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A Hermite Method for Maxwell's Equations

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Abstract: A mathematical formulation suitable for the application of a novel Hermite finite element method, to solve electro-magnetic field problems in two- and three-dimensional domains is studied. This approach offers the possibility to generate accurate approximations of Maxwell's equations in smooth domains, with a rather rough interpolation and without curved elements. Method's degrees of freedom are the normal derivative mean values of the electric field across the edges or the faces of a mesh consisting of N-simplices, in addition to the mean value in the mesh elements of the field itself. Second-order convergence of the electric field in the mean-square sense and first-order convergence of the magnetic field in the same sense are rigorously established, if the domain is a convex polytope. Numerical results for two-dimensional problems suggest however that second-order convergence can also be expected of the magnetic field. Both behaviors are shown to apply to the case of curved domains as well, provided a simple interpolated boundary condition technique is employed.

Keywords: Electric field, Finite elements, Hermite, Magnetic field, Maxwell's equations, Raviart-Thomas.

1 Introduction

The classical Lagrange family of finite elements is not suitable for the numerical solution of Maxwell's equations governing electro-magnetic field generation. One of the main reasons for this is the fact that boundary conditions applying to the tangential component of the electric field are not compatible with vertex-based methods. That is why to date well-established techniques to solve this kind of problems proposed several decades ago are widely in use, such as Nédélec edge elements (cf. [17]). Indeed in this kind of method mean values along triangle or tetrahedron edges of the tangential components of the electric field are method's degrees of freedom. This enables enforcement of the above boundary conditions avoiding inconsistencies at corner nodes. Moreover Nédélec elements are curl conforming, which makes them suitable to approximate the fields involved in the equations of electromagnetism. Some practitioners use divergence conforming methods to solve the equations under certain circumstances. One of the methods of the latter type is known in the specialized literature as the Rao-Wilton-Glisson triangular element [16]. It is based on an interpolation of the electric field, which is nothing but the one of the flux variable in the lowest-order Raviart-Thomas mixed element [20]. Whatever the case, this kind of method couples helplessly the components of the electric field, in such a way that algorithms allowing for an uncoupled solution of Maxwell's equations become difficult to implement.

In order to cope with the issues pointed out in the introductory considerations above, we present in this work a finite element method to solve Maxwell's equations in a bounded N-dimensional domain, for $N=2$ and $N=3$. The purpose of this proposal is three-fold. First of all our method takes boundary conditions on the tangential component into account as Neumann boundary conditions in a suitable variational framework. In this manner they are implicit in the formulation and by no means strongly enforced. A second method's interesting feature is the fact that the N components of the unknown fields are equally represented everywhere, which renders it well suited to uncoupling solution algorithms. Finally an important point is the use of a special interpolation to represent the unknown discrete electric field, which happens to be the Hermite counterpart of the Raviart-Thomas mixed finite element method of the lowest order. This allows to recover second-order approximations in the sense of L^2, in contrast to the mixed method, though at practically the same cost. This is because the total number of degrees of freedom is the same for a given mesh. In this respect the authors should clarify beforehand, that, as far as the spatial discretization is concerned, the new method can be viewed as a vector version of the Hermite finite element method introduced

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in [23], for the solution of scalar second-order elliptic boundary value problems. However we should point out that the method studied in this paper is mostly aimed at providing optimal approximations of solutions to Maxwell's equations at reasonable cost, in the case of domains which are at least as regular as convex polytopes. In short, singular behavior of solutions arising in the case of boundary re-entrant corners (see e.g. [9], [10], [18], [19] and references therein) cannot be dealt with by means of our numerical model. Actually our method's main goal is to provide second-order numerical solutions in sufficiently smooth domains, using a rather rough and inexpensive interpolation.
An outline of this paper is as follows. In Section 2 we first recall the equations to solve in the time-dependent case. Then we recast them in the form of a second-order hyperbolic system in terms of the electric-field components. This system is shown to be equivalent to the original system in non singular cases. In the same section we set up the stationary counterpart of the latter system, to be taken as a model problem in the subsequent numerical studies. In Section 3 we describe our numerical formulation of the model problem, together with the underlying Hermite finite-element solution method. In the same section method's numerical analysis is conducted and optimal error estimates are derived for the case of polygonal and polyhedral domains. In Section 4 results of numerical experiments in the two-dimensional case are reported, for both rectangular and smooth curved domains. For the latter case we describe an interpolated boundary condition technique, which prevents the numerical solution from any order erosion, owing to the approximation of the curved boundary by a polygon. We conclude in Section 5 with a few remarks on the extension of our numerical formulation to the case of variable coefficients, among other comments on the whole work.

2 Model equations

Henceforth we consider that the method studied in this work applies to the solution of Maxwell's equations in a bounded domain Ω of $\Re^N$, for $N = 2,3$ in a slightly simplified form. For more details on these equations we refer to [8].
For the purpose of the numerical analysis to be conducted hereafter it is convenient to restrict the presentation to the case where Ω is a polygon for $N = 2$ and a polyhedron for $N = 3$. As pointed out above, in order to avoid singular behaviors, we assume that Ω is convex. Irrespective of convexity, the case of smoother domains will be addressed afterward in the framework of numerical experimentation.
Letting Γ be the boundary of Ω, we denote by $\mathbf{n}$ the unit outer normal vector defined everywhere on Γ except at its vertices. Then given an electric field $\mathbf{e}_0$ and a solenoidal magnetic field $\mathbf{m}_0$ at the initial time $t = 0$, and $\gamma \in \Re^+$, we wish to determine the couple of electric and magnetic fields $(\mathbf{e},\mathbf{m})$ depending on the space variable $\mathbf{x}$ and time $t \in (0,T]$, where T is a final time, such that,

$$\begin{cases} \text{Given } \rho \in H^1(\Omega), \mathbf{j} \in \{H^1[(0,T);L^2(\Omega)]\}^N, \\ (\mathbf{e},\mathbf{m}) \text{ fulfills:} \\ div\, \mathbf{e} = \rho \text{ and } div\, \mathbf{m} = 0 \text{ in } \Omega \times (0,T], \\ \dfrac{\partial \mathbf{m}}{\partial t} + \gamma\, curl\, \mathbf{e} = \mathbf{0} \text{ in } \Omega \times (0,T], \\ \dfrac{\partial \mathbf{e}}{\partial t} - \gamma\, curl\, \mathbf{m} = -\mathbf{j} \text{ in } \Omega \times (0,T], \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \times (0,T]. \end{cases} \quad (1)$$

The assumption that the electric charge density ρ does not depend on t requires that the electric current density $\mathbf{j}$ be solenoidal, which is the main particularity of equation (1) as compared to the general form of Maxwell's equations. Notice that this assumption also requires that $div\, \mathbf{e}_0 = \rho$ in Ω
Taking into account that $\partial curl\, \mathbf{m}/\partial t = -\gamma\, curl\, curl\, \mathbf{e}$, setting $\lambda = \gamma^2$, we can manipulate system (1) to obtain :

$$\begin{cases} \text{Given } \rho \in H^1(\Omega),\ \mathbf{j} \in \{H^1[(0,T);L^2(\Omega)]\}^N \\ \text{with } div\, \mathbf{j}_{|t=0},\ (\mathbf{e},\mathbf{m}) \text{ fulfills:} \\ div\, \mathbf{e} = \rho \text{ in } \Omega \times (0,T], \\ \dfrac{\partial \mathbf{m}}{\partial t} = -\gamma\, curl\, \mathbf{e} \text{ in } \Omega \times (0,T], \\ \dfrac{\partial^2 \mathbf{e}}{\partial t^2} + \lambda\, curl\, curl\, \mathbf{e} = -\dfrac{\partial \mathbf{j}}{\partial t} \text{ in } \Omega \times (0,T], \\ \mathbf{e}(\cdot,0) = \mathbf{e}_0(\cdot),\ \dfrac{\partial \mathbf{e}}{\partial t}(\cdot,0) = \gamma\, curl\, \mathbf{m}_0 - \mathbf{j}_{|t=0} \\ \text{and } \mathbf{m}(\cdot,0) = \mathbf{m}_0(\cdot) \text{ in } \Omega, \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \times (0,T] \end{cases} \quad (2)$$

Notice that the assumption that $div\, \mathbf{j} = 0$ at $t = 0$, together with the third equation of (2) suffice to guarantee that $div\, \mathbf{j} = 0$ for every t. Otherwise stated the initial condition on $div\, \mathbf{j}$ is equivalent to requiring that $div\, \mathbf{j} = 0$ for all t.
Now we use the well-knon identity applying to the laplacian of a vector field, namely,

$$-\Delta \mathbf{e} = curl\, curl\, \mathbf{e} - grad\, div\, \mathbf{e}.$$

Plugging this relation into (2) and observing that $div\,\mathbf{e} = \rho$ on Γ, we rewrite (2) as follows:

$$\left\{\begin{array}{l} \text{Given } \rho \in H^1(\Omega),\ \mathbf{j} \in \{H^1[(0,T);L^2(\Omega)]\}^N \\ \text{with } div\,\mathbf{j} = 0\ \forall t,\ (\mathbf{e},\mathbf{m}) \text{ fulfills:} \\ \dfrac{\partial \mathbf{m}}{\partial t} = -\gamma\, curl\,\mathbf{e} \text{ in } \Omega \times (0,T], \\ \dfrac{\partial^2 \mathbf{e}}{\partial t^2} - \lambda \Delta \mathbf{e} = -\lambda grad\,\rho - \dfrac{\partial \mathbf{j}}{\partial t} \text{ in } \Omega \times (0,T], \\ \mathbf{e}(\cdot,0) = \mathbf{e}_0(\cdot),\ \dfrac{\partial \mathbf{e}}{\partial t}(\cdot,0) = \gamma\, curl\,\mathbf{m}_0 - \mathbf{j}_{|t=0} \\ \text{and } \mathbf{m}(\cdot,0) = \mathbf{m}_0(\cdot) \text{ in } \Omega, \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \times (0,T] \\ div\,\mathbf{e} = \rho \text{ on } \Gamma \times (0,T]. \end{array}\right. \quad (3)$$

