Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120512

A Symbolic Algorithm for Polynomial Interpolation with Integral Conditions

Srinivasarao Thota

Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Ethiopia

Received: 2 Jun. 2018, Revised: 27 Jul. 2018, Accepted: 10 Aug. 2018
Published online: 1 Sep 2018

Abstract: This paper presents a new symbolic algorithm for polynomial interpolation with integral conditions at arbitrary points. For expressing the integral conditions in the present algorithm, we employ the algebra of integro-differential operators. We also present another algorithm for computing the polynomial interpolation with Stieltjes conditions (combination of general, differential and integral conditions) as a quotient of two determinants. Error due to the formulation of a given function by the proposed interpolation is discussed and its symbolic formulation is presented. This algorithm helps OR would help to implement the manual calculations in commercial packages such as Maple, Mathematica, Matlab, Singular, etc. Certain numerical examples are presented to verify the proposed algorithms.

Keywords: Polynomial interpolation, integral conditions, error estimation, integro-differential operators

1 Introduction

In science and engineering, researchers often come up with data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate the value of that function for an intermediate value of the independent variable. There exist many interpolation techniques in the literature for general functional values, see for example, [1,2,3]. The purpose of this paper is to develop an algorithm to construct a polynomial interpolation with a finite set of integral conditions alone as well as the combination of general, differential and integral conditions, so-called *Stieltjes conditions* [4], via integro-differential operators.

The paper is organized as follows: In Section 1.1, we present the definitions and basic concepts of the polynomial interpolation, in Section 1.2, we recall the algebra of integro-differential operators and the operator representation of integral and Stieltjes functionals/conditions in terms of integro-differential operators. Proposed symbolic algorithm for the polynomial interpolation is presented in Section 2, and Section 2.1 provides the condition of existence and uniqueness of the solution of a given interpolation problem. In Section 3, we discuss various formulation of the error estimation. Selected examples are discussed to demonstrate the proposed algorithm.

1.1 Interpolation problem

We give, first, the general form of the interpolation problem as follows [3,5,6,7]: Suppose $\mathscr{S}$ is a normed linear space. For a finite linearly independent set $\Theta \subset \mathscr{S}$ of bounded functionals and associated values $\Omega = \{\alpha_\theta : \theta \in \Theta\} \subset \mathbb{R}$, the *interpolation problem* is to find a $\widetilde{f}_s(x) \in \mathscr{S}$ such that

$$\Theta(\widetilde{f}_s) = \Omega, \quad \text{i.e.} \quad \theta \widetilde{f}_s = \alpha_\theta, \quad \theta \in \Theta. \tag{1}$$

Here s is called the order of the interpolating function $\widetilde{f}_s(x)$. To describe the polynomial interpolation, let $\mathscr{S} = \mathbb{K}[x]$ be a polynomial ring over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. One can observe that the interpolation problem given in equation (1) may have many solutions if there is no restriction on the dimension of the problem. But we want a single interpolate polynomial which must satisfies the given conditions. Hence, for the unique solution of the problem, we must have finite dimensional subspace Θ of $\mathscr{S}$ having the dimension equal to the number of conditions.

* Corresponding author e-mail: srinithota@ymail.com, srinivasarao.thota@astu.edu.et

Definition 1. *We call the pair (Θ,Ω) a* polynomial interpolation problem, *where Θ is a finite linearly independent set of functionals with associated values $\Omega \subset \mathbb{K}$.*

Definition 2. *A polynomial interpolation problem (Θ,Ω) is called* regular *with respected to Θ if (Θ,Ω) has a unique solution for each choice of values of $\Omega \subset \mathbb{K}$ such that $\Theta(\widetilde{f_s}) = \Omega$. Otherwise, it is called* singular.

The following proposition gives the regularity test in terms of linear algebra.

Proposition 1. *[8,7] Let $M = \{m_0,\ldots,m_{t-1}\}$ be a basis for $\mathscr{M}$, a finite dimensional subspace of $\mathscr{S}$, and $\Theta = \{\theta_0,\ldots,\theta_{s-1}\} \subset \mathscr{S}^*$ with θ_i linearly independent. Then the following statements are equivalent:*

(i) The polynomial interpolation problem is regular for $\mathscr{M}$ with respected to Θ.
(ii) $t = s$, and the evaluation matrix,

$$\Theta M = \begin{pmatrix} \theta_0(m_0) & \cdots & \theta_0(m_{t-1}) \\ \vdots & \ddots & \vdots \\ \theta_{s-1}(m_0) & \cdots & \theta_{s-1}(m_{t-1}) \end{pmatrix} \quad (2)$$

is nonsingular. Denote the evaluation matrix ΘM by $\mathscr{E}$ for simplicity.
(iii) $\mathscr{S} = M \oplus \Theta^{\perp}$.

1.2 Algebra of integro-differential operators

In this section, we recall some basic concepts of integro-differential algebras and operators see, for example, [8,9,10,11,12,13] for further details. In this section, $\mathbb{K}$ denotes the field of characteristic zero.

