_{m_w-t}\}$ of $\tau \bigcup \tau' = \{j_{m_{w-1}+1}, j_{m_{w-1}+2}, \ldots, j_{m_w}\}$ and $\tau \bigcap \tau' = \emptyset$.

Proof. Eq. (1) can be expressed as

$$\begin{aligned}
&per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\ \underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]\\
&= \sum_P (\prod_{l=1}^{m_1} F_{j_l}(x_1)) \prod_{w=2}^{d+1} \prod_{l=m_{w-1}+1}^{m_w} \left[F_{j_l}(x_w) - F_{j_l}(x_{w-1})\right].
\end{aligned} \tag{3}$$

It can be written as

$$\begin{aligned}
&\prod_{l=m_{w-1}+1}^{m_w} \left[F_{j_l}(x_w) - F_{j_l}(x_{w-1})\right]\\
&= \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w-t} \sum_{n_\tau=t-m_{w-1}} (\prod_{l=1}^{t-m_{w-1}} F_{\tau_l}(x_w)) \prod_{l=1}^{m_w-t} F_{\tau'_l}(x_{w-1}),
\end{aligned} \tag{4}$$

and using Eq. (4) in Eq. (3), Eq. (2) is obtained.

It can be written as $C^{-1}\sum_{CP_{m_d,\ldots,m_2,m_1}}$ or $(n-m_d)!\sum_{P_{m_d}}$ instead of $\sum_P$ in Theorem 1 and Theorem 2. Here, $\sum_{CP_{m_d,\ldots,m_2,m_1}}$ denotes sum over all $n!$ permutations $(j_1, j_2,\ \ldots, j_n)$ of $(1, 2, \ldots, n)$ for which $j_1 < j_2 < \ldots < j_{m_1}$, $j_{m_1+1} < j_{m_1+2} < \ldots < j_{m_2}$, $\ldots$, $j_{m_d+1} < j_{m_d+2} < \ldots < j_n$, $C = \prod_{w=1}^{d+1}[(m_w - m_{w-1})!]^{-1}$ and $\sum_{P_{m_d}}$ denotes sum over all permutations $(j_1, j_2,\ \ldots, j_{m_d})$ of $(1, 2, \ldots, n)$.

Realize that $\sum_{CP_{m_d,\ldots,m_2,m_1}}$ includes $\frac{n!}{m_1!(m_2-m_1)!\ldots(n-m_d)!}$ terms, while $\sum_P$ includes $n!$ terms.

Theorem 2 can be expressed as Eq. (5) using expansion of permanent.

Theorem 3.

$$\begin{aligned}
&per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]\\
&= \sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \left[\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w-m_w}\right]\\
&\cdot \sum_{n_s=n-t_d+m_d} (t_d-m_d)! \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{d-1}}} \prod_{w=1}^{d} n_{s_w}! \prod_{l=1}^{n_{s_w}} F_{s_w^l}(x_w),
\end{aligned} \tag{5}$$

where $\sum_{n_{s_1},n_{s_2},...,n_{s_{d-1}}}$ denotes sum over $\bigcup_{w=1}^{d-1} s_w$ for which $s_\nu \bigcap s_\upsilon = \emptyset$ for $\nu \neq \upsilon$, $s = \bigcup_{w=1}^{d} s_w$, $s \subset \{1,2,...,n\}$, $s_w = \{s_w^1, s_w^2, ..., s_w^{n_{s_w}}\}$, $n_{s_w} = m_{w+1} - m_{w-1} - t_w + t_{w-1}$ and $t_0 = m_1$.

Proof.

$$per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}...\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_{t_d=m_d}^{n} ... \sum_{t_1=m_1}^{m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \left[\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w - m_w}\right]$$
$$\cdot \sum_{n_s=n-t_d+m_d} (t_d - m_d)!$$
$$\cdot per[\underset{m_2-t_1+m_1}{\mathbf{F}(x_1)} \underset{m_3-m_1-t_2+t_1}{\mathbf{F}(x_2)} ... \underset{n-m_{d-1}-t_d+t_{d-1}}{\mathbf{F}(x_d)}][s/.)$$
$$= \sum_{t_d,...,t_2,t_1}^{n,...,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \left[\prod_{w=1}^{d} \binom{m_{w+1}-m_w}{t_w - m_w}\right]$$
$$\cdot \sum_{n_s=n-t_d+m_d} (t_d - m_d)! \sum_{n_{s_1},n_{s_2},...,n_{s_{d-1}}}$$
$$\prod_{w=1}^{d} per[\underset{m_{w+1}-m_{w-1}-t_w+t_{w-1}}{\mathbf{F}(x_w)}][s_w/.).$$

Thus, the proof is completed.