Conversely, it is easy to see that if the pair $(\mathbf{e},\mathbf{m})$ fulfills (3) then it also satisfies (2). Indeed, taking the divergence of both sides of the second equation of (3) and denoting by d the divergence of $\mathbf{e}$, d is seen to satisfy the equation

$$\left\{\begin{array}{l} \text{Given } \rho \in H^1(\Omega), \text{ find } d \text{ such that:} \\ \dfrac{\partial^2 d}{\partial t^2} - \lambda \Delta d = -\lambda \Delta \rho \text{ in } \Omega \times (0,T], \\ d(\cdot,0) = \rho \text{ and } \dfrac{\partial d}{\partial t}(\cdot,0) = 0 \text{ in } \Omega, \\ d = \rho \text{ on } \Gamma \times (0,T]. \end{array}\right. \quad (4)$$

Obviously enough hyperbolic equation (4) has a unique solution given by $d = \rho$. It follows that $\lambda(-\Delta \mathbf{e} + grad\,\rho) = \lambda curl\ curl\ \mathbf{e}$ and we are done.

An interesting characteristic of system (3) is the fact that the N components of $\mathbf{e}$ are coupled only due to the boundary conditions. Hence, as long as we can deal with them in an uncoupled manner we can easily solve system (3) component-by-component. This is precisely one of the main features of the new numerical method to solve (1) - i.e. (2) - in the equivalent form (3), which we describe in the next section. Before doing so we introduce a vector Poisson problem, which is a sort of stationary counterpart of (3) for $\lambda = 1$:

$$\left\{\begin{array}{l} \text{Given } \mathbf{g} \in \mathbf{H}(curl,\Omega) \\ \text{and } \rho \in H^1(\Omega), \\ \text{find } \mathbf{e} \in [H^1(\Omega)]^N \text{ such that:} \\ -\Delta \mathbf{e} = curl\,\mathbf{g} - grad\,\rho \text{ in } \Omega, \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \\ div\,\mathbf{e} = \rho \text{ on } \Gamma. \end{array}\right. \quad (5)$$

Taking the divergence of the fist equation of (5), in the same way as for system (3), we conclude that $div\,\mathbf{e} = \rho$ a.e. in Ω. This implies that any solution of (5) solves the system

$$\left\{\begin{array}{l} \text{Given } \mathbf{j} \in \mathbf{H}(curl,\Omega) \cap Ker(div) \\ \text{and } \rho \in H^1(\Omega), \\ \text{find } \mathbf{e} \in [H^1(\Omega)]^N \text{ such that:} \\ curl\,\mathbf{e} = \mathbf{j} \text{ in } \Omega, \\ div\,\mathbf{e} = \rho \text{ in } \Omega, \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \end{array}\right. \quad (6)$$

It is possible to prove that $\mathbf{j} = \mathbf{g} + grad\ \eta$ where $\eta \in H^1(\Omega)$ is any solution of the Neumann problem $-\Delta\eta = div\,\mathbf{g}$ in Ω and $\partial\eta/\partial n = -\mathbf{g}\cdot\mathbf{n}$ on Γ.

We next introduce and study our finite element method applied to model problem (5). Its application to (3) is straightforward, as long as a suitable time-marching scheme is implemented to complete the underlying space discretization.

In the sequel we employ the following notations: For a strictly positive integer M, S being a bounded open set of $\Re^N$, $(\cdot|\cdot)_S$ denotes the standard inner product of $[L^2(S)]^M$. We denote the standard norm and semi-norm of Sobolev space $[H^m(S)]^M$ (cf. [1]), for any positive integer m by $\|\cdot\|_{m,S}$ and $|\cdot|_{m,S}$ respectively, including $[L^2(S)]^M$ taking $m = 0$. We drop the subscript S in all the above notations in case $S = \Omega$.

3 A Hermite solution method

We are given a finite element partition $\mathscr{T}_h$ of Ω, consisting of triangles or tetrahedra according to the value of N, and belonging to a regular family of partitions (cf. [7]). h will denote the maximum diameter of the elements of $\mathscr{T}_h$. We define a finite element space E_h associated with $\mathscr{T}_h$ as follows. Every function $e \in E_h$ is such that in each element $T \in \mathscr{T}_h$ it is expressed by $a\,\mathbf{x}\cdot\mathbf{x}/2 + \mathbf{b}\cdot\mathbf{x} + c$, where $\mathbf{x}$ represents the space variable, $\mathbf{b}$ is a constant vector of $\Re^N$ and a and c are two real constants. Now F being an edge if $N = 2$ or a face if $N = 3$ belonging to the boundary ∂T of an element $T \in \mathscr{T}_h$, and $\mathbf{n}_F$ being the unit normal vector on F oriented in a unique manner for the whole mesh, every function $e \in E_h$ is such that its restriction to any $T \in \mathscr{T}_h$ is defined by means of the following $N+1$ degrees of freedom:

1. The N mean values $\nu_F(e)$ of the the flux $grad\ e \cdot \mathbf{n}_F$ over $F \subset \partial T$;
2. The mean value $\mu_T(e)$ of e over T.

The degrees of freedom $\nu_F(\cdot)$ are required to coincide on both sides of every face F common to two elements of $\mathscr{T}_h$.

The canonical basis functions for this space corresponding to the above degrees of freedom can be determined as follows. First we note that $\forall e \in E_h$ the fluxes of $e_{/T}$ are constant over every edge or face F of any element T of the mesh. Indeed from the particular form of $e_{/T}$ we have $grad\ e_{/T} = a\mathbf{x} + \mathbf{b}$. Then from a

well-known property of the lowest order Raviart-Thomas mixed element, whose flux variable is locally defined by functions of the same form, the result follows. Incidentally this allows us to the determine a and $\mathbf{b}$ for each basis function φ corresponding to a given flux, in the same way as for the flux basis fields corresponding to the lowest order Raviart-Thomas element. Then the value of c is adjusted in such a way that $\int_T \varphi_{/T} dx = 0$, and the flux canonical basis fields are uniquely defined. Finally the basis function corresponding to degrees of freedom μ_T are given by $a = 0$, $\mathbf{b} = \mathbf{0}$ and $c = 1$.
Now let $\mathbf{E}_h$ be the space of N-component vector fields $\mathbf{e} = (e_1, \ldots, e_N)$ such that $e_i \in E_h$ for $i = 1, \ldots, N$ and $\mathbf{D}_h$ be the subspace of $\mathbf{E}_h$ of those fields $\mathbf{d}$ that fulfill $\nu_F(\mathbf{d} \cdot \mathbf{n}_F) = 0$ for every edge or face F of $\mathscr{T}_h$ contained in Γ. We further define a linear manifold $\mathbf{D}_h^\rho$ of $\mathbf{E}_h$ consisting of fields $\mathbf{d}$ such that $\nu_F(\mathbf{d} \cdot \mathbf{n}_F) = [meas(F)]^{-1} \int_F \rho dF$ for every edge or face F of $\mathscr{T}_h$ contained in Γ.
Next we set up the discrete variational problem (7) aimed at approximating (5), namely,

$$\begin{cases} \text{Find } \mathbf{e}_h \in \mathbf{D}_h^\rho \text{ such that} \\ a_h(\mathbf{e}_h, \mathbf{d}) = L(\mathbf{d}) \qquad \forall \mathbf{d} \in \mathbf{D}_h, \\ \text{where:} \\ a_h(\mathbf{e}, \mathbf{d}) := \sum_{T \in \mathscr{T}_h} [(grad\ \mathbf{e} | grad\ \mathbf{d})_T + \\ \qquad (\Delta \mathbf{e} | \mathbf{d})_T + (\mathbf{e} | \Delta \mathbf{d})_T]\ \forall \mathbf{e}, \mathbf{d} \in \mathbf{E}_h; \\ \text{and} \\ L(\mathbf{d}) := -(\mathbf{f} | \mathbf{d}) \qquad \forall \mathbf{d} \in [L^2(\Omega)]^N, \\ \text{with } \mathbf{f} := curl\ \mathbf{g} - grad\ \rho. \end{cases} \tag{7}$$

Defining $\Pi_h : [L^2(\Omega)]^N \longrightarrow [L^2(\Omega)]^N$ as the operator given by $\Pi_h[\mathbf{w}]_{/T} := \int_T \mathbf{w} dx / meas(T)\ \forall T \in \mathscr{T}_h$, and setting

$$\mathbf{E} := \{\mathbf{e} \,/\, \mathbf{e} \in [H^1(\Omega)]^N ; \Delta \mathbf{e} \in [L^2(\Omega)]^N\},$$

we extend a_h to $(\mathbf{E}_h + \mathbf{E}) \times (\mathbf{E}_h + \mathbf{E})$, and further introduce the functional $\| \cdot \|_h$: $\mathbf{E}_h + \mathbf{E} \longrightarrow \Re$ given by:

$$\| \mathbf{e} \|_h^2 := (\Pi_h[\mathbf{e}] | \Pi_h[\mathbf{e}]) + \sum_{T \in \mathscr{T}_h} [(grad\ \mathbf{e} | grad\ \mathbf{e})_T + (\Delta \mathbf{e} | \Delta \mathbf{e})_T]. \tag{8}$$

The expression $\| \cdot \|_h$ obviously defines a norm over $\mathbf{E} + \mathbf{E}_h$.

