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Applications of Fourier Series and Zeta Functions to Genocchi Polynomials

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Abstract: In this paper, we firstly consider the properties of Genocchi polynomials, Fourier series and Zeta functions. In the special cases, we see that the Fourier series yield Zeta functions. From here, we show that zeta functions for some special values can be computed by Genocchi polynomials. Secondly, we consider the Fourier series of periodic Genocchi functions. For odd indexes of Genocchi functions, we construct good links between Genocchi functions and Zeta function. Finally, since Genocchi functions reduce to Genocchi polynomials over the interval $[0,1)$, we see that Zeta functions have integral representations in terms of Genocchi polynomials.

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1 Introduction

Fourier series plays an important role in the principal methods of analysis for mathematical physics, engineering, and signal processing. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis. Hurwitz found the Fourier expansion of the Bernoulli polynomials over a century ago. In general, Fourier analysis can be fruitfully employed to obtain properties of the Bernoulli polynomials and related functions in a simple manner. Very recently, the Fourier series expansions of some special polynomials have been studied in details, see [1]-[8].

The Riemann zeta function is useful in number theory for investigating properties of prime numbers including special function of mathematics and physics that arises in definite integration given by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s \in \mathbb{C}; \Re e(s) > 1) \tag{1}$$

and

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{d}t$$

where $\Gamma(s)$ is Gamma function for $\Re e(s) > 0$ with $s \in \mathbb{C}$ known as

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} \mathrm{d}t \quad \text{(see [11])}.$$

Srivastava [13,14,18] developed the family of rapidly converging series representations for the Riemann Zeta function $\zeta(s)$ at $s = 2n+1$. We also note that the Bernoulli numbers are interpolated by the Riemann zeta function at negative integers, which plays an important role in analytic number theory and has applications in physics, probability theory and applied statistics, as follows:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

where B_n stands for Bernoulli numbers defined by means of the following generating function:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \quad (|t| < 2\pi).$$

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A few Bernoulli numbers are listed below:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\cdots$$

and

$$B_{2n+1} = 0 \text{ for } n \geq 1$$

(see [5],[6],[11], [18], [22] for details).

Euler also gave the following interesting equality

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}}{2(2n)!}B_{2n}, \quad n = 0,1,2,\cdots.$$

From here one may see that Riemann zeta functions at even natural numbers can be computed by using this equality (see [4],[5],[11],[13],[14],[18]), for example,

$$\zeta(0) = -\frac{1}{2},\ \zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\cdots.$$

λ and β functions are also defined, respectively, by

$$\lambda(s) := \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} = \frac{2^s-1}{2^s}\zeta(s) \quad (s \in \mathbb{C}; \Re e(s) > 1) \tag{2}$$

and

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \quad (s \in \mathbb{C}; \Re e(s) > 0) \tag{3}$$

(see [4],[5]).

A further generalization of Riemann zeta function is Hurwitz zeta function given by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (s \in C; \Re e(s) > 1; a \neq 0,-1,-2,\cdots).$$

Obviously that $\zeta(s,1) := \zeta(s)$ (see [11]). Digamma function is also known as

$$\Psi(s) = \frac{d}{ds}\log\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)} \quad (s \in C; \Re e(s) > 0)$$

which has integral and series representations, respectively, as follows:

$$\Psi(s) = \int_0^{\infty}\left(\frac{e^{-x}}{x} + \frac{e^{-sx}}{e^{-x}-1}\right)\mathrm{d}x$$

and

$$\Psi(s) = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+s}\right) \tag{4}$$

where $\gamma := 0,57721...$ is Euler-Mascheroni constant (see [11]).

Apostol Frobenius-Euler polynomials are known as

$$\sum_{n=0}^{\infty} H_n(x,u,\lambda)\frac{t^n}{n!} = \frac{1-u}{\lambda e^t - u}e^{xt} \quad (u \neq 1, \lambda \neq 1, u \neq \lambda)$$

(see [9],[10],[16],[17],[19],[20],[21]). In the special cases, we have

$$H_n(x,-1,1) := E_n(x) = \frac{G_{n+1}(x)}{n+1}$$

where $E_n(x)$ and $G_n(x)$ are called, respectively, Euler polynomials and Genocchi polynomials. These polynomials in the value $x = 0$ reduce to Euler numbers and Genocchi numbers denoted by $E_n(0) := E_n$ ve $G_n(0) := G_n$, *cf.* [8],[22].