We express three theorems to obtain *pdf* of order statistics of *innid* continuous random variables.

Let us consider Eq. (1) in Theorem 1. We establish the following theorem using expansion of permanent.

Theorem 4.

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}...\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_P \left[\prod_{w=1}^{d+1} \prod_{l=r_{w-1}+1}^{r_w-1} \Delta F_{j_l}(x_w)\right] \prod_{w=1}^{d} f_{j_{r_w}}(x_w), \tag{6}$$

where $x_1 < x_2 < ... < x_d$, $\mathbf{f}(x_w) = (f_1(x_w), f_2(x_w), ..., f_n(x_w))'$ is column vector, $r_0 = 0$ and $r_{d+1} = n+1$.

Proof. Using expansion of permanent, it can be written

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}...\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_P \Delta F_{j_1}(x_1)...\Delta F_{j_{r_1-1}}(x_1) f_{j_{r_1}}(x_1) \Delta F_{j_{r_1+1}}(x_2)...$$
$$\cdot \Delta F_{j_{r_2-1}}(x_2)... f_{j_{r_d}}(x_d) \Delta F_{j_{r_d+1}}(x_{d+1})...\Delta F_{j_n}(x_{d+1})$$
$$= \sum_P \left[\prod_{l=1}^{r_1-1} \Delta F_{j_l}(x_1)\right] f_{j_{r_1}}(x_1) \left[\prod_{l=r_1+1}^{r_2-1} \Delta F_{j_l}(x_2)\right]$$
$$\cdot f_{j_{r_2}}(x_2)... f_{j_{r_d}}(x_d) \prod_{l=r_d+1}^{n} \Delta F_{j_l}(x_{d+1}).$$

Thus, Eq. (6) is obtained.

Theorem 4 can be expressed as follows.

Theorem 5.

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}...\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_P \left(\prod_{l=1}^{r_1-1} F_{j_l}(x_1)\right) \left[\prod_{w=2}^{d+1} \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t}\right.$$
$$\left.\cdot \sum_{n_\tau=t-r_{w-1}} \left(\prod_{l=1}^{t-r_{w-1}} F_{\tau_l}(x_w)\right) \prod_{l=1}^{r_w-1-t} F_{\tau'_l}(x_{w-1})\right] \prod_{w=1}^{d} f_{j_{r_w}}(x_w), \tag{7}$$

where $\sum_{n_\tau=t-r_{w-1}}$ denotes sum over all $\binom{r_w-r_{w-1}}{t-r_{w-1}}$ subsets $\tau = \{\tau_1, \tau_2, ..., \tau_{t-r_{w-1}}\}$, $\tau' = \{\tau'_1, \tau'_2, ..., \tau'_{r_w-1-t}\}$ of $\tau \bigcup \tau' = \{j_{r_{w-1}+1}, j_{r_{w-1}+2}, ..., j_{r_w-1}\}$ and $\tau \bigcap \tau' = \emptyset$.

Proof. Eq. (6) can be expressed as

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}...\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_P \left[\prod_{l=1}^{r_1-1} F_{j_l}(x_1)\right] \left[\prod_{w=2}^{d+1} \prod_{l=r_{w-1}+1}^{r_w-1} (F_{j_l}(x_w) - F_{j_l}(x_{w-1}))\right]$$
$$\cdot \prod_{w=1}^{d} f_{j_{r_w}}(x_w). \tag{8}$$

By similar expansion of Eq. (4), Eq. (9) can be written as

$$\prod_{l=r_{w-1}+1}^{r_w-1} [F_{j_l}(x_w) - F_{j_l}(x_{w-1})]$$
$$= \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \sum_{n_\tau=t-m_{w-1}} \left[\prod_{l=1}^{t-r_{w-1}} F_{\tau_l}(x_w)\right] \prod_{l=1}^{r_w-1-t} F_{\tau'_l}(x_{w-1}), \tag{9}$$

and using Eq. (9) in Eq. (8), Eq. (7) is obtained.

