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Generalized Fuzzy Soft Continuity

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Abstract: In this paper we introduce the concept of generalized fuzzy soft mappings on families of generalized fuzzy soft sets and study the properties of generalized fuzzy soft images (inverse images) of generalized fuzzy soft sets. Furthermore, generalized fuzzy soft continuous mappings, generalized fuzzy soft open (closed) mappings and generalized fuzzy soft homeomorphisms are introduced.

Keywords: Soft set, fuzzy soft set, generalized fuzzy soft set, generalized fuzzy soft topology, generalized fuzzy soft mapping, generalized fuzzy soft continuity, generalized fuzzy soft open (closed) mapping

1 Introduction

The concept of soft sets was first introduced by Molodtsov [1] as a general mathematical tool for dealing with uncertain objects. Maji et al.[2] introduced the concept of fuzzy soft set and some of its properties. Tanay and Kandemir [3] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [4] gave the definition of fuzzy soft topology over the initial universe set. Majumdar and Samanta [5] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and studied some of its basic properties. Chakraborty and Mukherjee [6] gave the topological structure of generalized fuzzy soft sets. Kharal and Ahmad [7,8] defined the notion of a mapping on classes of soft (fuzzy soft) sets.

In this paper, we define the notion of mappings on families of generalized fuzzy soft sets. We also define and study the properties of generalized fuzzy soft images (inverse images) of generalized fuzzy soft sets, and support them with examples and counterexamples. Also we introduce generalized fuzzy soft continuity of mappings. Furthermore, we use the notion generalized soft quasi-coincidence to characterize fundamental concepts of generalized fuzzy soft topological spaces such as generalized fuzzy soft closures and generalized fuzzy soft continuity. Finally, generalized fuzzy soft open (closed) mappings and generalized fuzzy soft homeomorphism for generalized fuzzy soft topological spaces are investigated.

2 Preliminaries

First we recall basic definitions and results.

Definition 2.1. ([9]) Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the 'grade of membership' of x in A for $x \in X$. The set of all fuzzy sets in a set X is denoted by I^X, where I is the closed unit interval $[0,1]$.

Theorem 2.2. ([9]) If $A, B \in I^X$, then, we have:
(1) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \ \forall x \in X$.
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \ \forall x \in X$.
(3) $C = A \vee B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X$.
(4) $D = A \wedge B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X$.
(5) $E = A^C \Leftrightarrow \mu_E(x) = 1 - \mu_A(x), \forall x \in X$.

Definition 2.3. ([1]) Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subseteq E$. A pair (f,A) is called a soft set over X if f is a mapping from A into $P(X)$, i.e., $f : A \longrightarrow P(X)$. In other words, a soft set is a parameterized family of subsets of the set X. For $e \in A$, $f(e)$ may be considered as the set of $e-$approximate elements of the soft set (f,A).

Definition 2.4. ([4]) Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq \underline{0}$ if $e \in A \subset E$, and $f_A(e) = \underline{0}$ if $e \notin A$, where $\underline{0}$ denoted empty fuzzy set in X

Definition 2.5. ([5]) Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \longrightarrow$

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I^X and μ be a fuzzy subset of E, i.e., $\mu : E \longrightarrow I$. Let F_μ be the mapping $F_\mu : E \longrightarrow I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalised fuzzy soft set ($GFSS$ in short) over (X,E).

Definition 2.6. ([5]) Let F_μ and G_δ be two $GFSSs$ over (X,E). F_μ is said to be a GFS subset of G_δ or G_δ is said to be a GFS super set of F_μ, denoted by $F_\mu \sqsubseteq G_\delta$, if

(1) μ is a fuzzy subset of δ;

(2) $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.

Definition 2.7. ([5]) Let F_μ be a $GFSS$ over (X,E). The complement of F_μ, denoted by F^c_μ, is defined by $F^c_\mu = G_\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e)$, $\forall e \in E$. Obviously $(F^c_\mu)^c = F_\mu$.

Definition 2.8. ([6]) Let F_μ and G_δ be two $GFSSs$ over (X,E). The union of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is The $GFSSH_\nu$, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e)$, $\forall e \in E$.