Definition 3. *[8] Let $\mathscr{S}$ be a commutative algebra over a field $\mathbb{K}$. The structure $(\mathscr{S},D,A)$ is called an* integro-differential algebra *if $(\mathscr{S},D)$ is a commutative differential algebra over $\mathbb{K}$ and the* differential Baxter axiom

$$(ADf)(ADg) + AD(fg) = (ADf)g + f(ADg)$$

holds. Where $D : \mathscr{S} \to \mathscr{S}$ and $A : \mathscr{S} \to \mathscr{S}$ are two maps such that D is a derivation and A is a $\mathbb{K}$-linear right inverse of D, i.e. $DA = 1$ (the identity map). The map A is called an integral *for D. An integro-differential algebra over $\mathbb{K}$ is called* ordinary *if $Ker(D) = \mathbb{K}$.*

For example [8], we have $\mathscr{S} = C^\infty(\mathbb{R})$, the set of smooth functions over the field of real numbers with $\mathsf{D} = \frac{d}{dx}$ and $\mathsf{A} = f \mapsto \int_0^x f(\xi)\ d\xi$. Here A is the right inverse of D, i.e. $\mathsf{DA} = 1$, but $\mathsf{AD} = 1 - \mathsf{E}$. Indeed, $(\mathsf{AD})f(x) = f(x) - f(0) = f(x) - \mathsf{E}f(x) \neq f(x)$.

The operator $\mathsf{E} = 1 - \mathsf{AD}$, called the *evaluation* of $\mathscr{S}$, evaluates at the initial point of the integral. The evaluation operator $\mathsf{E} : \mathscr{S} \to \mathbb{K}$ is a $\mathbb{K}$-linear map, also called as *character*. In the above standard example $\mathscr{S} = C^\infty(\mathbb{R})$, evaluate f at the initial point of the integral, i.e. $\mathsf{E}f(x) = f(0)$. It is shown in [8, Section 3] that the evaluation $\mathsf{E} = 1 - \mathsf{AD}$ is multiplicative linear functional (character), it means that $\mathsf{E}fg = (\mathsf{E}f)(\mathsf{E}g)$. The evaluation allows to formulate the initial value problems, see [8,11]. For treating boundary value problems, we need another character $\mathsf{E}_c : \mathscr{S} \to \mathbb{K}$, the evaluation operator at various points $c \in \mathbb{R}$, i.e. $\mathsf{E}_c : f \mapsto f(c)$. Let $\Phi \subseteq \mathscr{S}^*$ be a set of all multiplicative linear functionals including E.

Definition 4. *[8] Let $(\mathscr{S},D,A)$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathscr{S}^*$. The* integro-differential operators *$\mathscr{S}[D,A]$ are defined as the $\mathbb{K}$-algebra generated by the symbols D and A, the functions $f \in \mathscr{S}$ and the characters (functionals) $E_c \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table 1.*

Table 1: Rewrite rules for integro-differential operators

$fg \to f \cdot g$	$\mathsf{D}f \to f\mathsf{D} + f'$	$\mathsf{A}f\mathsf{A} \to (\mathsf{A}f)\mathsf{A} - \mathsf{A}(\mathsf{A}f)$
$\chi\phi \to \phi$	$\mathsf{D}\phi \to 0$	$\mathsf{A}f\mathsf{D} \to f - \mathsf{A}f' - (\mathsf{E}f)\mathsf{E}$
$\phi f \to (\phi f)\phi$	$\mathsf{DA} \to 1$	$\mathsf{A}f\phi \to (\mathsf{A}f)\phi$

The following lemma shows that every integro-differential operator can be expresses as linear combination of monomials of the form $f\phi\mathsf{A}g\psi\mathsf{D}^i$.

Lemma 1. *[8] Every integro-differential operator in $\mathscr{S}[D,A]$ can be reduced to a linear combination of monomials $f\phi Ag\psi D^i$, where $i \geq 0$ and each of f,ϕ,A,g,ψ may also be absent.*

Definition 5. *[8] The elements of the right ideal $|\Phi) = \Phi \cdot \mathscr{S}[D,A]$ are called* Stieltjes conditions *over $\mathscr{S}$, and the elements of the two-sided ideal (Φ) of $\mathscr{S}[D,A]$ generated by Φ are called* Stieltjes operators.

Since the rewrite system of Table 1 is Noetherian and confluent (see, for example, [8] for further details), every integro-differential operator has a unique *normal form*. Moreover, every monomial is either a differential operator or an integral operator or a Stieltjes operator, so the normal form of integro-differential operators can be expressed as a sum of differential, integral and Stieltjes operators. The normal form of differential operators is as usual, the normal form of integral operators are itself and linear combinations of terms of the form $f\mathsf{A}g$, and the normal form of Stieltjes operators is of the following form [8, Proposition 25]

$$\sum_{\phi\in\Phi}\left(\sum_{i\in\mathbb{N}} a_{\phi,i}\phi\mathsf{D}^i + \phi\mathsf{A}f_\phi\right) \quad (3)$$

with $a_{\phi,i} \in \mathbb{K}$ and $f_\phi \in \mathscr{S}$ almost all zero. These operators act on $\mathscr{S}$ as linear functions in $\mathscr{S}^*$. The following proposition states the above fact.