Using arguments in all similar to those in [23] applying to the scalar Poisson equation, we can prove existence and uniqueness results, together with a priori error estimates for problem (7). The main difference relies on the fact that we must extend to (5) the theoretical background in [6], as applied to the discrete mixed formulation of the (scalar) Poisson equation. Let us work this out.

*Remark.*In the two-dimensional case we could as well adapt to the our vector problem the technique employed in [20], [21] for its scalar analog. Essentially the only tool that is lacking for such a purpose is a vector counterpart of the discrete Friedrichs-Poincaré inequality (6.18) of [21] for the space of weakly continuous piecewise linear functions used in the primal hybrid finite element method. Actually we supply in Appendix 1 the proof of such an inequality.

To begin with we prove

Lemma 1.*Let $\mathbf{S}_h$ be the space of fields which are constant in every $T \in \mathscr{T}_h$. Referring to [20] we denote by $\mathscr{R}_h$ the subspace of $[\mathbf{H}(div, \Omega)]^N$ consisting of $N \times N$ tensors $Q = \{q_{i,j}\}$ for $1 \leq i,j \leq N$ such that $[q_{i,j}]_{|T}$ is a function of the form $a\mathbf{x} + \mathbf{b}$, with $a \in \Re$ and $\mathbf{b} \in \Re^N$. Let also $\mathscr{Q}_h$ be the subspace of $\mathscr{R}_h$ consisting of those fields Q such that $(Q\mathbf{n}) \cdot \mathbf{n} = 0$ on Γ. Then given $\mathbf{s} \in \mathbf{S}_h$ there exists $Q \in \mathscr{Q}_h$ satisfying for a constant C_S independent of h and $\mathbf{s}$ such that:*

$$\begin{cases} \mathbf{div}\ Q = \mathbf{s}\ a.e.\ in\ \Omega, \\ \{\| Q \|_0^2 + \| \mathbf{div}\ Q \|_0^2\}^{1/2} \leq C_S \| \mathbf{s} \|_0, \end{cases} \tag{9}$$

where $\mathbf{div}\ Q$ stands for the N-component vector field $\mathbf{d} = [d_1, \ldots, d_N]^{\mathscr{T}}$ given by $d_i = \sum_{j=1}^N \frac{\partial Q_{ij}}{\partial x_j}$.

PROOF. The subset of $\mathscr{Q}_h$ consisting of those tensors Q such that $(\mathbf{div}\ Q | \mathbf{s}) = 0\ \forall \mathbf{s} \in \mathbf{S}_h$ is contained in $[\mathbf{H}(div, \Omega) \cap Ker(div)]^N$. Thus we can resort to Proposition II.2.8 of [6], combined with relations analogous to IV.1.24 in the same reference that apply to our vector case, taking $k = 1$. This is because the following *inf-sup* condition [6] holds:

$$\exists \beta > 0 \text{ such that } \forall \mathbf{s} \in [L^2(\Omega)]^N \\ \sup_{Q \in \mathscr{Q} \setminus \{\mathscr{O}\}} \frac{(\mathbf{div}\ Q | \mathbf{s})}{[\| Q \|_0^2 + \| \mathbf{div}\ Q \|_0^2]^{1/2}} \geq \beta \| \mathbf{s} \|_0. \tag{10}$$

(10) can be established by taking $Q = -grad\ \mathbf{u}$, where $\mathbf{u}$ is the solution of the vector Poisson problem.

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{s} \in [L^2(\Omega)]^N \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ and } div\ \mathbf{u} = 0 \text{ on } \Gamma. \end{cases} \tag{11}$$

Indeed since Ω is convex $\mathbf{u} \in [H^1(\Omega)]^N$ [9]. Moreover the following Friedrichs-Poincaré inequality holds for a constant C_N (cf. [11]):

$$\| \mathbf{v} \|_0 \leq C_N \| grad\ \mathbf{v} \|_0\ \forall \mathbf{v} \in \mathbf{V}$$

where

$$\mathbf{V} := \{\mathbf{v} | \ \mathbf{v} \in [H^1(\Omega)]^N,\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

Therefore $(-\Delta \mathbf{u} | \mathbf{u}) \leq C_N \| \mathbf{s} \|_0 \| grad\ \mathbf{u} \|_0$ and since $\partial[\mathbf{u} \cdot \mathbf{n}] / \partial n = 0$ on plane faces of Γ, a classical density argument yields $\| \, grad \, \mathbf{u} \, \|_0 \leq C_N \, \| \, \mathbf{s} \, \|_0$. It follows that (10) holds with $\beta = [1 + C_N^2]^{-1/2}$. ■

Now we can establish that

Proposition 1*Problem (7) has a unique solution.*

PROOF. Thanks to Lemma 1 the proof of this result is based on exactly the same arguments as in Proposition 2.1 of [23], in the particular case of the scalar Poisson equation (i.e. in case $\mathcal{K}$ is the identity tensor). ■

Next we prove

Theorem 2.*Provided $\mathbf{e} \in [H^2(\Omega)]^N$ and $\mathbf{f} \in [H^1(\Omega)]^N$ there exists a mesh independent constant C such that,*

$$\| \, \mathbf{e} - \mathbf{e}_h \, \|_h \leq Ch \, [|\mathbf{e}|_2 + |\mathbf{f}|_1]. \tag{12}$$

PROOF. The proof of this error estimate is entirely analogous to the one of Theorem 2.2 of [23] in the case of the scalar Poisson equation. ■

To close this section we derive a second-order error estimate for our method in the norm of $L^2(\Omega)$. This estimate applies to the case where the solution to (11) belongs to $H^2(\Omega)]^N \cap \mathbf{V}$ for every $\mathbf{s} \in [L^2(\Omega)]^N$. To the best of our knowledge, up to now it has not been formally demonstrated that such a regularity applies to arbitrary convex polygons or polyhedra. Notice that this is known to be true for domains of the class $C^{2,1}$ [2], and eventually to some domains of the $C^{1,1}$-class. Let us then simply consider that our estimate holds for polytopes belonging to a class of domains denoted by $\mathcal{P}$ such that the H^2-regularity does apply, and moreover the following inequality holds for a constant C_P independent of $\mathbf{s}$:

$$|\mathbf{u}|_2 \leq C_P \, \| \, \mathbf{s} \, \|_0 \, . \tag{13}$$

Among domains of the $\mathcal{P}$-class lie rectangular ones, according to,

Proposition 3*Let Ω be a rectangle if $N = 2$ or a rectangular parallelepiped if $N = 3$. Then for every $\mathbf{s} \in [L^2(\Omega)]^N$ the solution of (11) belongs to $\mathbf{U}$ where*

$$\mathbf{U} := \{\mathbf{u} | \ \mathbf{u} \in [H^2(\Omega)]^N \cap \mathbf{V} \textit{ and } div \, \mathbf{u} \in H_0^1(\Omega)\}.$$

Moreover $|\mathbf{u}|_2 = \| \, \mathbf{s} \, \|_0$.

PROOF. First we note that for a rectangular domain $\Omega = (0,L_1) \times \ldots \times (0,L_N)$, problem (11) reduces to N uncoupled Poisson equations $-\Delta u_i = s_i \in L^2(\Omega)$ with mixed Dirichlet-Neumann boundary conditions for the components u_i and s_i of $\mathbf{u}$ and $\mathbf{s}$, $i = 1,\ldots,N$. For example for u_1 we have $u_1 = 0$ on the edges or faces given by $x_i = 0$ and $x_i = L_i$ with $i \neq 1$, and $\partial u_1/\partial n = 0$ on the faces given by $x_1 = 0$ and $x_1 = L_1$. Obviously enough, by symmetry the analogous conditions that hold for the other components of $\mathbf{u}$ can be written down by simple permutation of coordinate subscripts. In any case, at least for $N = 2$, we can resort to results for the Poisson equation in [14] among other works. Nevertheless, since the existence of a solution of the Poisson equation satisfying these boundary conditions which do not belong to $H^2(\Omega)$ is not explicitly ruled out in previous studies, we give in Appendix 2 a proof of the fact that in this specific case $u_i \in H^2(\Omega)$ for $i = 1,\ldots,N$, $N = 2$ or 3. As a by-product we also prove that (13) holds, for $C_P = 1$, with an equality instead of an inequality. ■

Finally we prove,

Theorem 4.*Let Ω be a polytope of the $\mathcal{P}$-class. Provided h is sufficiently small and assuming that $\mathbf{e} \in [H^2(\Omega)]^N$ and $\mathbf{f} \in [H^1(\Omega)]^N$, there exists a mesh independent constant C_0 such that,*

$$\| \, \mathbf{e} - \mathbf{e}_h \, \|_0 \leq C_0 h^2 \, [|\mathbf{e}|_2 + |\mathbf{f}|_1]. \tag{14}$$

The proof of estimate (14) is quite similar to the one of Theorem 2.2 of [23], taking the isotropic case, i.e. the case of the scalar Poisson equation. That is why we highlight only the differences inherent to problem (5).
Similarly to [23] we apply the usual Aubin-Nitsche argument (cf. [7]). Recalling the above defined space $\mathbf{U}$ we have:

$$\| \, \mathbf{e} - \mathbf{e}_h \, \|_0 = \sup_{\mathbf{u} \in \mathbf{U} \setminus \{\mathbf{0}\}} \frac{-(\mathbf{e} - \mathbf{e}_h, \Delta \mathbf{u})}{\| \, \Delta \mathbf{u} \, \|_0}. \tag{15}$$