Let $\{(F_\mu)_\lambda, \lambda \in \Lambda\}$, where Λ is an index set, be a family of $GFSSs$. The union of these family, denoted by $\sqcup_{\lambda \in \Lambda}(F_\mu)_\lambda$, is The $GFSS$ H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = \bigvee_{\lambda \in \Lambda}(F(e))_\lambda$, and $\nu(e) = \bigvee_{\lambda \in \Lambda}(\mu(e))_\lambda$, $\forall e \in E$.

Definition 2.9. ([6]) Let F_μ and G_δ be two $GFSSs$ over (X,E). The Intersection of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is the $GFSS$ M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.

Let $\{(F_\mu)_\lambda, \lambda \in \Lambda\}$, where Λ is an index set, be a family of $GFSSs$. The Intersection of these family, denoted by $\sqcap_{\lambda \in \Lambda}(F_\mu)_\lambda$, is the $GFSS$ M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = \bigwedge_{\lambda \in \Lambda}(F(e))_\lambda$, and $\sigma(e) = \bigwedge_{\lambda \in \Lambda}(\mu(e))_\lambda$, $\forall e \in E$.

Definition 2.10. ([5]) A$GFSS$ is said to be a generalized null fuzzy soft set, denoted by $\widetilde{0}_\theta$, if $\widetilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\widetilde{0}_\theta(e) = (\widetilde{0}(e), \theta(e))$ where $\widetilde{0}(e) = \overline{0}$ $\forall e \in E$ and $\theta(e) = 0$ $\forall e \in E$ (Where $\overline{0}(x) = 0$, $\forall x \in X$).

Definition 2.11. ([5]) A $GFSS$ is said to be a generalized absolute fuzzy soft set, denoted by $\widetilde{1}_\triangle$, if $\widetilde{1}_\triangle : E \longrightarrow I^X \times I$, where $\widetilde{1}_\triangle(e) = (\widetilde{1}(e), \triangle(e))$ is defined by $\widetilde{1}(e) = \overline{1}, \forall e \in E$ and $\triangle(e) = 1, \forall e \in E$ (Where $\overline{1}(x) = 1, \forall x \in X$).

Definition 2.12. ([6]) Let T be a collection of generalized fuzzy soft sets over (X,E). Then T is said to be a generalized fuzzy soft topology ($GFST$, in short) over (X,E) if the following conditions are satisfied:

(1) $\widetilde{0}_\theta$ and $\widetilde{1}_\triangle$ are in T.

(2) Arbitrary unions of members of T belong to T.

(3) Finite intersections of members of T belong to T.

The triplet (X,T,E) is called a generalized fuzzy soft topological space ($GFST$- space, in short) over (X,E). The members of T are called GFS open sets in (X,T,E). and complements of them are called a GFS- closed sets in (X,T,E). The family of all GFS- closed sets in (X,T,E) is denoted by T'.

Definition 2.13. ([6]) Let (X,T,E) be a $GFST$-space and F_μ be a $GFSS$ over (X,E). Then the generalized fuzzy soft closure of F_μ, denoted by $\overline{F_\mu}$, is the intersection of all GFS- closed supper sets of F_μ. Clearly, $\overline{F_\mu}$ is the smallest GFS- closed set over (X,E) which contains F_μ.

Definition 2.14. ([6]) A $GFSS$ F_μ in a $GFST$-space (X,T,E) is called a generalized fuzzy soft neighborhood [GFS-nbd, in short] of the $GFSS$ G_δ if there exists a GFS open set H_ν such that $G_\delta \sqsubseteq H_\nu \sqsubseteq F_\mu$.

Definition 2.15. ([6]) Let (X,T,E) be a $GFST$-space and F_μ be a $GFSS$ over (X,E). Then the generalized fuzzy soft interior of F_μ, denoted by F°_μ, is the union of all GFS open subsets of F_μ. Clearly, F°_μ is the largest GFS open set over (X,E) which is contained in F_μ.

Definition 2.16. ([10]) The generalized fuzzy soft set $F_\mu \in GFS(X,E)$ is called a generalized fuzzy soft point (GFS point in short) if there exists the element $e \in E$ and $x \in X$ such that $F(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $F(e)(y) = 0$ for all $y \in X - \{x\}$ and $\mu(e) = \lambda$ $(0 < \lambda \leq 1)$. We denote this generalized fuzzy soft point $F_\mu = (x_\alpha, e_\lambda)$.