Proposition 2. *[8] For an ordinary integro-differential algebra $\mathscr{S}$ and characters $\Phi \subseteq \mathscr{S}^*$, we have $\mathscr{S}[\mathsf{D},\mathsf{A}] = \mathscr{S}[\mathsf{D}] \oplus \mathscr{S}[\mathsf{A}] \oplus (\Phi)$, where $\mathscr{S}[\mathsf{D}]$ is the differential operators and $\mathscr{S}[\mathsf{A}]$ is the bimodule generated by A and monomials of the form $f\mathsf{A}g$, and (Φ) is an ideal of Stieltjes operators.*

Definition 6. *[3] A set of integro-differential operators $\Gamma = \{\gamma_0,\dots,\gamma_s\}$ is called a* Tchebycheff system, *or simply a* T-system *for a finite linearly independent set $M = \{m_0,\dots,m_s\}$, if the evaluation matrix ΓM is nonsingular for all sets of $s+1$ evaluation points $c \in \mathbb{R}$. The operator $\gamma_0,\dots,\gamma_i$ form a* complete Tchebycheff system, *or simply $CT-$system, if $\{\gamma_0,\dots,\gamma_i\}$ is a T-system for each $i = 0,\dots,s$.*

2 Symbolic formulation of polynomial interpolation

Consider the interpolation problem (Θ,Ω) given in Section 1.1 where $\Theta = \{\theta_0,\dots,\theta_{s-1}\}$ of the form $\{\mathsf{E}_c\mathsf{A} : c \in \mathbb{R}\}$, the monomials of integral conditions. From Proposition 1, the polynomial interpolation problem is regular with respect to Θ if and only if there exists a finite linearly independent set $M = \{m_0,\dots,m_{s-1}\}$ of $\mathscr{S}$ such that the evaluation matrix $\mathscr{E}$ defined as in equation (2) is regular. Indeed, there exists a unique $\widetilde{f_s}(x) \in \mathscr{S}$ satisfying Θ if and only if there exists a set $M = \{1,\dots,x^{s-1}\} \subset \mathscr{S}$ such that the evaluation matrix $\mathscr{E}$ is regular.

The classification of the interpolation problem depends on the type of the functionals $\theta \in \Theta$ that have to be matched with the polynomial. If Θ is a finite set of monomials of the form $\{\mathsf{E}_c : c \in \mathbb{K}\}$ with associated values $\Omega = \{\alpha_\theta : \theta \in \Theta\}$, then such type of interpolation is called *Lagrange or Newton interpolation* and the points $c \in \mathbb{K}$ are called *nodes*. If Θ is of the form $\{\mathsf{E}_c\mathsf{D}^i : c \in \mathbb{K}, i \in \mathbb{N}\}$ with Ω, then it is called *Hermite or Birkhoff interpolation* depending on i consecutive or not. It is seen that, there is no existing method available for Θ of the form $\{\mathsf{E}_c\mathsf{A} : c \in \mathbb{K}\}$ with given Ω to find a polynomial $\widetilde{f_s}(x)$. Hence, we develop an algorithm to construct polynomial $\widetilde{f_s}(x)$ with a special choice of the conditions of the form $\Theta = \{\mathsf{E}_{x_i}\mathsf{A} : x_i \in \mathbb{K}, i \in \mathbb{N}\}$.

2.1 Existence and uniqueness of the solution

The interpolation problem (Θ,Ω), i.e. $\widetilde{f_s}(x) = a_0 + a_1x + \dots + a_{s-1}x^{s-1}$ such that $\Theta\widetilde{f_s} = \Omega$, can be expressed as a linear system

$$\mathscr{E}u = \sigma, \tag{4}$$

where $u = (a_0,\dots,a_{s-1})^T$ and $\sigma = (\alpha_{\theta_0},\dots,\alpha_{\theta_{s-1}})^T$. Existence of the solution depends upon nature of the evaluation matrix $\mathscr{E}$. As mentioned at beginning of Section 2, there exists a solution for (Θ,Ω) if and only if $\mathscr{E}$ is regular. For uniqueness of the solution, suppose $\widetilde{f_s}(x)$ and $\widetilde{f_s'}(x)$ are two solutions, then $\theta_i\widetilde{f_s}(x) = \alpha_{\theta_i} = \theta_i\widetilde{f_s'}(x)$ implies that $\widetilde{f_s}(x) = \widetilde{f_s'}(x)$ for each $\theta_i \in \Theta$ is injective.

The following Examples 1 and 2 are given to calculate the evaluation matrix $\mathscr{E}$ with integral conditions at arbitrary point on the interval $[1,2]$ and a generalization for $\{x_0,\dots,x_{s-1}\} \subset \mathbb{R}$ respectively.

*Example 1.*Consider the integral conditions $\int_0^1 f(x)\,dx = 2$, $\int_0^{1.25} f(x)\,dx = 1$, $\int_0^{1.4} f(x)\,dx = 1$, $\int_0^{1.6} f(x)\,dx = 2$, $\int_0^{1.85} f(x)\,dx = 2$ and $\int_0^{2} f(x)\,dx = 3$. In symbolic notations, we have $\Theta = \{\mathsf{E}_1\mathsf{A}, \mathsf{E}_{1.25}\mathsf{A}, \mathsf{E}_{1.4}\mathsf{A}, \mathsf{E}_{1.6}\mathsf{A}, \mathsf{E}_{1.85}\mathsf{A}, \mathsf{E}_2\mathsf{A}\}$ and its associated values $\Omega = \{2,1,1,2,2,3\}$. Then the following evaluation matrix is calculated as in (2):

$$\mathscr{E} \approx \begin{pmatrix} 1.00 & 0.50 & 0.33 & 0.25 & 0.20 & 0.17 \\ 1.25 & 0.78 & 0.65 & 0.61 & 0.61 & 0.64 \\ 1.40 & 0.98 & 0.91 & 0.96 & 1.08 & 1.25 \\ 1.60 & 1.28 & 1.37 & 1.64 & 2.10 & 2.80 \\ 1.85 & 1.71 & 2.11 & 2.93 & 4.33 & 6.68 \\ 2.00 & 2.00 & 2.67 & 4.00 & 6.40 & 10.67 \end{pmatrix}.$$

Since $\det(\mathscr{E}) \approx -1.40 \times 10^{-8} \neq 0$, there exists a unique interpolating polynomial with respected to Θ.