We can write $D^{-1}\sum_{DP_{r_d,...,r_2,r_1}}$ or $(n-r_d)!\sum_{P_{r_d}}$ instead of $\sum_P$ in Theorem 4 and Theorem 5. Here, $\sum_{DP_{r_d,...,r_2,r_1}}$ denotes sum over all $n!$ permutations $(j_1, j_2, ..., j_n)$ of $(1,2,...,n)$ for which $j_1 < j_2 < ... < j_{r_1-1}$, $j_{r_1+1} < j_{r_1+2} < ... < j_{r_2-1}$, ..., $j_{r_d+1} < j_{r_d+2} < ... < j_n$, $D = \prod_{w=1}^{d+1}[(r_w - r_{w-1} - 1)!]^{-1}$ and $\sum_{P_{r_d}}$ denotes sum over all permutations $(j_1, j_2, ..., j_{r_d})$ of $(1,2,...,n)$.

The following theorem can be written using expansion of permanent.

Theorem 6.

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}...\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= \sum_{t_d,...,t_2,t_1}^{n,...,r_3-1,r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \left[\prod_{w=1}^{d} \binom{r_{w+1}-r_w-1}{t_w - r_w}\right]$$
$$\cdot \sum_{n_s=n+r_d-t_d} (t_d - r_d)! \sum_{n_{s_1},n_{s_2},...,n_{s_{d-1}}} \prod_{w=1}^{d} \sum_{n_{\varsigma_w}} n_{\varsigma_w}! \left(\prod_{l=1}^{n_{\varsigma_w}} F_{\varsigma_w^l}(x_w)\right)$$
$$\cdot f_{\varsigma_w'^{w}}(x_w), \tag{10}$$

where $\sum_{n_{\mathbf{s}_1},n_{\mathbf{s}_2},\ldots,n_{\mathbf{s}_{d-1}}}$ denotes sum over $\bigcup_{w=1}^{d-1}\mathbf{s}_w$ for which $\mathbf{s}_v\bigcap\mathbf{s}_\upsilon=\emptyset$ for $v\neq\upsilon$, $\mathbf{s}=\bigcup_{w=1}^{d}\mathbf{s}_w$, $\mathbf{s}\subset\{1,2,\ldots,n\}$, $n_{\mathbf{s}_w}=r_{w+1}-r_{w-1}-t_w+t_{w-1}$, $t_0=r_1-1$, $\mathbf{s}_w=\varsigma_w\bigcup\varsigma'_w$, $\varsigma_w\bigcap\varsigma'_w=\emptyset$, $\varsigma_w=\{\varsigma_w^1,\varsigma_w^2,\ldots,\varsigma_w^{n_{\varsigma_w}}\}$, $\varsigma'_w=\{\varsigma_w'^{\,w}\}$ and $n_{\varsigma_w}=r_{w+1}-r_{w-1}-1-t_w+t_{w-1}$.

Proof.