Next denoting by $\mathbf{n}_T$ the unit outer normal vector on ∂T for $T \in \mathcal{T}_h$, and representing by F a generic edge of ∂T for $N = 2$ or a generic face of ∂T for $N = 3$, we observe that $\forall \mathbf{u} \in \mathbf{U}$,

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \{ & (grad \, [\mathbf{e}_h - \mathbf{e}], grad \, \mathbf{u})_T + (\Delta [\mathbf{e}_h - \mathbf{e}], \mathbf{u})_T \\ & - \sum_{F \subset \partial T} (grad [\mathbf{e}_h - \mathbf{e}] \mathbf{n}_T | \mathbf{u})_F \} = 0. \end{aligned} \tag{16}$$

Owing to the trace properties of $\mathbf{e}$ and $\mathbf{u}$, together with the construction of $\mathbf{E}_h$, we can assert that the summation of all the integrals on $F \not\subset \Gamma$ in (16) cancel out. As for the integrals on $F \subset \Gamma$ they reduce to $(\{grad[\mathbf{e}_h - \mathbf{e}]\mathbf{n}\} \cdot \mathbf{n} | \mathbf{u} \cdot \mathbf{n})$. Recalling that $\mathbf{e} \times \mathbf{n} = \mathbf{0}$ on Γ, we infer that for a polygon or a polyhedron $[grad \, \mathbf{e} \, \mathbf{n}] \cdot \mathbf{n} = div \, \mathbf{e}$ on $F \subset \Gamma$. On the other hand by construction $[grad \, \mathbf{e}_h \, \mathbf{n}] \cdot \mathbf{n} = 0$ on every $F \subset \Gamma$. Since we are considering the case where $div \, \mathbf{e} = 0$ on Γ, it follows that the whole summations over F and T in (16) vanish. Thus we readily derive,

$$-(\mathbf{e} - \mathbf{e}_h, \Delta \mathbf{u})_0 = a_h(\mathbf{e}_h - \mathbf{e}, \mathbf{u}) \ \forall \mathbf{u} \in \mathbf{U}. \tag{17}$$

By a straightforward calculation using the properties of $\mathbf{E}_h$ we derive $a_h(\mathbf{e}, \mathbf{d}) = L(\mathbf{d}) = a_h(\mathbf{e}_h, \mathbf{d}) \ \forall \mathbf{d} \in \mathbf{E}_h$. Hence combining this with (15), (16) and (17) we obtain:

$$\| \, \mathbf{e}_h - \mathbf{e} \, \|_0 \leq \sup_{\mathbf{u} \in \mathbf{U} \setminus \{\mathbf{0}\}} \frac{a_h(\mathbf{e}_h - \mathbf{e}, \mathbf{u} - \mathbf{d})}{\| \, \Delta \mathbf{u} \, \|_0} \ \forall \mathbf{d} \in \mathbf{E}_h. \tag{18}$$

If $\mathbf{e} = \mathbf{e}_h$ (14) trivially holds. Let then $\mathbf{e} \neq \mathbf{e}_h$ and $B_U(\mathbf{0},1) := \{\mathbf{u} / \ \mathbf{u} \in \mathbf{U}, \ \| \, \Delta \mathbf{u} \, \|_0 = 1\}$. In this case it is easy to see that $\exists \mathbf{u}_0 \in B_U(\mathbf{0},1)$ such that

$$\sup_{\mathbf{u}\in\mathbf{U}\setminus\{\mathbf{0}\}} \frac{a_h(\mathbf{e}_h-\mathbf{e},\mathbf{u}-\mathbf{d})}{\parallel \Delta\mathbf{u}\parallel_0} = a_h(\mathbf{e}_h-\mathbf{e},\mathbf{u}_0-\mathbf{d})\ \forall \mathbf{d}\in\mathbf{E}_h.$$

Indeed we may take for instance $\mathbf{u}_0$ such that $\Delta\mathbf{u}_0 = [\mathbf{e}_h - \mathbf{e}]/\parallel \mathbf{e}_h - \mathbf{e}\parallel_0$, since $a_h(\mathbf{e}_h-\mathbf{e},\mathbf{u}) = (\mathbf{e}_h-\mathbf{e},\Delta\mathbf{u})\ \forall \mathbf{u}\in\mathbf{U}$.
Let now $I_h\mathbf{u}$ denote the standard $\mathbf{E}_h$-interpolate of $\mathbf{u}\in\mathbf{U}$. According to standard results (cf. [7] and [20]) there exists a constant C_I such that,

$$\parallel [I_h\mathbf{u}]_{|T}\parallel_{2,T} \leq C_I \parallel \mathbf{u}_{|T}\parallel_{2,T}\ \ \forall \in H^2(\Omega) \text{ and } \forall T\in\mathscr{T}_h. \tag{19}$$

Since the space $\mathbf{W} := [H^3(\Omega)]^N \cap \mathbf{U}$ is dense in $\mathbf{U}$, setting $\varepsilon = 2\sqrt{2N(1+C_I^2)}$, there exists $\mathbf{u}_\varepsilon \in \mathbf{W}$ such that $\parallel \mathbf{u}_0 - \mathbf{u}_\varepsilon\parallel_2 < \varepsilon/2$.
Taking $\mathbf{d} = I_h\mathbf{u}_0$, owing to (12) and the interpolation theory [7], there exist C_5 and C_6 such that,

$$\begin{aligned} a_h(\mathbf{e}_h-\mathbf{e},\mathbf{u}_0-I_h\mathbf{u}_0) \leq & \\ \sum_{T\in\mathscr{T}_h} (\mathbf{e}_h-\mathbf{e},\Delta[\mathbf{u}_0-I_h\mathbf{u}_0\}])_T & \\ +C_P(C_5|u|_{2,\Omega}+C_6h|f|_{1,\Omega})h^2, & \end{aligned} \tag{20}$$

where C_P is the constant of (13). On the other hand we have

$$\begin{aligned} &\sum_{T\in\mathscr{T}_h} (\mathbf{e}_h-\mathbf{e},\Delta[\mathbf{u}_0-I_h\mathbf{u}_0\}])_T \leq \\ &\parallel \mathbf{e}_h-\mathbf{e}\parallel_0 \left\{ \left[\sum_{T\in\mathscr{T}_h} \parallel \Delta(\mathbf{w}_1+\mathbf{w}_3)]\parallel_{0,T}^2\right]^{1/2} \right. \\ &\left. + \left[\sum_{T\in\mathscr{T}_h} \parallel \Delta\mathbf{w}_2]\parallel_{0,T}^2\right]^{1/2} \right\}, \end{aligned} \tag{21}$$

where $\mathbf{w}_1 = \mathbf{u}_0 - \mathbf{u}_\varepsilon$;
$\mathbf{w}_2 = \mathbf{u}_\varepsilon - I_h\mathbf{u}_\varepsilon$;
$\mathbf{w}_3 = I_h(\mathbf{u}_\varepsilon - \mathbf{u}_0)$.

Owing to (19) and to the basic property of $\mathbf{u}_\varepsilon$, it is then clear that

$$\left\{\sum_{T\in\mathscr{T}_h} \parallel \Delta[\mathbf{w}_1+\mathbf{w}_3)]\parallel_{0,T}^2\right\}^{1/2} \leq \sqrt{2N(C_I^2+1)}\varepsilon/2 \tag{22}$$

Moreover since $\mathbf{u}_\varepsilon$ is sufficiently smooth, there must exist h_0 such that $\forall h \leq h_0$ it holds that

$$\left\{\sum_{T\in\mathscr{T}_h} \parallel \Delta\mathbf{w}_2]\parallel_{0,T}^2\right\}^{1/2} \leq \sqrt{2N(C_I^2+1)}\varepsilon/2 \tag{23}$$

Finally taking into account the choice of ε together with (20)-(21)-(22)-(23), it follows that $\forall h \leq h_0$, $\parallel \mathbf{e}-\mathbf{e}_h\parallel_0 \leq 2C_P(C_5|\mathbf{e}|_2 + C_6h_0|\mathbf{f}|_1)h^2$, which proves the Theorem. ■

4 Numerical validation

In this section we assess our method's accuracy by solving some academic test-problems. The subscript h next to a differential operator indicates that it is defined on an element-by-element basis, while G_T denotes the centroid of a mesh element T.

4.1 Test-problems for a square domain

In order to validate the a priori error estimates (12) and (14) we approximated problem (5) in the square $\Omega = (1,0)\times(0,1)$ by (7), using meshes with $2K^2$ triangles, obtained from a first partition of Ω into K^2 equal squares, each one of them being subdivided into two triangles by the diagonal parallel to the line $x_1 = x_2$.

Test-problem 1: Here ρ vanishes identically in Ω and $\mathbf{f} = 2(cos[\pi x_1]sin[\pi x_2], -cos[\pi x_2]sin[\pi x_1])$. The exact solution is $\mathbf{e} = \mathbf{f}/(2\pi^2)$. In Sub-figures 4.1.1 and 4.1.2 of Figure 4.1 we display the resulting errors in the indicated senses, for $K = 2^m$, $m = 1,2,\ldots,5$.