Recently, Araci and Acikgoz have derived Fourier expansion of Apostol Frobenius-Euler polynomials in the Laurent series form, as follows:

$$H_n(x,u,\lambda) = \frac{u-1}{u}n!\left(\frac{u}{\lambda}\right)^x \sum_{k\in\mathscr{Z}} a_{k,n}(\lambda,u)z^k \tag{5}$$

where

$$z = e^{2\pi ix} \text{ and } a_{k,n}(\lambda,u) = \frac{1}{\left(2\pi ik - \log\left(\frac{\lambda}{u}\right)\right)^{n+1}} \quad (\text{see [8]}).$$

In the special cases of the parameters u and λ, one can easily derive that

Fourier expansion of Euler polynomials ([3],[8]):

$$H_n(x,-1,1) := E_n(x) = \frac{2n!}{(2\pi i)^{n+1}} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi i\left(k+\frac{1}{2}\right)x}}{\left(k+\frac{1}{2}\right)^{n+1}}.$$

Fourier expansion of Genocchi polynomials ([1],[8]):

$$nH_{n-1}(x,-1,1) := G_n(x) = \frac{2n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi i\left(k+\frac{1}{2}\right)x}}{\left(k+\frac{1}{2}\right)^n}. \tag{6}$$

In this paper, our main focus is going to be Genocchi numbers and polynomials whose history can be traced back to Italian mathematician Angelo Genocchi (1817–1889). From Genocchi's time to the present, Genocchi numbers and polynomials have been extensively studied in many different context in such branches of Mathematics as, for instance, elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions), quantum physics (quantum groups). The works of Genocchi numbers and their combinatorial relations have received much attention [1,3,7,12,15]. For showing the value of this type of numbers and polynomials, we list some properties known in the literature:

–

–Series representations of Genocchi polynomials arising from its Fourier expansion (see [1]):

$$G_{2n}(x) = \frac{4(-1)^n (2n)!}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{\cos((2m+1)\pi x)}{(2m+1)^{2n}}, \quad (7)$$

$$G_{2n+1}(x) = \frac{4(-1)^n (2n+1)!}{\pi^{2n+1}} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x)}{(2m+1)^{2n+1}} \quad (8)$$

– Generating function of Genocchi polynomials (see [7], [15]):

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{xt}.$$

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– A few Genocchi polynomials (see [7],[15]):

$$G_0(x) = 0,\ G_1(x) = 1,\ G_2(x) = 2x-1,\ G_3(x) = 3x^2-3x, \cdots.$$

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–The symmetric property for Genocchi polynomials (see [7],[15]):

$$G_n(1-x) = (-1)^n G_n(x).$$

–

–A recurrence relation for Genocchi polynomials (see [7] ,[15]):

$$(-1)^{n+1} G_n(-x) = G_n(x) - 2nx^{n-1}.$$

The $n-$th periodic Genocchi function may be introduced in the following way:

$$\widetilde{G}_n(x) := G_n(x), (0 \le x < 1) \text{ and } \widetilde{G}_n(x+1) = -\widetilde{G}_n(x) \quad (x \in R)$$

(see [1]). Note that the period of $\widetilde{G}_n$ is 2 since
$$\widetilde{G}_n(x+2) = -\widetilde{G}_n(x+1) = \widetilde{G}_n(x).$$

2 Main Results

Now we are in a position to state and prove the zeta functions and the uniform integral representations for Genocchi polynomials as follows.

Theorem 1. *The following equality holds*

$$\zeta(2n) = \frac{1}{4^{n+1}-4} \frac{(2\pi)^{2n}(-1)^n}{(2n)!} G_{2n}$$

which seems to be a similar formula given by Euler as follows

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}}{2(2n)!} B_{2n}.$$

Proof. Substituting $x=0$ into Eq. (8) yields

$$\frac{\pi^{2n}(-1)^n}{4(2n)!} G_{2n} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = \lambda(2n).$$

By using well-known equality $G_{2n} = 2(1-4^n) B_{2n}$ in [14], we have

$$\lambda(2n) = \frac{\pi^{2n}(-1)^n}{2(2n)!} (1-4^n) B_{2n}. \quad (9)$$

Thus, from ([5]) and (9), we complete the proof.

Theorem 2. *The following equality holds true*

$$\beta(2n+1) = \frac{(-1)^n \pi^{2n+1}}{4(2n+1)!} G_{2n+1}\left(\frac{1}{2}\right).$$

Proof. It is proved by using (7) for the value $x=\frac{1}{2}$. So we omit the proof.

Theorem 2 shows that β functions at odd positive integers can be computed by Genocchi polynomials at $x=\frac{1}{2}$, for example,

$$\beta(1) = \frac{\pi}{4},\ \beta(3) = \frac{\pi^3}{32}, \cdots.$$

Recall from Eq. (8) that for $x \in \mathbb{R}$

$$\tilde{G}_{2n+1}(x) = 4(-1)^n (2n+1)! \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x]}{((2m+1)\pi)^{2n+1}}. \quad (10)$$

When $x>0$ we can derive uniformly convergent series for $\frac{\tilde{G}_{2n+1}(x)}{x}$ from (10) and integration from zero to infinity using the well-known formula in complex analysis

$$\int_0^{\infty} \frac{\sin(ax)}{x} \mathrm{d}x = \frac{\pi}{2} \quad (a>0). \quad (11)$$

Theorem 3. *For $n \in \mathbb{N}$, we have*

$$\zeta(2n+1) = \frac{4^n(-1)^n \pi^{2n}}{(2n+1)!(2^{2n+1}-1)} \int_0^{\infty} \frac{\tilde{G}_{2n+1}(x)}{x} \mathrm{d}x. \quad (12)$$