(x,e) and (α,λ) are called respectively, the support and the value of (x_α, e_λ).

Definition 2.17. ([11]) For any two $GFSSs$ F_μ and G_δ over (X,E). F_μ is said to be a generalised soft quasi-coincident with G_δ, denoted by $F_\mu q G_\delta$, if there exist $e \in E$ and $x \in X$ such that $F(e)(x) + G(e)(x) > 1$ and $\mu(e) + \delta(e) > 1$.

If F_μ is not generalised soft quasi-coincident with G_δ, then we write $F_\mu q G_\delta \Leftrightarrow$ For every $e \in E$ and $x \in X$, $F(e)(x) + G(e)(x) \leq 1$ or for every $e \in E$ and $x \in X$, $\mu(e) + \delta(e) \leq 1$.

Definition 2.18. ([11]) Let (x_α, e_λ) be a generalized fuzzy soft point and F_μ be a $GFSS$ over (X,E). (x_α, e_λ) is said to be generalised soft quasi-coincident with F_μ, denoted by $(x_\alpha, e_\lambda) q F_\mu$, if and only if there exists an element $e \in E$ such that $\alpha + F(e)(x) > 1$ and $\lambda + \mu(e) > 1$.

Definition 2.19. ([11]) Let F_μ and G_δ are $GFSSs$ over (X,E). Then the followings are hold:

(1) $F_\mu \sqsubseteq G_\delta \Leftrightarrow F_\mu \overline{q} (G_\delta)^c$;

(2) $F_\mu q G_\delta \Rightarrow F_\mu \sqcap G_\delta \neq \widetilde{0}_\theta$;

(3) $(x_\alpha, e_\lambda) \overline{q} F_\mu \Leftrightarrow (x_\alpha, e_\lambda) \widetilde{\in} (F_\mu)^c$;

(4) $F_\mu \overline{q} (F_\mu)^c$.

Theorem 2.20. ([6]) Let (X,T,E) be a $GFST$-space and F_μ be a $GFSS$ over (X,E). Then

(1) $(\overline{F_\mu})^c = (F^c_\mu)^\circ$;

(2) $(F^\circ_\mu)^c = \overline{(F^c_\mu)}$.

Definition 2.21. ([11]) Let (X,T,E) be a $GFST$-space. Let F_μ be a $GFSS$ over (X,E). Then the generalized fuzzy soft boundray of F_μ, denoted by F^b_μ, is defined as $F^b_\mu = \overline{F_\mu} \sqcap \overline{F^c_\mu}$. clearly, F^b_μ is the smsllest GFS closed set over (X,E) which contains F_μ.

Theorem 2.22. ([11]) Let (X,T,E) be a $GFST$-space. Let F_μ be a $GFSS$ over (X,E). Then

(1) $(F_\mu^b)^c = F_\mu^0 \sqcap (F_\mu^c)^0$.

(2) $F_\mu^b = \overline{F_\mu} \sqcap \overline{F_\mu^c} = \overline{F_\mu} \setminus F_\mu^0$.

Definition 2.23. ([8]) Let $FS(X,E)$ and $FS(Y,K)$ be the familes of all fuzzy soft sets over X and Y, respectivly. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be two functions. Then a mapping $f_{up} : FS(X,E) \longrightarrow FS(Y,K)$ is defined as follows: for a fuzzy soft set $f_A \in FS(X,E), \forall k \in K$ and $y \in Y$. Then

$$f_{up}(f_A)(k)(y) = \begin{cases} \bigvee_{x\in u^{-1}(y)}(\bigvee_{e\in p^{-1}(k)\cap A}) f_A(e))(x), \\ \qquad if u^{-1}(y) \neq \phi, p^{-1}(k)\cap A \neq \phi, \\ 0, \qquad \text{otherwise.} \end{cases}$$

$f_{up}(f_A)$ is called a fuzzy soft image of a fuzzy soft set f_A.

Definition 2.24. ([8]) Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings.