$$per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}\ldots\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$=\sum_{t_d=0}^{n-r_d}(-1)^{n-r_d-t_d}\binom{n-r_d}{t_d}\ldots\sum_{t_1=0}^{r_2-r_1-1}(-1)^{r_2-r_1-1-t_1}$$
$$\cdot\binom{r_2-r_1-1}{t_1}per[\underset{r_2-2-t_1}{\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\ldots\underset{1}{\mathbf{f}(x_d)}\underset{t_d}{1}\underset{n-r_d-t_d+t_{d-1}}{\mathbf{F}(x_d)}]$$
$$=\sum_{t_d=r_d}^{n}\cdots\sum_{t_2=r_2}^{r_3-1}\sum_{t_1=r_1}^{r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}$$
$$\cdot\left[\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}\right]\sum_{n_\mathbf{s}=n+r_d-t_d}(t_d-r_d)!\sum_{n_{\mathbf{s}_1},n_{\mathbf{s}_2},\ldots,n_{\mathbf{s}_{d-1}}}$$
$$per[\underset{r_2-1-t_1+r_1}{\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}][\mathbf{s}_1/.)per[\underset{r_3-r_1-1-t_2+t_1}{\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}][\mathbf{s}_2/.)$$
$$\cdot\ldots per[\underset{n-r_{d-1}-1-t_d+t_{d-1}}{\mathbf{F}(x_d)}\underset{1}{\mathbf{f}(x_d)}][\mathbf{s}_d/.)$$
$$=\sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\left[\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}\right]$$
$$\cdot\sum_{n_\mathbf{s}=n+r_d-t_d}(t_d-r_d)!\sum_{n_{\mathbf{s}_1},n_{\mathbf{s}_2},\ldots,n_{\mathbf{s}_{d-1}}}\prod_{w=1}^{d}\sum_{n_{\varsigma_w}}$$
$$per[\underset{r_{w+1}-r_{w-1}-1-t_w+t_{w-1}}{\mathbf{F}(x_w)}][\varsigma_w/.)per[\underset{1}{\mathbf{f}(x_w)}][\varsigma'_w/.).$$

Thus, the proof is completed.

In the above theorem, the expansion of the permanent is established as the special sum which is a direct way for *innid* continuous random variables.

3 Results for distribution and probability density functions

In this section, some results related to *df* and *pdf* of $X_{r_1:n},X_{r_2:n},\ldots,X_{r_d:n}$ are given.

In general, distribution theory for order statistics is complicated when random variables are *innid*. However, *df* and *pdf* of order statistics of *innid* continuous random variables can be obtained easily from the identities in the above theorems.

We now obtain four expressions for *df* in Result 1-2.

In Result 1, joint *df* of $X_{r_1:n},X_{r_2:n},\ldots,X_{r_d:n}$ is expressed as Eq. (11).

Result 1. The above identity can be expressed as

$$F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)$$
$$=\sum_{m_d,\ldots,m_2,\,m_1}^{n,\ldots,m_3,\,m_2}C\,per[\underset{m_1}{\Delta\mathbf{F}(x_1)}\underset{m_2-m_1}{\Delta\mathbf{F}(x_2)}\ldots\underset{n-m_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$=\sum_{m_d,\ldots,m_2,\,m_1}^{n,\ldots,m_3,\,m_2}C\sum_{P}\prod_{w=1}^{d+1}\prod_{l=m_{w-1}+1}^{m_w}\Delta F_{j_l}(x_w)$$
$$=\sum_{m_d,\ldots,m_2,\,m_1}^{n,\ldots,m_3,\,m_2}C\sum_{P}\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\prod_{w=2}^{d+1}\sum_{t=m_{w-1}}^{m_w}(-1)^{m_w-t}$$
$$\cdot\sum_{n_\tau=t-m_{w-1}}\left[\prod_{l=1}^{t-m_{w-1}}F_{\tau_l}(x_w)\right]\prod_{l=1}^{m_w-t}F_{\tau'_l}(x_{w-1})$$
$$=\sum_{m_d,\ldots,m_2,\,m_1}^{n,\ldots,m_3,\,m_2}C\sum_{t_d,\ldots,t_2,\,t_1}^{n,\ldots,m_3,\,m_2}(-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)}\left[\prod_{w=1}^{d}\binom{m_{w+1}-m_w}{t_w-m_w}\right]$$
$$\cdot\sum_{n_s=n-t_d+m_d}(t_d-m_d)\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{d-1}}}\prod_{w=1}^{d}n_{s_w}!\prod_{l=1}^{n_{s_w}}F_{s_w^l}(x_w), \tag{11}$$

where $x_1<x_2<\ldots<x_d$.

Proof. It can be written as

$$F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=$$
$$P\left\{X_{r_1:n}\le x_1,X_{r_2:n}\le x_2,\ldots,X_{r_d:n}\le x_d\right\}.$$

From Eq. (1), Eq. (2) and Eq. (5), Eq. (11) is obtained.

In Result 2, *df* of $X_{r_1:n}$ is obtained from Eq. (11).