*Proof.*By (10) and (11), we see that

$$\begin{aligned}\int_0^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x &= 4(-1)^n(2n+1)!\sum_{m=0}^\infty \frac{1}{((2m+1)\pi)^{2n+1}}\int_0^\infty \frac{\sin[(2m+1)\pi x]}{x}\mathrm{d}x \\ &= \frac{4(-1)^n(2n+1)!}{\pi^{2n+1}}\sum_{m=0}^\infty \frac{1}{(2m+1)^{2n+1}}\frac{\pi}{2} \\ &= \frac{2(-1)^n(2n+1)!}{\pi^{2n}}\lambda(2n+1).\end{aligned}$$

which, from (2), gives the following relation:

$$\zeta(2n+1) = \frac{4^n(-1)^n\pi^{2n}}{(2n+1)!\,(2^{2n+1}-1)}\int_0^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x.$$

Thus our assertion is proven.

Theorem 4.*The following equality holds true*

$$\int_0^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x = \int_0^1 G_{2n+1}(x)\left(\frac{1}{x}-\sum_{k=1}^\infty \frac{1}{(x+2k)(x+2k-1)}\right)\mathrm{d}x. \tag{13}$$

*Proof.*From the definition of periodic Genocchi function, we have

$$\begin{aligned}\int_0^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x &= \int_0^1 \frac{G_{2n+1}(x)}{x}\mathrm{d}x + \int_1^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x \\ &= \int_0^1 \frac{G_{2n+1}(x)}{x}\mathrm{d}x + \sum_{k=1}^\infty \int_{2k-1}^{2k+1}\frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x.\end{aligned}$$

Integrating by substitution $x = t+2k$ in the second integral on the right hand side of the above equality yields

$$\int_0^\infty \frac{\tilde{G}_{2n+1}(x)}{x}\mathrm{d}x = \int_0^1 \frac{G_{2n+1}(x)}{x}\mathrm{d}x + \sum_{k=1}^\infty \int_{-1}^1 \frac{\tilde{G}_{2n+1}(t)}{t+2k}\mathrm{d}t. \tag{14}$$

Now it is sufficient to analyse the integral $\int_{-1}^1 \frac{\tilde{G}_{2n+1}(t)}{t+2k}\mathrm{d}t$ as follows:

$$\begin{aligned}\int_{-1}^1 \frac{\widetilde{G}_{2n+1}(t)}{t+2k}\mathrm{d}t &= \int_{-1}^0 \frac{\widetilde{G}_{2n+1}(t)}{t+2k}\mathrm{d}t + \int_0^1 \frac{G_{2n+1}(t)}{t+2k}\mathrm{d}t \\ &= \int_0^1 \frac{G_{2n+1}(u-1)}{u-1+2k}\mathrm{d}u + \int_0^1 \frac{G_{2n+1}(t)}{t+2k}\mathrm{d}t \\ &= -\int_0^1 \frac{G_{2n+1}(u)}{u-1+2k}\mathrm{d}u + \int_0^1 \frac{G_{2n+1}(t)}{t+2k}\mathrm{d}t \\ &= -\int_0^1 \frac{G_{2n+1}(x)}{(x+2k)(x+2k-1)}\mathrm{d}x.\end{aligned}$$

Thus, the proof is completed.

Theorem 5.*For $n \in \mathbb{N}$, we have*

$$\zeta(2n+1) = \frac{4^n(-1)^n\pi^{2n}}{(2n+1)!\,(2^{2n+1}-1)}\int_0^1 G_{2n+1}(x)\left(\frac{1}{x-1}+\Psi\left(\frac{x-1}{2}\right)-\Psi\left(\frac{x}{2}\right)\right)\mathrm{d}x.$$

*Proof.*From (4), we derive

$$\begin{aligned}\Psi\left(\frac{x-1}{2}\right)-\Psi\left(\frac{x}{2}\right) &= \frac{1}{x}-\frac{1}{x-1}+\sum_{k=1}^\infty \frac{1}{x+2k}-\frac{1}{x+2k-1} \\ &= \frac{1}{x}-\frac{1}{x-1}-\sum_{k=1}^\infty \frac{1}{(x+2k)(x+2k-1)}.\end{aligned}$$

One may see that this equality is closely related to Eq. (13). So we derive that

$$\Psi\left(\frac{x-1}{2}\right)-\Psi\left(\frac{x}{2}\right)+\frac{1}{x-1} = \frac{1}{x}-\sum_{k=1}^\infty \frac{1}{(x+2k)(x+2k-1)}.$$

Thus, our assertion follows from Eq.(13) and last equality on the above.

3 Perspective

We have derived new identities related to zeta functions. From these identities, we have computed some values of zeta functions using Genocchi polynomials. After that, we have considered the Fourier series of periodic Genocchi functions. For odd indexes of Genocchi functions, we have constructed some new relations between Genocchi functions and Zeta functions. Since Genocchi functions reduce to Genocchi polynomials over the interval $[0,1)$, we see that Zeta functions for odd positive integers have integral representations in terms of Genocchi polynomials.

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