Let $f_{up} : FS(X,E) \longrightarrow FS(Y,K)$ be mapping and $g_B \in FS(Y,K)$. Then $f_{up}^{-1}(g_B)$, is a fuzzy soft set in $FS(X,E)$, defined by

$$f_{up}^{-1}(g_B)(e)(x) = g_B(p(e)(u(x))), \ \forall e \in E, x \in X.$$

$f_{up}^{-1}(G_\delta)$ is called a fuzzy soft inverse image of G_δ.

If u and p are injective then the fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the fuzzy soft mapping f_{up} is said to be surjective. The fuzzy soft mapping f_{up} is constant, if u and p are constant.

3 Generalized fuzzy soft mappings

Definition 3.1. Let $GFS(X,E)$ and $GFS(Y,K)$ be the familes of all $GFSSs$ over (X,E) and (Y,K), respectivly. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then a mapping $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ is defined as follows: for a $GFSSF_\mu \in GFS(X,E), \forall k \in K$ and $y \in Y$, then

$$f_{up}(F_\mu)(k)(y) = \begin{cases} (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} F(e)(x), \bigvee_{e\in p^{-1}(k)} \mu(e)), \\ \qquad if u^{-1}(y) \neq \phi, p^{-1}(k) \neq \phi, \\ (0,0), \qquad \text{otherwise.} \end{cases}$$

f_{up} is called a generalized fuzzy soft mapping [GFS mapping for short] and $f_{up}(F_\mu)$ is called the GFS image of a $GFSS$ F_μ.

Definition 3.2. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping and $G_\delta \in GFS(Y,K)$. Then $f_{up}^{-1}(G_\delta) \in GFS(X,E)$ is defined as follows:

$f_{up}^{-1}(G_\delta)(e)(x) = (G(p(e)(u(x)), \delta(p(e)),$ for $e \in E, x \in X$.

$f_{up}^{-1}(G_\delta)$ is called the GFS inverse image of G_δ.

If u and p are injective then the generalized fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the generalized fuzzy soft mapping f_{up} is said to be surjective. The generalized fuzzy soft mapping f_{up} is called constant, if u and p are constant.

Example 3.3.

Let $X = \{a,b,c\}, Y = \{x,y,z\}, E = \{e_1,e_2,e_3,e_4\}$ and $K = \{e_1^{'},e_2^{'},e_3^{'}\}$. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be tow mappings defined as

$$u(a) = z \qquad u(b) = y \qquad u(c) = y,$$

$$p(e_1) = e_1^{'} \quad p(e_2) = e_1^{'}, \quad p(e_3) = e_3^{'}, \quad p(e_4) = e_2^{'}.$$

Let $F_\mu \in GFS(X,E)$ and $G_\delta \in GFS(Y,K)$ where.

$F_\mu = \{(e_1 = \{\frac{a}{0.5}, \frac{b}{0.7}, \frac{c}{0.6}\}, 0.3),$
$(e_2 = \{\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.1}\}, 0.8), (e_3 = \{\frac{a}{0.9}, \frac{b}{0.1}, \frac{c}{0.5}\}, 0.1)\},$
$G_\delta = \{(e_1^{'} = \{\frac{x}{0.1}, \frac{y}{0.9}, \frac{z}{0.5}\}, 0.2),$
$(e_2^{'} = \{\frac{x}{0.4}, \frac{y}{0.8}, \frac{z}{0.6}\}, 0.4), (e_3^{'} = \{\frac{x}{0.5}, \frac{y}{0.9}, \frac{z}{0.6}\}, 0.8)\}.$

Then the GFS image of F_μ under $f_{pu} : GFS(X,E) \longrightarrow GFS(Y,K)$ is obtained as

$f_{up}(F_\mu)(e_1^{'})(x) = (\bigvee_{s\in u^{-1}(x)} \bigvee_{e\in p^{-1}(e_1^{'})} F(e)(s), \bigvee_{e\in p^{-1}(e_1^{'})} \mu(e))$
$= (0, \bigvee_{e\in\{e_1,e_2\}} \mu(e))$ (as $u^{-1}(x) = \phi$)
$= (0, \mu(e_1) \bigvee \mu(e_2))$
$= (0, 0.3 \bigvee 0.8) = (0, 0.8),$