*Example 2.*Consider the monomials $\Theta = \{\mathsf{E}_{x_0}\mathsf{A}, \mathsf{E}_{x_1},\dots,\mathsf{E}_{x_{s-1}}\mathsf{A}\}$. Then evaluation matrix is given by

$$\mathscr{E} = \begin{pmatrix} x_0 & \frac{1}{2}x_0^2 & \cdots & \frac{1}{s}x_0^s \\ x_1 & \frac{1}{2}x_1^2 & \cdots & \frac{1}{s}x_1^s \\ \vdots & \vdots & \ddots & \vdots \\ x_{s-1} & \frac{1}{2}x_{s-1}^2 & \cdots & \frac{1}{s}x_{s-1}^s \end{pmatrix}, \tag{5}$$

and the determinant of the evaluation matrix (5) is

$$\det(\mathscr{E}) = \frac{x_0x_1\cdots x_{s-1}}{s!}\prod_{0\le j<i\le(s-1)}(x_j - x_i).$$

$\det(\mathscr{E}) \neq 0$ if $x_i \neq x_j$, for all $i, j, i \neq j$.

From the generalized evaluation matrix (5) in Example 2, the problem of constructing a polynomial $\widetilde{f_s}(x)$ satisfying the given integral conditions at arbitrary points of the form $\Theta = \{\mathsf{E}_{x_i}\mathsf{A} : x_i \in \mathbb{R}, i \in \mathbb{N}\}$ is reduced to the construction of a polynomial of the form

$$\widetilde{\widetilde{f}}_s(x) = \widetilde{a}_0 + \widetilde{a}_1x + \dots + \widetilde{a}_{s-1}x^{s-1} \tag{6}$$

at given arbitrary nodes $\{\mathsf{E}_{x_i} : x_i \in \mathbb{R}\}$ with associated values $\{\widetilde{\alpha}_{\theta_0}, \ldots, \widetilde{\alpha}_{\theta_{s-1}}\}$, where $\widetilde{a}_i = \frac{a_i}{i+1}$ and $\widetilde{\alpha}_{\theta_i} = \frac{\alpha_{\theta_i}}{x_i}$, for $i = 0, 1, \ldots, s-1$. Following Newton's divided difference interpolations formula, we have

$$\begin{aligned} f[x_r, x_{r+1}] &= \frac{x_r[\mathsf{E}_{r+1} f(x)] - x_{r+1}[\mathsf{E}_r f(x)]}{x_{r+1}x_r[x_{r+1} - x_r]} \\ &= \frac{x_r \alpha_{\theta_{r+1}} - x_{r+1}\alpha_{\theta_r}}{x_{r+1}x_r[x_{r+1} - x_r]} \\ &= \frac{\alpha_{\theta_r}}{x_r(x_{r+1} - x_r)} + \frac{\alpha_{\theta_{r+1}}}{x_{r+1}(x_r - x_{r+1})} \\ &= \sum \frac{\alpha_{\theta_r}}{x_r(x_{r+1} - x_r)}, \end{aligned}$$

similarly, we have the $(s-1)$th divided differences

$$f[x_0, x_1, \ldots, x_{s-1}] = \sum \frac{\alpha_{\theta_0}}{x_0(x_0 - x_1)\cdots(x_0 - x_{s-1})}. \quad (7)$$

Theorem 1 *The $(s-1)$th divided differences can be expressed as the quotient of two determinants of order s each.*

Proof. From equation (7), we can write

$$f[x_0, x_1, \ldots, x_{s-1}] = \frac{\sum \alpha_{\theta_0} \prod_{1 \le j < i \le s-1}(x_j - x_i)}{(x_0 x_1 \cdots x_{s-1}) \prod_{0 \le j < i \le s-1}(x_j - x_i)},$$

by Vandermonde determinant,

$$\begin{aligned} & f[x_0, x_1, \ldots, x_{s-1}] \\ &= \frac{\sum \left(\alpha_{\theta_0} \begin{vmatrix} 1 & x_1 & \cdots & x_1^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{s-1} & \cdots & x_{s-1}^{s-1} \end{vmatrix} \right)}{(x_0 x_1 \cdots x_{s-1}) \begin{vmatrix} 1 & x_0 & \cdots & x_0^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{s-1} & \cdots & x_{s-1}^{s-1} \end{vmatrix}} \\ &= \left(\frac{1}{\prod_{i=0}^{s-1} x_i} \right) \frac{\begin{vmatrix} \alpha_{\theta_0} & 1 & x_0 & \cdots & x_0^{s-1} \\ \alpha_{\theta_1} & 1 & x_1 & \cdots & x_1^{s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{\theta_{s-1}} & 1 & x_{s-1} & \cdots & x_{s-1}^{s-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & \cdots & x_0^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{s-1} & \cdots & x_{s-1}^{s-1} \end{vmatrix}}. \end{aligned}$$