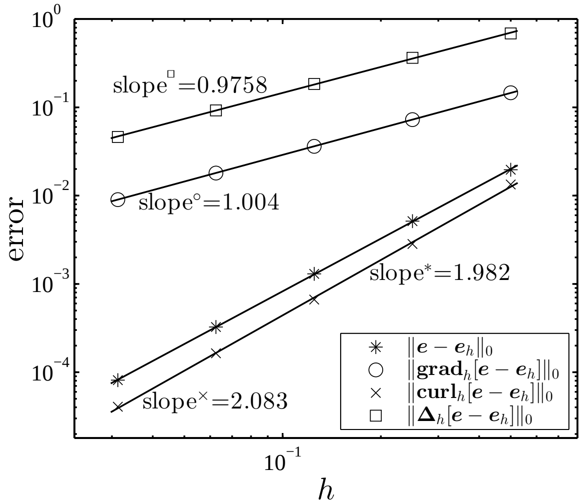
Sub-figure 4.1.1 confirms the predicted orders of convergence, while Sub-figure 4.1.2 shows that the errors of method's degrees of freedom of both types decrease roughly at a quadratic rate as the mesh is refined. Particularly noteworthy is the observation that the error of *curl* $\mathbf{e}$ is also diminishing at the same rate as the error of $\mathbf{e}$. The laplacian of the solution is also being approximated at the same rate in the point-wise sense.

Test-problem 2: Take $\mathbf{f} \equiv -grad\ \rho$ where ρ is given by $\rho = 2(x_1^2+x_2^2-x_1-x_2)$. The exact solution is $\mathbf{e} = ([1-2x_1][x_2-x_2^2],[1-2x_2][x_1-x_1^2])$. Instead of the theoretical mean value of ρ at boundary edges we prescribe the value of ρ at those edge mid-points. In Figure 4.2 we display the resulting errors in the same senses as in the previous test-problem, for increasing values of K.

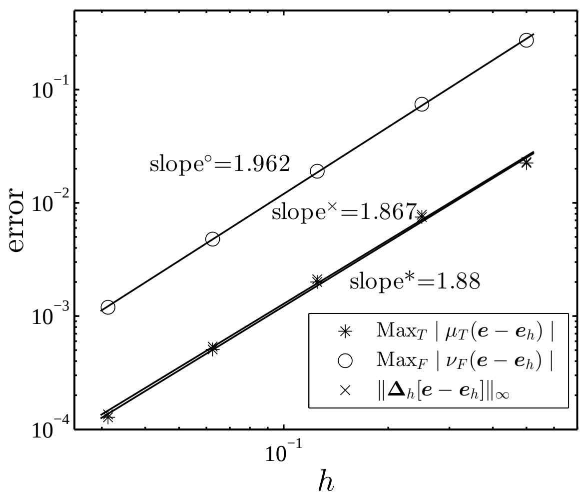
Sub-figures 4.2.1 and 4.2.2 of Figure 4.2 show that in the presence on inhomogeneous (Dirichlet) flux boundary conditions prescribed pointwise, practically the same behavior as in Test-problem 1 is observed. Notice that here the laplacian of the solution at element centroids is exact up to machine precision, but probably this is only because it is a linear field in this test case.

4.2 Test-problems for curved domains

The purpose of the experiments reported in this sub-section is to show that, provided some simple modifications are implemented, the method studied in this work together with its qualitative properties, extend to the case of sufficiently smooth curved domains.
First of all we denote by Ω_h the polygon with vertices on Γ, that approximate the curved domain Ω in the usual manner, that is, the union of the triangles in the mesh $\mathscr{T}_h$.
The modifications in view are motivated by the fact that we cannot, akin to the polygonal case, simply apply the

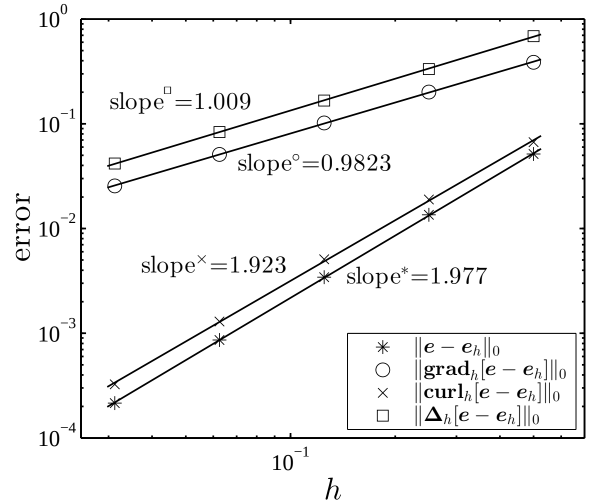


4.1.1: Mean-square errors

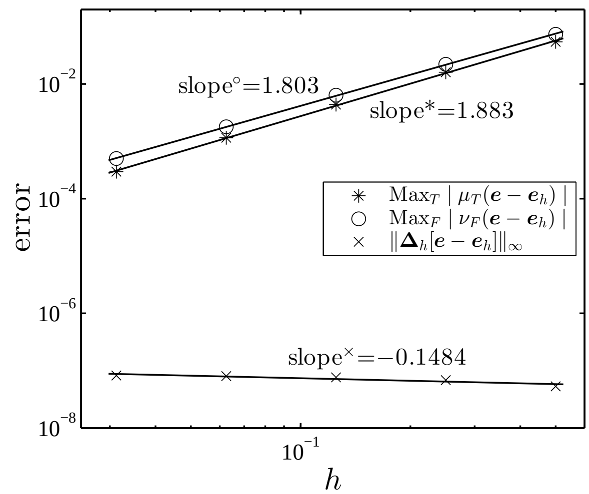


4.1.2: Maximum-value errors

Fig. 4.1: Numerical errors in different senses for Test-problem 1



4.2.1: Mean-square errors



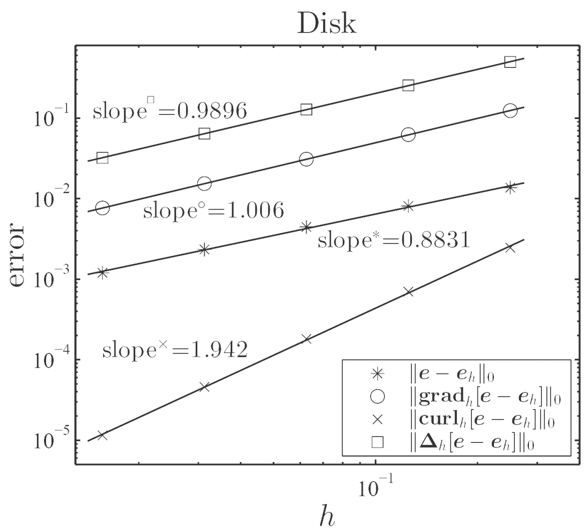
4.2.2: Maximum-value errors

Fig. 4.2: Numerical errors in different senses for Test-problem 2

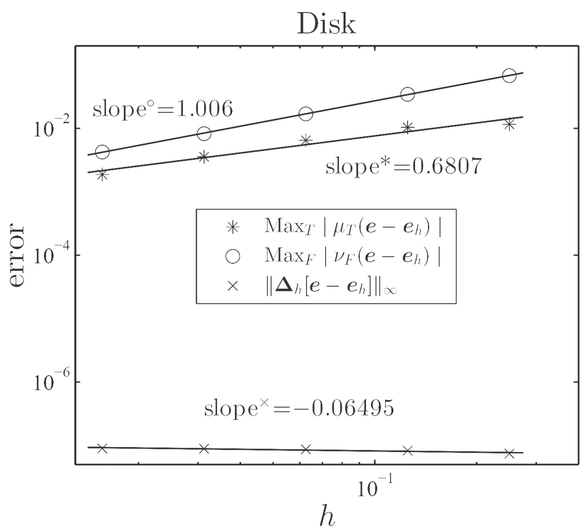
numerical counterpart of the boundary condition $div\ e = \rho$ on the (mid-points of the) edges of Ω_h. This is because such an approximation erodes the second-order error estimate in the L^2-norm for the electric field established in Theorem 4 in the polygonal case. In order to illustrate this assertion we consider

Test-problem 3: Ω being the unit disk with center at the origin of the cartesian coordinates x_1 and x_2 we take $\rho = 4(x_1^2+x_2^2)$ and $\mathbf{f} \equiv -grad\ \rho$. $\mathbf{e} = (x_1[x_1^2+x_2^2], x_2[x_1^2+x_2^2])$ is the exact solution. The meshes used in our computations to solve this problem consist of $2K^2$ triangles, where $K > 1$ is an integer. They are generated by the transformation of cartesian into polar coordinates of the nodes of standard symmetric uniform meshes of the square $(-1,1)\times(-1,1)$ by a procedure described in [22]. Denoting by $\|\cdot\|_{0,h}$ the norm of $L^2(\Omega_h)$, we show in Sub-figures 4.3.1 and 4.3.2 the evolution of the absolute errors in logarithm scale, in the indicated senses, for increasing values of K, where $h = 1/K$.