Result 2.

$$F_{r_1:n}(x_1)=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}per[\underset{m_1}{\mathbf{F}(x_1)}\underset{n-m_1}{\mathbf{1}-\mathbf{F}(x_1)}]$$
$$=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_{P}\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\prod_{l=m_1+1}^{n}[1-F_{j_l}(x_1)]$$
$$=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_{P}\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\sum_{t=m_1}^{n}(-1)^{n-t}$$
$$\cdot\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}F_{\tau'_l}(x_1)$$
$$=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_{t_1=m_1}^{n}(-1)^{n-t_1}\binom{n-m_1}{t_1-m_1}$$
$$\cdot\sum_{n_s=n-t_1+m_1}(t_1-m_1)!(n-t_1+m_1)!\prod_{l=1}^{n-t_1+m_1}F_{s_1^l}(x_1). \tag{12}$$

Proof. In Eq. (11), if $d=1$, Eq. (12) is obtained.

We now obtain four expressions for *pdf* in Result 3-6.

In Result 3, joint *pdf* of $X_{r_1:n},X_{r_2:n},\ldots,X_{r_d:n}$ is expressed as Eq. (13).

Result 3.

$$f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)$$
$$= D per[\underset{r_1-1}{\Delta\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{r_2-r_1-1}{\Delta\mathbf{F}(x_2)}\underset{1}{\mathbf{f}(x_2)}\ldots\underset{1}{\mathbf{f}(x_d)}\underset{n-r_d}{\Delta\mathbf{F}(x_{d+1})}]$$
$$= D\sum_P(\prod_{w=1}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}\Delta F_{j_l}(x_w))\prod_{w=1}^{d}f_{j_{r_w}}(x_w)$$
$$= D\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right]\left[\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\right.$$
$$\left.\cdot\sum_{n_\tau=t-r_{w-1}}(\prod_{l=1}^{t-r_{w-1}}F_{\tau_l}(x_w))\prod_{l=1}^{r_w-1-t}F_{\tau_l'}(x_{w-1})\right]\prod_{w=1}^{d}f_{j_{r_w}}(x_w)$$
$$= D\sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}$$
$$\cdot\left[\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}\right]\sum_{n_s=n+r_d-t_d}(t_d-r_d)!$$
$$\cdot\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{d-1}}}\prod_{w=1}^{d}\sum_{n_{\varsigma_w}}n_{\varsigma_w}!\left[\prod_{l=1}^{n_{\varsigma_w}}F_{\varsigma_w^l}(x_w)\right]f_{\varsigma_w'^{w}}(x_w), \tag{13}$$

where $x_1 < x_2 < \ldots < x_d$.

Proof. Consider

$$P\{x_1 < X_{r_1:n} \le x_1+\delta x_1, x_2 < X_{r_2:n} \le x_2+\delta x_2, \ldots, x_d < X_{r_d:n} \le x_d+\delta x_d\}.$$

Dividing the above identity by $\prod_{w=1}^{d}\delta x_w$ and then letting $\delta x_1,\delta x_2,\ldots,\delta x_d$ tend to zero, we obtain $f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)$. From Eq. (6), Eq. (7) and Eq. (10), Eq. (13) is obtained.

In Result 4, *pdf* of $X_{r_1:n}$ is obtained from Eq. (13).

Result 4.

$$f_{r_1:n}(x_1) = \frac{1}{(r_1-1)!(n-r_1)!}per[\underset{r_1-1}{\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_1)}\underset{n-r_1}{\mathbf{1}-\mathbf{F}(x_1)}]$$
$$= \frac{1}{(r_1-1)!(n-r_1)!}\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right]$$
$$\cdot\left[\prod_{l=r_1+1}^{n}[1-F_{j_l}(x_1)]\right]f_{j_{r_1}}(x_1)$$
$$= \frac{1}{(r_1-1)!(n-r_1)!}\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right]$$
$$\cdot\left[\sum_{t=r_1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}F_{\tau_l'}(x_1)\right]f_{j_{r_1}}(x_1)$$
$$= \frac{1}{(r_1-1)!(n-r_1)!}\sum_{t_1=r_1}^{n}(-1)^{n-t_1}\binom{n-r_1}{t_1-r_1}\sum_{n_s=n+r_1-t_1}(t_1-r_1)!$$
$$\cdot\sum_{n_{\varsigma_1}=n-t_1+r_1-1}(n-t_1+r_1-1)!\left[\prod_{l=1}^{n-t_1+r_1-1}F_{\varsigma_1^l}(x_1)\right]f_{\varsigma_1'^1}(x_1) \tag{14}$$

Proof. In Eq. (13), if $d=1$, Eq. (14) is obtained.