The required interpolating polynomial satisfying the given integral conditions is constructed similar to Newton's divided difference formula as follows

$$\begin{aligned} \widetilde{\widetilde{f}}_s(x) &= \frac{\alpha_{\theta_0}}{x_0} + (x - x_0) \sum \frac{\alpha_{\theta_0}}{x_0(x_1 - x_0)} \\ &\quad + (x - x_0)(x - x_1) \sum \frac{\alpha_{\theta_0}}{x_0(x_0 - x_1)(x_0 - x_1)} \\ &\quad + \cdots \\ &\quad + (x - x_0)(x - x_1)\cdots(x - x_{s-2}) \\ &\qquad \times \sum \frac{\alpha_{\theta_0}}{x_0(x_0 - x_1)(x_0 - x_1)\cdots(x - x_{s-1})} \\ &= \widetilde{a}_0 + \widetilde{a}_1 x + \cdots + \widetilde{a}_{s-1} x^{s-1}. \end{aligned} \quad (8)$$

Hence, we have $\widetilde{f}_s(x) = a_0 + a_1 x + \cdots + a_{s-1}x^{s-1}$, where coefficients a_i, for $i = 0, \ldots, s-1$, are calculated as $a_i = (i+1)\widetilde{a}_i$.

The following theorem gives generalization of above formulation.

Theorem 2 *Let $\Theta = \{\theta_0, \ldots, \theta_{s-1}\}$ be a finite set of the form $\{\mathsf{E}_c A : c \in \mathbb{R}\} \subset \mathscr{S}^*$ with associated values $\Omega = \{\alpha_{\theta_i} : \theta_i \in \Theta\} \subset \mathbb{R}$. Then there exists a unique polynomial $\widetilde{f}_s(x) = a_0 + a_1 x + \cdots + a_s x^s$ satisfying Θ if and only if the evaluation matrix $\mathscr{E}$ is regular, and $\widetilde{f}_s(x)$ is given by,*

$$\widetilde{f}_s(x) = a_0 + (2\widetilde{a}_1)x + \cdots + (s\widetilde{a}_{s-1})x^{s-1}, \quad (9)$$

where $\widetilde{a}_i = \frac{a_i}{i+1}$.

To verify the algorithm in Theorem 2, we provide a test example to construct a polynomial interpolation with integral conditions.

Example 3. Consider the conditions $\Theta = \{\mathsf{E}_1 \mathsf{A}, \mathsf{E}_{\frac{3}{2}} \mathsf{A}, \mathsf{E}_{\frac{5}{3}} \mathsf{A}, \mathsf{E}_2 \mathsf{A}\}$ with $\Omega = \{2, \frac{1}{2}, 0, -1\}$. Now we construct $\widetilde{f}_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that $\widetilde{f}_4(x)$ satisfies the given conditions Θ. From Theorem 2, we can compute a polynomial

$$\widetilde{\widetilde{f}}_4(x) = \frac{65}{6} - 14x + \frac{37}{6}x^2 - x^3,$$

and hence the required interpolating polynomial for (Θ, Ω) is

$$\widetilde{f}_4(x) = \frac{65}{6} - 28x + \frac{37}{2}x^2 - 4x^3.$$

One can easily check that $\Theta(\widetilde{f}_4) = \Omega$.

If we choose $\Omega = \{c_0, c_1, c_2, c_3\}$, then we have

$$\begin{aligned} \widetilde{\widetilde{f}}_4(x) &= \left(15c_0 - \frac{160}{3}c_1 + \frac{243}{5}c_2 - \frac{15}{2}c_3\right) \\ &\quad + \left(-\frac{53}{2}c_0 + 112c_1 - \frac{1053}{10}c_2 + 17c_3\right)x \\ &\quad + \left(\frac{31}{2}c_0 - \frac{224}{3}c_1 + \frac{729}{10}c_2 - \frac{25}{2}c_3\right)x^2 \\ &\quad + \left(-3c_0 + 16c_1 - \frac{18}{5}c_2 + 3c_3\right)x^3, \end{aligned}$$

and hence, we have

$$\begin{aligned}\widetilde{f}_4(x) &= \left(15-53x+\frac{93}{2}x^2-12x^3\right)c_0\\ &+\left(-\frac{160}{3}+224x-224x^2+64x^3\right)c_1\\ &+\left(\frac{243}{5}-\frac{1053}{5}x+\frac{2187}{10}x^2-\frac{324}{5}x^3\right)c_2\\ &+\left(-\frac{15}{2}+34x-\frac{72}{2}x^2+12x^3\right)c_3.\end{aligned}$$

Now, we present another algorithm to find a polynomial that satisfies the given Stieltjes conditions, i.e. not only on the function values and its derivatives, but also for arbitrary integral conditions, as a quotient of two determinants in the following theorem. This quotient represents a particular case of general remainder theorem [14, p. 75].