Observation of both sub-figures shows that errors $\| \mathbf{e}-\mathbf{e}_h \|_{0,h}$ and $\max_{T\in\mathscr{T}_h} |\mu_T(\mathbf{e}-\mathbf{e}_h)|$ no longer decrease roughly like an $O(h^2)$ as h goes to zero, but rather like an $O(h^\alpha)$ with $\alpha = 0.8831$ and $\alpha = 0.6807$, respectively.
In order to remedy this we next describe the modifications to be carried out in our method in the framework of two-dimensional problems posed in curved domains. Of course the main issue to be addressed is the prescription of the divergence boundary condition. In order to avoid the erosion of method's (second-order) convergence in the L^2-norm, in case this boundary condition is applied to flux degrees of freedom on the edges forming the boundary Γ_h of Ω_h, we use a variant of the technique known as the interpolated boundary condition (cf. [5]). This variant is similar to the procedure proposed by the first author in [24] to solve Poisson's equation in curved domains with Dirichlet boundary conditions. The idea is



4.3.1: Mean-square errors for the classical approach



4.3.2: Maximum-value errors for the classical approach

Fig. 4.3: Errors for Test-problem 3 shifting the boundary conditions from Γ to Γ_h

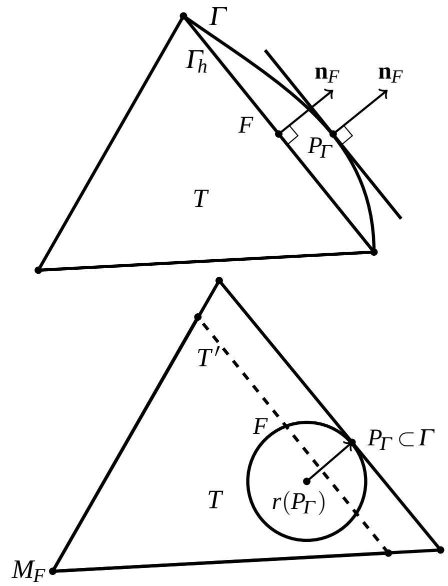


Fig. 4.4: Element $T \in \mathscr{T}_h$ with an edge $F \subset \Gamma_h$ and triangle T' (below) with an edge parallel to F tangent to Γ

to apply the condition

$$div\,\mathbf{e}(P_F) = \rho(P_F) \tag{24}$$

at the points $P_F \in \Gamma$ determined as follows for each edge F contained in Γ_h. Still denoting by $\mathbf{n}_F$ the unit outer normal vector to Γ_h upon F, and referring to Figure 4.4, P_F is the boundary point at which the outer normal vector coincides with $\mathbf{n}_F$, located at the smallest distance from F measured in the direction of $\mathbf{n}_F$.

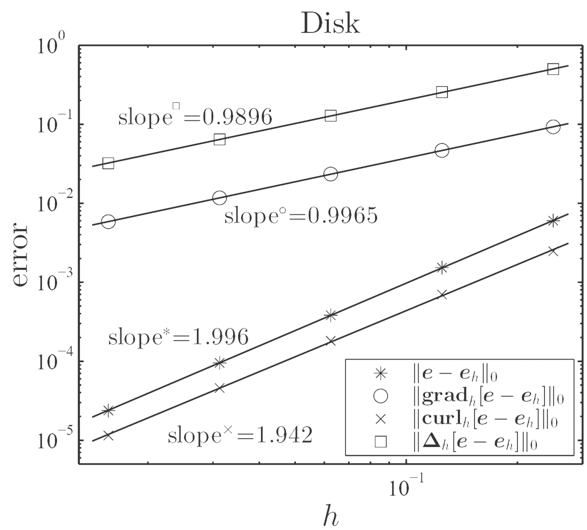
Now let $r(P)$ be the curvature radius of Γ at a point $P \in \Gamma$ with normal $\mathbf{n}$. At this point we recall standard representations of differential operators in general coordinate systems (see e.g. [15]). From straightforward calculations the divergence of a field $\mathbf{u}$ at P, expressed in terms of a local orthogonal frame $(\mathbf{n};\mathbf{t})$ with associated curvilinear coordinates, where $\mathbf{t}$ is the unit tangent vector along Γ, is given by $[grad\,\mathbf{u}\,\mathbf{t}]\cdot\mathbf{t} + [grad\,\mathbf{u}\,\mathbf{n}]\cdot\mathbf{n} + \mathbf{u}\cdot\mathbf{n}/r$. Notice that the first term in the above expression is nothing but the derivative with respect to the curvilinear abscissa along Γ, of the tangential component of $\mathbf{u}$ on Γ, i.e. $\mathbf{u}\cdot\mathbf{t}$. If $\mathbf{u} = \mathbf{e}$ then $\mathbf{u}\cdot\mathbf{t}$ vanishes on Γ. Hence $div\,\mathbf{e}$ at P_F (cf. Figure 4.4) is given by,

$$div\,\mathbf{e}(P_F) = [grad\,\mathbf{e}(P_F)\,\mathbf{n}_F]\cdot\mathbf{n}_F + \mathbf{e}(P_F)\cdot\mathbf{n}_F/r(P_F). \tag{25}$$

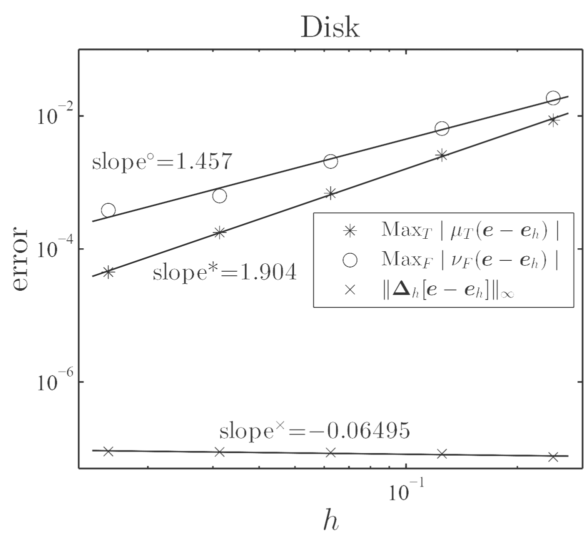
At the level of our finite element method, the modification consists of replacing the flux degrees of freedom related to edges F on Γ_h, for every triangle T having F as an edge opposite to its vertex, say M_F. More precisely they are replaced by the corresponding degree of freedom for triangle T' also having M_F as a vertex, an edge parallel to F passing through P_F, and the other two edges aligned with the edges of T intersecting at M_F (cf. Figure 4.4). In doing so the quadratic field $\mathbf{e}_h$ restricted to such a boundary element T is extended to T' in case $T \subset T'$. On the other hand the test field $\mathbf{d}$ is still defined in the same manner as in the case of a polygonal Ω, and is not extended to such T's if applicable. The purpose of extending only $\mathbf{e}_h$ to the T's is to enable the application of the natural counterpart of (24)-(25). Now denoting by $\nu_F(v)$ the mean value of the normal derivative along F of a function v defined in T' we prescribe,

$$\left[\nu_F(\mathbf{e}_h\cdot\mathbf{n}_F) + \frac{\mathbf{e}_h\cdot\mathbf{n}_F}{r}\right](P_F) = \rho(P_F)\ \forall \text{ edge } F \subset \Gamma_h. \tag{26}$$

Results of the computations using the above formula and the same meshes as in the experiments illustrated in Figure 4.3 are shown in Figure 4.5

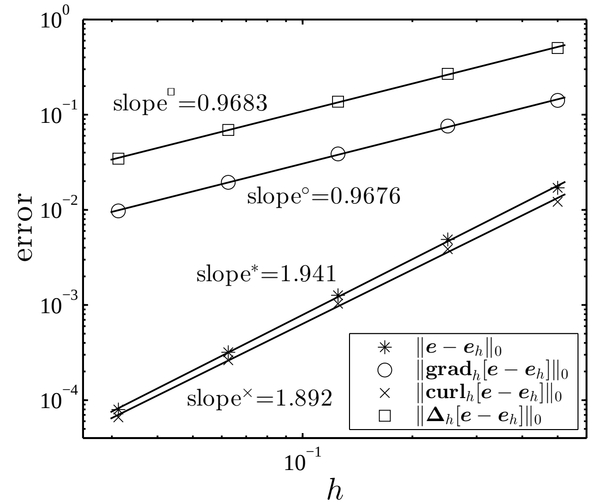


4.5.1: Mean-square errors using b.c. interpolation

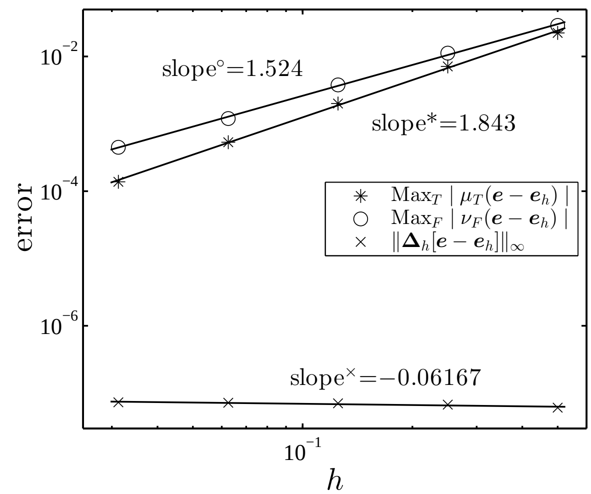


4.5.2: Maximum-value errors using b.c. interpolation

Fig. 4.5: Errors for Test-problem 3 using curved boundary condition interpolation



4.6.1: Mean-square errors



4.6.2: Maximum-value errors

Fig. 4.6: Numerical errors in different senses for Test-problem 4

These results show that in the presence on inhomogeneous (Dirichlet) flux boundary conditions, practically the same behavior as in Test-problem 1 is observed for the modified method. Notice that in all cases the laplacian of the solution at element centroids is exact up to machine precision, but probably this is only because it is a linear field in this test-problem, as it happens in the next one as well.

Test-problem 4 In order to rule out any particularity of Test-problem 3 in a disk, here Ω is the ellipse whose equation is $x_1^2 + c^2 x_2^2 = c^2$, c being taken equal to $1/2$. $\rho = (3+c^2)x_1^2 + (c^2+3c^4)x_2^2$ and $\mathbf{f} \equiv -grad\ \rho$. The exact solution is $\mathbf{e} = (x_1^2 + c^2 x_2^2) \times (x_1; c^2 x_2)$. Here again the meshes used in the computations consist of $2K^2$ triangles, K being an integer greater than one, and are generated by the transformation of cartesian into polar coordinates of the nodes of standard symmetric uniform meshes of the square $(-1,1)\times(-1,1)$ by the procedure described in [22]. In Figure 4.6 the errors in the same senses as in Test-problems 1, 2, 3 are supplied, for increasing values of K.