In Result 5, joint *pdf* of $X_{1:n}$ and $X_{n:n}$ is expressed as Eq. (15).

Result 5.

$$f_{1,n:n}(x_1,x_2) = \frac{1}{(n-2)!}per[\underset{1}{\mathbf{f}(x_1)}\underset{n-2}{\mathbf{F}(x_2)-\mathbf{F}(x_1)}\underset{1}{\mathbf{f}(x_2)}]$$
$$= \frac{1}{(n-2)!}\sum_P\left[\prod_{l=2}^{n-1}(F_{j_l}(x_2)-F_{j_l}(x_1))\right]f_{j_1}(x_1)f_{j_n}(x_2)$$
$$= \frac{1}{(n-2)!}\sum_P\left[\sum_{t=1}^{n-1}(-1)^{n-1-t}\sum_{n_\tau=t-1}\left(\prod_{l=1}^{t-1}F_{\tau_l}(x_2)\right)\right.$$
$$\left.\cdot\prod_{l=1}^{n-1-t}F_{\tau_l'}(x_1)\right]f_{j_1}(x_1)f_{j_n}(x_2)$$
$$= \frac{1}{(n-2)!}\sum_{t_1=1}^{n-1}(-1)^{n-1-t_1}\binom{n-2}{t_1-1}\sum_{n_s=n}\sum_{n_{s_1}=n-t_1}$$
$$\prod_{w=1}^{2}\sum_{n_{\varsigma_w}}n_{\varsigma_w}!\left(\prod_{l=1}^{n_{\varsigma_w}}F_{\varsigma_w^l}(x_w)\right)f_{\varsigma_w'^{w}}(x_w). \tag{15}$$

Proof. In Eq. (13), if $d=2$and $r_1=1$, $r_2=n$, Eq. (15) is obtained.

Proceeding similarly, joint *pdf* of $X_{1:n},X_{2:n},\ldots,X_{k:n}$ can be expressed as Eq. (16).

Result 6.

$$f_{1,2,\ldots,k:n}(x_1,x_2,\ldots,x_k) = \frac{1}{(n-k)!}per[\underset{1}{\mathbf{f}(x_1)}\underset{1}{\mathbf{f}(x_2)}\ldots\underset{1}{\mathbf{f}(x_k)}\underset{n-k}{\mathbf{1}-\mathbf{F}(x_k)}]$$
$$= \frac{1}{(n-k)!}\sum_P\left[\prod_{l=k+1}^{n}(1-F_{j_l}(x_k))\right]f_{j_1}(x_1)f_{j_2}(x_2)\ldots f_{j_k}(x_k)$$
$$= \frac{1}{(n-k)!}\sum_P\left[\sum_{t=k}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}F_{\tau_l'}(x_k)\right]f_{j_1}(x_1)f_{j_2}(x_2)\ldots f_{j_k}(x_k)$$
$$= \frac{1}{(n-k)!}\sum_{t_1=k}^{n}(-1)^{n-t_1}\binom{n-k}{t_1-k}\sum_{n_s=n+k-t_1}(t_1-k)!$$
$$\cdot\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{k-1}}}\prod_{w=1}^{k}\sum_{n_{\varsigma_w}}n_{\varsigma_w}!\left[\prod_{l=1}^{n_{\varsigma_w}}F_{\varsigma_w^l}(x_w)\right]f_{\varsigma_w'^{w}}(x_w). \tag{16}$$

Proof. In Eq. (13), if $d=k$ and $r_1=1$, $r_2=2,\ldots,r_k=k$, Eq. (16) is obtained.

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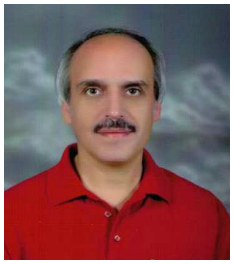

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