Theorem 3 *Let $\Theta = \{E_{x_0},\cdots,E_{x_{l-1}}, E_{x_l}D, E_{x_{l+1}}D,\cdots,E_{x_{l+m-1}}D, E_{x_{l+m}}A, E_{x_{l+m+1}}A,\ldots,E_{x_{s-1}}A : x_i\in\mathbb{R}; 0<l<l+m<s-1; l,m\in\mathbb{N}\}\subset\mathscr{S}^*$ with associated values $\Sigma=\{\alpha_{\theta_i}:\theta_i\in\Theta\}\subset\mathbb{R}$. Then there exists a unique polynomial $\widetilde{f}_s(x)$ satisfying Θ if and only if the evaluation matrix $\mathscr{E}$ is regular, and $\widetilde{f}_s(x)$ is given by,*

$$\widetilde{f}_s(x) = 1-\frac{D_1}{D_2}, \tag{10}$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & x & x^2 & \cdots & x^{s-1}\\ \alpha_{\theta_0} & 1 & x_0 & x_0^2 & \cdots & x_0^s\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \alpha_{\theta_{l-1}} & 1 & x_{l-1} & x_{l-1}^2 & \cdots & x_{l-1}^s\\ \overline{\alpha}_{\theta_l} & 0 & x_l & 2x_l^2 & \cdots & (s-1)x_l^{s-1}\\ \overline{\alpha}_{\theta_{l+1}} & 0 & x_{l+1} & 2x_{l+1}^2 & \cdots & (s-1)x_{l+1}^{s-1}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \overline{\alpha}_{\theta_{l+m-1}} & 0 & x_{l+m-1} & 2x_{l+m-1}^2 & \cdots & (s-1)x_{l+m-1}^{s-1}\\ \widetilde{\alpha}_{\theta_{l+m}} & 1 & \frac{x_{l+m}}{2} & \frac{x_{l+m}^2}{3} & \cdots & \frac{x_{l+m}^{s-1}}{s}\\ \widetilde{\alpha}_{\theta_{l+m+1}} & 1 & \frac{x_{l+m+1}}{2} & \frac{x_{l+m+1}^2}{3} & \cdots & \frac{x_{l+m+1}^{s-1}}{s}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \widetilde{\alpha}_{\theta_{s-1}} & 1 & \frac{x_{s-1}}{2} & \frac{x_{s-1}^2}{3} & \cdots & \frac{x_{s-1}^{s-1}}{s} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^s\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{l-1} & x_{l-1}^2 & \cdots & x_{l-1}^s\\ 0 & x_l & 2x_l^2 & \cdots & (s-1)x_l^{s-1}\\ 0 & x_{l+1} & 2x_{l+1}^2 & \cdots & (s-1)x_{l+1}^{s-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & x_{l+m-1} & 2x_{l+m-1}^2 & \cdots & (s-1)x_{l+m-1}^{s-1}\\ 1 & \frac{x_{l+m}}{2} & \frac{x_{l+m}^2}{3} & \cdots & \frac{x_{l+m}^{s-1}}{s}\\ 1 & \frac{x_{l+m+1}}{2} & \frac{x_{l+m+1}^2}{3} & \cdots & \frac{x_{l+m+1}^{s-1}}{s}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \frac{x_{s-1}}{2} & \frac{x_{s-1}^2}{3} & \cdots & \frac{x_{s-1}^{s-1}}{s} \end{vmatrix}$$

and $\overline{\alpha}_{\theta_i}=x_i\alpha_{\theta_i}$, $\widetilde{\alpha}_{\theta_j}=\frac{\alpha_{\theta_j}}{x_j}$.
The denominator D_2 is non-zero, for the evaluation matrix is regular.

In the following example, we find a polynomial with Stieltjes conditions using the algorithm presented in Theorem 3.

Example 4. Consider the conditions $\Theta=\{\mathsf{E}_0,\mathsf{E}_{\frac{1}{2}},\mathsf{E}_1\mathsf{D},\mathsf{E}_{\frac{4}{3}}\mathsf{A},\mathsf{E}_2\mathsf{A}\}$ with $\Omega=\{1,2,0,3,-1\}$. By Theorem 3, we have

$$\widetilde{f}_5(x)=1-\frac{-\frac{136}{3645}+\frac{136}{3645}+\frac{28}{1215}x+\frac{167}{1944}x^2+\frac{1373}{14580}x^3-\frac{29}{243}x^4}{-\frac{136}{3645}},$$

after simplification, we have

$$\widetilde{f}_5(x)=1+\frac{21}{34}x+\frac{2505}{1088}x^2+\frac{1373}{544}x^3-\frac{435}{136}x^4.$$

If we choose $\Omega=\{c_0,c_1,c_2,c_3,c_4\}$, then we get

$$\begin{aligned}\widetilde{f}_5(x) &= \left(1-\frac{39}{17}x+\frac{9}{272}x^2+\frac{181}{136}x^3-\frac{15}{34}x^4\right)c_0\\ &+\left(\frac{384}{17}x-\frac{1104}{17}x^2+\frac{928}{17}x^3-\frac{240}{17}x^4\right)c_1\\ &+\left(\frac{62}{17}x-\frac{849}{68}x^2+\frac{413}{34}x^3-\frac{60}{17}x^4\right)c_2\\ &+\left(-\frac{243}{17}x+\frac{48843}{1088}x^2-\frac{20169}{544}x^3+\frac{1215}{136}x^4\right)c_3\\ &+\left(-\frac{21}{34}x+\frac{687}{272}x^2-\frac{441}{136}x^3+\frac{45}{34}x^4\right)c_4.\end{aligned}$$