These results confirm that for arbitrary smooth curved domains, practically the same behavior as in the polygonal case can be expected, as long as the modification advocated in this subsection is implemented, to treat (Dirichlet) flux boundary conditions.

5 Extensions and final comments

To conclude we briefly consider a relevant extension of the studies carried out so far, and highlight some of the related ongoing research aimed at completing the analysis and the numerical experiments presented in this work.

5.1 The case of variable coefficients

Throughout the previous sections we considered only equations with constant coefficients. However in several important applications of Maxwell's equations of electromagnetism some variable coefficients must be handled. Actually in this case, unlike (1), the original system to solve involves two strictly positive real coefficients μ and ε assumed to vary only in space, and to belong to $L^{\infty}(\Omega)$. Then instead of (1) we have to solve

$$\begin{cases} \text{Given } \sigma \in H^1(\Omega) \text{ and } \mathbf{k} \in \{H^1[(0,T);L^2(\Omega)]\}^N \\ \text{with } div\, \mathbf{k} = 0 \; \forall t,, \; (\mathbf{e},\mathbf{m}) \text{ fulfills:} \\ div(\varepsilon \mathbf{e}) = \sigma \text{ and } div(\mu \mathbf{m}) = 0 \text{ in } \Omega \times (0,T], \\ \mu \dfrac{\partial \mathbf{m}}{\partial t} + curl\; \mathbf{e} = \mathbf{0} \text{ in } \Omega \times (0,T], \\ \dfrac{\partial \mathbf{e}}{\partial t} - \varepsilon\, curl\; \mathbf{m} = -\mathbf{k} \text{ in } \Omega \times (0,T], \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \times (0,T]. \end{cases} \tag{27}$$

Akin to the case of (1), we can combine the three equations in (27) to derive a second-order hyperbolic system for $\mathbf{e}$, namely:

$$\begin{aligned} &\frac{\partial^2 \mathbf{e}}{\partial t^2} + \varepsilon grad\; \mu \times curl\; \mathbf{e} \\ &-\varepsilon\mu\{\Delta \mathbf{e} + grad[\mathbf{e} \cdot grad(log\varepsilon)]\} = \\ &-\frac{\partial \mathbf{k}}{\partial t} - \mu[\varepsilon grad\; \sigma - grad(log\varepsilon)\; \sigma] \end{aligned} \tag{28}$$

supplemented with the initial and boundary conditions derived from (27), together with the condition $div(\varepsilon \mathbf{e}) = \sigma$ in Ω. Now restricting the latter condition to the boundary, like in the constant coefficient case, is more tricky. However if the coefficients are piecewise constant this can be accomplished in a similar manner. We skip details since the justification of this assertion is the same as the one applying to the stationary counterpart of (28). Assuming that Ω is simply connected, the latter is an equation analogous to (5), namely,

$$\begin{cases} \text{Given } \mathbf{g} \in \mathbf{H}(curl,\Omega) \text{ and } \sigma \in H^1(\Omega), \\ \text{find } \mathbf{e} \in [H^1(\Omega)]^N \text{ such that:} \\ div(\varepsilon \mathbf{e}) = \sigma \text{ on } \Omega \\ \varepsilon grad\; \mu \times curl\; \mathbf{e} \\ -\varepsilon\mu\{\Delta \mathbf{e} + grad[\mathbf{e} \cdot grad(log\varepsilon)]\} = \\ curl\; \mathbf{g} - \mu[grad\; \sigma - grad(log\varepsilon)\; \sigma] \text{ in } \Omega, \\ \mathbf{e} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \end{cases} \tag{29}$$

The extension to (29) of the Hermite method to solve (5) described in Section 3, though perfectly possible, involves several additional technicalities in case the coefficients μ and ε vary continuously. That is why we consider only piecewise constant coefficients, which is a very important case in practice. For example, in a coefficient identification problem for Maxwell's equations, this is the situation one often has to deal with. In this respect we refer to [3] for the solution of the underlying inverse problem.
Application of our discretization method becomes rather simple in this case, as long as the mesh is such that discontinuities lines or planes of $\lambda := \mu\varepsilon$ coincide with inter-element boundaries. Indeed, in these circumstances the problem can be viewed as much like the solution of porous media flow equations with the scalar counterpart of our Hermite method, as described in [23]: it suffices to require continuity of $\lambda \partial \mathbf{e}_h \cdot \mathbf{n}_F$ on every interface F common to two mesh elements, and the rest functions exactly like in the case of constant coefficients after straightforward modifications.

5.2 Conclusions and perspectives

To summarize, in this paper we have rewritten the classical Maxwell's equations in the form of a second-order hyperbolic system for the electric field. This system can be solved in an uncoupled manner for the components of the electric field. This uncoupling is immediate if problem's domain is a rectangle. Otherwise classical uncoupling algorithms based on the influence matrix technique can be used (cf. [13]).
Further we have introduced a quasilinear finite element method to solve these equations in rather smooth domains, based on Hermite interpolation, to be combined with a particular non coercive variational form. This variational form mimics in a vector framework, the well-known mixed formulation of the (scalar) Poisson equation. A priori error estimates were derived, showing that optimal first-order convergence in the L^2-norm of both the gradient and the laplacian of the electric holds, assuming that the domain is a convex polytope. Moreover second-order convergence of the electric field in the L^2-norm is to be expected for the same type of domain. Numerical examples validated these estimates in the two-dimensional case, and showed that the curl of the electric field is also approximated to the second order in the L^2-norm. Further numerical experiments were reported for test-problems posed in curved domains. Using a modification of the original method in order to take into account the prescribed divergence of the electric field at points situated on the true curved boundary, we observed that the same convergence properties that hold in the polygonal case are recovered.

In future work the authors intend to prove lacking error estimates, which would corroborate some of the observations reported for the numerical experiments.
First of all we will attempt to give a formal justification for the observed second-order convergence in the L^2-norm of the curl of the electric field. A priori this

makes sense, since in two dimensions the space spanned by the curl of fields in the space $\mathbf{E}_h$ contains all piecewise linear functions in terms of the two space variables. Notice that this statement is false in three-dimension space.
We will also turn our attention to the modification of the numerical scheme advocated in the case of curved domains. Formal proofs that this leads to the observed second-order approximations of the electric field in the L^2-norm are underway.
Finally the authors are planning to test their method in real life problems with variable coefficients.

Appendix 1

A discrete Friedrichs-Poincaré inequality

We prove below the natural extension of the Friedrichs-Poincaré inequality for the space $\mathbf{V} := \{\mathbf{v} | \ \mathbf{v} \in [H^1(\Omega)]^2 \ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \Gamma\}$ (cf. [11]), to the space of weakly continuous piecewise polynomial fields, whose tangential trace is equal to zero on the boundary of Ω in a certain weak sense.

Proposition 5 *P_k being the space of polynomials of degree less than or equal to k in a given subset of $\Re^2$, let $\mathbf{W}_h := \{\mathbf{v} \in [L^2(\Omega)]^2, \mathbf{v}_{|T} \in [P_k(T)]^2 \ \forall T \in \mathscr{T}_h\}$. Further, e being an edge of an element $T \in \mathscr{T}_h$, we denote by $\Pi_{e,T}(v_\tau)$ (resp. $\Pi_{e,T}(v_n)$) the mean value of the tangential component $v_\tau := \mathbf{v}_{|T}\cdot\mathbf{t}$ (resp. outer normal component $\mathbf{v}_n := \mathbf{v}_{|T}\cdot\mathbf{n}$) of $\mathbf{v}_{|T}$ along e, for $\mathbf{v} \in \mathbf{W}_h$, the pair of unit vectors $(\mathbf{n};\mathbf{t})$ being oriented in the direct (counterclockwise) sense, in such a way that $\mathbf{n}$ points outwards T. Now T_1 and T_2 being an arbitrary pair of elements in $\mathscr{T}_h$ having a common edge e we introduce the following subspace of $\mathbf{W}_h$:*

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{W}_h | \ \Pi_{e,T_1}(v_\tau) + \Pi_{e,T_2}(v_\tau) = 0 \text{ and}$$
$$\Pi_{e,T_1}(v_n) + \Pi_{e,T_2}(v_n) = 0 \ \forall T_1, T_2 \in \mathscr{T}_h \ s.t. T_1 \cap T_2 = e$$
$$\text{and } \Pi_{e,T}(v_\tau) = 0 \text{ if } e \subset \Gamma \ \forall T \ s.t. T \cap \Gamma \neq \emptyset\}.$$

Then, defining the discrete gradient operator $grad_h$ over $\mathbf{W}_h$ by $[grad_h \mathbf{v}]_{|T} = grad\ \mathbf{v}_{|T} \ \forall T \in \mathscr{T}_h$, there exists a mesh independent constant C_V such that

$$\| \mathbf{v} \|_0 \leq C_V \| grad_h \mathbf{v} \|_0 \ \ \forall \mathbf{v} \in \mathbf{V}_h. \tag{30}$$

PROOF. In this proof the *curl* of a two-dimensional vector field $\mathbf{w}$ sufficiently smooth is to be understood as the function $\partial w_2/\partial x_1 - \partial w_1/\partial x_2$ and the *curl* of a function $w \in H^1(\Omega)$ represents the field $(\partial w/\partial x_2, -\partial w/\partial x_1)$. Moreover we denote by $\mathbf{t}$ the unit tangential vector defined on Γ except at its vertices, oriented in the counterclockwise sense.
Let us recall the space

$$\mathbf{U} = \{\mathbf{u} | \ \mathbf{u} \in [L^2(\Omega)]^2, \ div\,\mathbf{u} \in H^1_0(\Omega), \ \Delta\mathbf{u} \in [L^2(\Omega)]^2 \text{ and } \mathbf{u}\cdot\mathbf{t} = 0 \text{ on } \Gamma\},$$

together with

$$\mathbf{V} = \{\mathbf{v} | \ \mathbf{v} \in [L^2(\Omega)]^2, \ div\,\mathbf{v} \in L^2(\Omega), \ curl\,\mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v}\cdot\mathbf{t} = 0 \text{ on } \Gamma\}.$$