3 Error estimation

The error involved due to the approximation of a function $f(x)$ by a polynomial interpolant $\widetilde{f}_s(x)$ is calculated as follows

$$E_s(f,\Theta)=f(x)-\widetilde{f}_s(x). \tag{11}$$

3.1 Error term as a quotient of determinants

The error term in equation (11) can be represented by a quotient of determinants as given in equation (10) for integral conditions, i.e.

$$E_s(f,\Theta) = \frac{\begin{vmatrix} f(x) & 1 & x & \cdots & x^{s-1} \\ \alpha_{\theta_0} & x_0 & \frac{x_0}{2} & \cdots & \frac{x_0^s}{s} \\ \alpha_{\theta_1} & x_1 & \frac{x_1}{2} & \cdots & \frac{x_1^s}{s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{\theta_{s-1}} & x_{s-1} & \frac{x_{s-1}}{2} & \cdots & \frac{x_{s-1}^s}{s} \end{vmatrix}}{\begin{vmatrix} x_0 & \frac{x_0}{2} & \cdots & \frac{x_0^s}{s} \\ x_1 & \frac{x_1}{2} & \cdots & \frac{x_1^s}{s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{s-1} & \frac{x_{s-1}}{2} & \cdots & \frac{x_{s-1}^s}{s} \end{vmatrix}}.$$

In other words, the quotient is a linear combination of $\{f(x),1,\ldots,x^{s-1}\}$ in which the coefficient of $f(x)$ is 1.

In the following section, we provide a symbolic formulation of the error $E_s(f,\Theta)$ over integro-differential algebras.

3.2 Symbolic formulation of error estimation

To formulate the error term $E_s(f,\Theta)$ as in equation (11) in symbolic form, we first choose a differential operator T, whose fundamental system is $\{1,x,\ldots,x^{s-1}\}$. Such differential operator T is of the form

$$T = \frac{d^{s+1}}{dx^{s+1}}. \tag{12}$$

The polynomial interpolant satisfies the differential equation

$$T\widetilde{f} = 0,$$
$$\Theta\widetilde{f_s} = \Theta f.$$

Therefore, we have *semi-inhomogeneous boundary value problem* (inhomogeneous differential equation and homogeneous boundary conditions) as follows

$$TE_s(f,\Theta) = Tf, \tag{13}$$
$$\Theta E_s(f,\Theta) = 0.$$

Using the symbolic analysis for boundary value problems developed by Rosenkranz et al. [8], we have following symbolic solution for the boundary value problem given in equation (13) over integro-differential algebras

$$E_s(f,\Theta) = (1-P)T^{\Diamond}Tf, \tag{14}$$

where $T^{\Diamond} \in \mathscr{S}[\mathsf{D},\mathsf{A}]$ is the fundamental right inverse of T, computed from the given fundamental system, i.e. $TT^{\Diamond} = 1$ and $T^{\Diamond}T = 1-P$; and $P \in \mathscr{S}[\mathsf{D},\mathsf{A}]$ is the projector onto $\mathrm{Ker}(T)$ along $\Theta^{\perp}$, computed as

$$P = \sum_{i=0}^{s-1} x^i \widetilde{\theta}_i, \tag{15}$$

here $(\widetilde{\theta}_0,\widetilde{\theta}_1,\ldots,\widetilde{\theta}_{s-1})^T = \mathscr{E}^{-1}(\theta_0,\theta_1,\ldots,\theta_{s-1})^T$.

Simplifying the equation (14) using the fact that $1-P$ is projector, we get

$$E_s(f,\Theta) = (1-P)f,$$

as an operator to calculate the error, we have $E_s = 1-P \in \mathscr{S}[\mathsf{D},\mathsf{A}]$, P is a projector computed as in equation (15). The following theorem gives the generalization of above observation.

Theorem 4*Given any finite set Θ of Stieltjes conditions such that the evaluation matrix is regular, the error induced due to the approximation of a function $f(x)$ by a polynomial interpolation is computed as*

$$E_s(f,\Theta) = (1-P)f(x),$$

where P is the projector operator as given in equation (15).

The following two examples 5 and 6 show the computation of the error evaluate operator $E = 1-P$ and errors at various points on the interval $[1,2]$ with integral and Stieltjes conditions respectively.

*Example 5.*Consider the data of integral conditions of the form $\Theta = \{\mathsf{E}_1\mathsf{A},\mathsf{E}_{1.2}\mathsf{A},\mathsf{E}_{1.5}\mathsf{A},\mathsf{E}_2\mathsf{A}\}$ at arbitrary points from the interval $[1,2]$. Then by equation (15) the projector operator is given by

$$\begin{aligned} P =& (36-144x+141x^2-40x^3)\mathsf{E}_1\mathsf{A} \\ &+(-52.08+225.69x-234.38x^2+69.44x^3)\mathsf{E}_{1.2}\mathsf{A} \\ &+(21.33-99.56x+112x^2-35.56x^3)\mathsf{E}_{1.5}\mathsf{A} \\ &+(-2.25+11.25x-13.88x^2+5x^3)\mathsf{E}_2\mathsf{A}. \end{aligned}$$

For simplicity, if $f(x) = e^x$, then using the algorithm in Theorem 4, the error involved due to the approximation of $f(x)$ by the polynomial interpolation is

$$\begin{aligned} E_4(f,\Theta) &= (1-P)f(x) \\ &= e^x - 0.919 - 1.46x + 0.197x^2 - 0.540x^3. \end{aligned}$$

Error at $x = 1.4$ is calculated as $E_4(f(1.4),\Theta) = -6.152\times 10^{-4}$. Similarly $E_4(f(1.9),\Theta) = 6.209\times 10^{-3}$.
If $f(x) = \sin(0.2x)e^{0.6x}$, then

$$\begin{aligned} E_4(f,\Theta) &= (1-P)f(x) \\ &= \sin(0.2x)e^{0.6x} + 0.00856 \\ &\quad -0.249x - 0.042x^2 - 0.0789x^3. \end{aligned}$$

*Example 6.*Consider a set of Stieltjes conditions of the form $\Theta = \{\mathsf{E}_1, \mathsf{E}_{1.2}\mathsf{D}, \mathsf{E}_{1.5}\mathsf{A}, \mathsf{E}_2\}$ at arbitrary points of $[1,2]$. Then error evaluate operator, computed similar to Example 5, is given by

$$\begin{aligned} E_4 = 1 &- (-3.244 + 8.169x - 4.577x^2 + 0.651x^3)\mathsf{E}_1 \\ &- (0.489 - 3.339x + 4.153x^2 - 1.303x^3)\mathsf{E}_{1.2}\mathsf{D} \\ &- (3.058 - 6.671x + 4.656x^2 - 1.042x^3)\mathsf{E}_{1.5}\mathsf{A} \\ &- (-0.342 + 1.837x - 2.407x^2 + 0.912x^3)\mathsf{E}_2. \end{aligned}$$

If $f(x) = \cos(0.4x)$, then error evaluate function is

$$E_4(f,\Theta) = \cos(0.4x) - 1.108 + 2.023x - 2.594x^2 + 0.706x^3.$$

Using this error evaluate function, we can compute errors at various points of the interval $[1,2]$.

4 Conclusion

In this paper, we developed an algorithm to construct a polynomial interpolation with a finite set of integral conditions, also discussed an algorithm to compute the polynomial interpolation with Stieltjes conditions as a quotient of two determinants using integro-differential operators. The symbolic formation of the error involved due to the approximation of a given function by the proposed interpolation is discussed. Several examples are presented to illustrate the proposed algorithms.

References

[1] A. Chakrabarti and Hamsapriye, Derivation of a general mixed interpolation formula, J. Comput. Appl. Math., Vol. 70, pp. 161–172 (1996).
[2] H. de Meyer, J. Vanthournout and G. Vanden Berghe, On a new type of mixed interpolation, J. Comput. Appl. Math. Vol. 30 pp. 55–69 (1990).
[3] J. P. Coleman, Mixed interpolation methods with arbitrary nodes, Journal of Computational and Applied Mathematics, Vol. 92, pp. 69–83 (1998).
[4] R. C. Brown and A. M. Krall, Ordinary differential operators under Stieltjes boundary conditions, Trans. American Mathematical Society, Vol. 198, pp. 73–92 (1974).
[5] R. A. Lorentz, Multivariate Hermite interpolation by algebraic polynomials: A survey, Journal of Computational and Applied Mathematics, Vol. 122, pp. 167–201 (2000).
[6] S. Thota and S. D. Kumar, On a mixed interpolation with integral conditions at arbitrary nodes, Cogent Mathematics, 3:115161 (2016).
[7] T. Sauer, Polynomial interpolation, ideals and approximation order of refinable functions, Proc. Amer. Math. Soc., (1997).
[8] M. Rosenkranz, G. Regensburger, L. Tec, and B. Buchberger, Symbolic analysis for boundary problems: From rewriting to parametrized Grbner bases. In: U. Langer and P. Paule (eds.), Numerical and Symbolic Scientific Computing: Progress and Prospects, Texts and Monographs in Symbolic Computation. Springer, Wien Heidelberg London New-York (2011).
[9] S. Thota and S. D. Kumar, Symbolic method for polynomial interpolation with Stieltjes conditions, Proceedings of International Conference on Frontiers in Mathematics, pp. 225–228 (2015).
[10] S. Thota and S. D. Kumar, Solving system of higher-order linear differential equations on the level of operators, International Journal of Pure and Applied Mathematics, Vol. 106, No. 1, pp. 11–21 (2016).
[11] S. Thota, On a new symbolic method for initial value problems for systems of higher-order linear differential equations, International Journal of Mathematical Models and Methods in Applied Sciences, Vol. 12, pp. 194–202 (2018).
[12] S. Thota and S. D. Kumar, Symbolic algorithm for a system of differential-algebraic equations, Kyungpook Mathematical Journal, Vol. 56, No. 4, pp. 1141–1160 (2016).
[13] S. Thota and S. D. Kumar, A new method for general solution of system of higher-order linear differential equations, International Conference on Inter Disciplinary Research in Engineering and Technology, Vol. 1, pp. 240–243 (2015).
[14] P. J. Davis, Interpolation and Approximation, Blaisdell, Waltham, MA. (1963).

Srinivasarao Thota completed his M.Sc. in Mathematics from Indian Institute of Technology Madras, India and Ph.D. in Mathematics from Motilal Nehru National Institute of Technology Allahabad, India. Srinivasarao Thota's area of research interest is Computer Algebra, precisely symbolic methods for differential equations. Presently Working as Assistant Professor at Department of Applied Mathematics, Adama Science and Technology University, Ethiopia.