Since $\mathbf{V}$ is a Hilbert space for the inner product given by $(curl\ \cdot | curl\ \cdot) + (div\ \cdot | div\ \cdot)$ (cf. [12]), from the Lax-Milgram Theorem the following problem has a unique solution for every $\mathbf{f} \in [L^2(\Omega)]^2$:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V} \text{ such that} \\ (curl\ \mathbf{u} | curl\ \mathbf{v}) + (div\ \mathbf{u} | div\ \mathbf{v}) = (\mathbf{f}|\mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}. \end{cases}$$

It can be easily established that $\mathbf{u} \in \mathbf{U}$, $-\Delta\mathbf{u} = \mathbf{f}$ a.e. in Ω and $grad\ curl\ \mathbf{u} \in [L^2(\Omega)]^2$. Moreover, denoting by $< \cdot|\cdot >_1$ the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, the obvious relation $< div\ curl\ curl\ \mathbf{u}|\varphi >_1 = 0$ $\forall \varphi \in H^1_0(\Omega)$ implies that $(curl\ curl\ \mathbf{u}|grad\ div\mathbf{u}) = 0$. Then we have

$$\sqrt{|curl\ \mathbf{u}|_1^2 + |div\ \mathbf{u}|_1^2} = \| \Delta\mathbf{u} \|_0 . \tag{31}$$

Owing to (31) we may write for every $\mathbf{v} \in \mathbf{V}_h$:

$$\begin{aligned} \| \mathbf{v} \|_0 &= \sup_{\mathbf{f}\in[L^2(\Omega)]^2\setminus\{0\}} \frac{(\mathbf{v}|\mathbf{f})}{\| \mathbf{f} \|_0} \\ &= \sup_{\mathbf{u}\in\mathbf{U}\setminus\{0\}} \frac{-(\mathbf{v}|\Delta\mathbf{u})}{[\ |curl\ \mathbf{u}|_1^2 + |div\ \mathbf{u}|_1^2\ ]^{1/2}} \end{aligned} \tag{32}$$

Now denoting by ∂T the boundary of $T \in \mathscr{T}_h$ and by e a generic edge of a mesh triangle, we observe that

$$\begin{aligned} -(\mathbf{v}|\Delta\mathbf{u}) = \sum_{T\in\mathscr{T}_h} \Bigg[\int_T (curl\ \mathbf{v}\ curl\ \mathbf{u} + div\ \mathbf{v}\ div\ \mathbf{u}) \\ + \sum_{e\subset\partial T} \oint_e (v_\tau\ curl\ \mathbf{u} + v_n\ div\ \mathbf{u}) \Bigg]. \end{aligned} \tag{33}$$

Owing to the properties of $\mathbf{V}_h$, the Trace Theorem applied to $curl\ \mathbf{u}$ and $div\ \mathbf{u}$ and to the fact that $div\ \mathbf{u} = 0$ on Γ, we can write:

$$\begin{aligned} &\sum_{T\in\mathscr{T}_h} \sum_{e\subset\partial T} \oint_e (v_\tau\ curl\ \mathbf{u} + v_n\ div\ \mathbf{u}) \\ &= \sum_{T\in\mathscr{T}_h} \sum_{e\subset\partial T} \oint_e \{[v_\tau - \Pi_{T,e}(v_\tau)]curl\ \mathbf{u} \\ &+ [v_n - \Pi_{T,e}(v_n)]div\ \mathbf{u}\} \end{aligned} \tag{34}$$

Now for every $T \in \mathscr{T}_h$ and $\forall e \subset \partial T$ we define the linear functional $\sigma_{T,e}$ on $\mathbf{V}_h$ equipped with the discrete H^1-(semi)norm $\| grad_h \cdot \|_0$, where

$$\sigma_{T,e}(\mathbf{v}) = \oint_e \{[v_\tau - \Pi_{T,e}(v_\tau)]curl\ \mathbf{u} + [v_n - \Pi_{T,e}(v_n)]div\ \mathbf{u}\}.$$

Owing to the Trace Theorem the functional $\sigma_{T,e}(\cdot)$ is bounded with a constant proportional to $\sqrt{\| curl\ \mathbf{u} \|_{1,T}^2 + \| div\ \mathbf{u} \|_{1,T}^2}$, Since $\sigma_{T,e}(\mathbf{v})$ vanishes if $\mathbf{v}$ is constant in T, after a standard transformation of T into the unit master triangle $\hat{T}$, a simple application of the Bramble-Hilbert Lemma (see e.g. [7]) yields $\forall e$ and $\forall T$,

$$\sigma_{T,e}(\mathbf{v}) \leq C_V \sqrt{\| \, curl \, \mathbf{u} \, \|_{1,T}^2 + \| \, div \, \mathbf{u} \, \|_{1,T}^2} \, \| \, grad \, \mathbf{v} \, \|_{0,T}, \tag{35}$$

where C_V is a mesh independent constant.
Now since $div\, \mathbf{u} \in H_0^1(\Omega)$ and $\int_\Omega curl\, \mathbf{u} = \oint_\Gamma [\mathbf{u} \cdot \mathbf{t}] = 0$, the following inequality of the Friedrichs-Poincaré holds with a constant C_{FP} $\forall \mathbf{u} \in \mathbf{U}$ (cf. [4]):

$$\sqrt{\| \, curl \, \mathbf{u} \, \|_1^2 + \| \, div \, \mathbf{u} \, \|_1^2} \leq C_{FP} \sqrt{|curl \, \mathbf{u}|_1^2 + |div \, \mathbf{u}|_1^2}. \tag{36}$$

Combining (32), (33), (34), (35) and (36) we readily derive (30). ■

*Remark.*A complete proof of the discrete Friedrichs-Poincaré inequality (6.18) omitted in [21] can be easily inferred from Proposition 5. ■

Appendix 2

Proof that (11) implies (13) for rectangular domains

For the sake of simplicity we consider only the case $N = 2$, for the case $N = 3$ can be viewed as a mere variant of the former. Let then u satisfy in the rectangle $\Omega = (0,L_x) \times (0,L_y)$:

$$\begin{cases} -\Delta u = f \in L^2(\Omega) \\ u(x,0) = u(x,L_y) = 0 \text{ for } x \in (0,L_x) \\ \{\partial u/\partial x\}(0,y) = \{\partial u/\partial x\}(L_x,y) = 0 \text{ for } y \in (0,L_y). \end{cases} \tag{37}$$

Taking in (37) $g \in H_0^1(\Omega)$ instead of f, let v be the corresponding solution. Then $v_y = \partial v/\partial y$ satisfies $-\Delta v_y = \partial g/\partial y$ in $L^2(\Omega)$ and $\partial v_y/\partial n = 0$ almost everywhere on Γ. While these boundary conditions are obvious on the edges $x = 0$ and $x = L_x$, on the edges $y = 0$ and $y = L_y$ they are a consequence of the fact that $\partial^2 v/\partial x^2 = 0$ and $\Delta v = 0$. The solution of the resulting Poisson problem for v_y with homogeneous Neumann boundary conditions in a rectangle with a right hand side in $L_0^2(\Omega)$ is known to belong to $H^2(\Omega)$ (cf. [14]). Since by symmetry the same conclusion applies to v_x we have $v \in H^3(\Omega)$. It follows that the trace of all second order derivatives of v on Γ are well defined in $L^2(\Gamma)$. Hence, denoting by $H(v)$ the hessian of v, we can apply first Green's identity twice to derive:

$$\int_\Omega |\Delta v|^2 = \oint_\Gamma \Delta v \partial v/\partial n - \oint_\Gamma grad\, v \cdot \partial(grad\, v)/\partial n + \int_\Omega |H(v)|^2 = \int_\Omega |H(v)|^2. \tag{38}$$

It follows that the H^2-seminorm of v (in fact a norm in this case) equals $\| \, g \, \|_0$. Let now $\{f_n\}_n \subset H_0^1(\Omega)$ be a sequence converging to f in $L^2(\Omega)$ by density (cf. [11]), and u_n be the corresponding solution of (37). Since $u_n - u_m$ is the solution of the same problem for the right hand side $f_n - f_m$ $\forall m,n$, from (38) $\{u_n\}_n$ is a Cauchy sequence of $H^2(\Omega)$. Indeed, the seminorm $|\cdot|_2$ is a norm on the subspace $V := \{v | \; v \in H^2(\Omega), u = 0 \text{ for } y = 0 \text{ and } y = L_y, \partial u/\partial x = 0 \text{ for } x = 0 \text{ and } x = L_x\}$ equivalent to $\| \cdot \|_2$ by Peetre's Lemma (cf. [11]), as one can easily check. Therefore by continuity $\exists u \in V$ solution of (37) such that $|u|_2 = \| \, f \, \|_0$, and hence a solution $\mathbf{u} \in \mathbf{V} \cap [H^2(\Omega)]^2$ to (11) satisfying (13) with $C_P = 1$. ■

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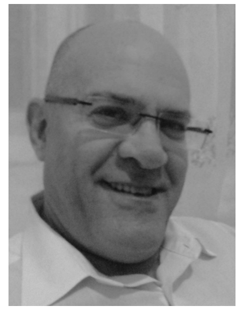

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