$f_{up}(F_\mu)(e_1^{'})(y) = (\bigvee_{s\in u^{-1}(y)} \bigvee_{e\in p^{-1}(e_1^{'})} F(e)(s), \bigvee_{e\in p^{-1}(e_1^{'})} \mu(e))$
$= (\bigvee_{s\in\{b,c\}} \bigvee_{e\in\{e_1,e_2\}} F(e)(s), 0.8)$
$= (\bigvee_{s\in\{b,c\}}(F(e_1) \bigvee F(e_2))(s), 0.8)$
$= (\bigvee_{s\in\{b,c\}}(\{\frac{a}{0.5}, \frac{b}{0.7}, \frac{c}{0.6}\})(s), 0.8)$
$= (0.7 \bigvee 0.6, 0.8) = (0.7, 0.8)$

$f_{up}(F_\mu)(e_1^{'})(z) = (0.5, 0.8)$. By similar calculations, we get $f_{up}(F_\mu) = \{(e_1^{'} = \{\frac{x}{0}, \frac{y}{0.7}, \frac{z}{0.5}\}, 0.8), (e_2^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0.1), (e_3^{'} = \{\frac{x}{0}, \frac{y}{0.5}, \frac{z}{0.9}\}, 0)\}$. Next, for $p(e_i)$, $i = 1,2,3,4$, $p(e_i) \in p(E) = K$, we calculate

$f_{up}^{-1}(G_\delta)(e_1)(a) = (G(p(e_1))(u(a)), \delta(p(e_1)))$
$= (G(e_1^{'})(z), \delta(e_1^{'}))$
$= (\{\frac{x}{0,1}, \frac{y}{0.9}, \frac{z}{0.5}\}(z), 0.2))$
$= (0.5, 0.2),$

$f_{up}^{-1}(G_\delta)(e_1)(b) = (G(p(e_1))(u(b)), 0.2))$
$= (G(e_1^{'})(y), \delta(e_1^{'}))$
$= (\{\frac{x}{0,1}, \frac{y}{0.9}, \frac{z}{0.5}\}(y), 0.2)$
$= (0.9, 0.2),$

$f_{up}^{-1}(G_\delta)(e_1)(c) = (0.9, 0.2)$. By similar calculations, we get

$f_{up}^{-1}(G_\delta) = \{(e_1 = \{\frac{a}{0.5}, \frac{b}{0.9}, \frac{c}{0.9}\}, 0.2), (e_2 = \{\frac{a}{0.5}, \frac{b}{0.9}, \frac{c}{0.9}\}, 0.2), (e_3 = \{\frac{a}{0.6}, \frac{b}{0.9}, \frac{c}{0.9}\}, 0.8), (e_4 = \{\frac{a}{0.6}, \frac{b}{0.8}, \frac{c}{0.8}\}, 0.4)\}.$

Definition 3.4. Let $f_{u_1p_1} : GFS(X,E) \longrightarrow GFS(Y,K)$ and $g_{u_2p_2} : GFS(Y,K) \longrightarrow GFS(Z,D)$ be GFS mappings and $F_\mu \in GFS(X,E)$.

Then $g_{u_2p_2}$ o $f_{u_1p_1} : GFS(X,E) \longrightarrow GFS(Z,D)$ is GFS mapping defined as follows: $\forall d \in D, \forall z \in Z$, then

$(g_{u_2p_2}\ o\ f_{u_1p_1})(F_\mu)(d)(z)$

$$= \begin{cases} (\bigvee_{x\in(u_2\ o\ u_1)^{-1}(z)} \bigvee_{e\in(p_2\ o\ p_1)^{-1}(d)} F(e)(x), \\ \qquad \bigvee_{e\in(p_2\ o\ p_1)^{-1}(d)} \mu(e)), \\ if (u_2\ o\ u_1)^{-1}(z) \neq \phi, (p_2\ o\ p_1)^{-1}(d) \neq \phi, \\ (0,0), \qquad \text{otherwise.} \end{cases}$$

If $M_\sigma \in GFS(Z,D)$. Then $(g_{u_2p_2}\ o\ f_{u_1p_1})^{-1}(M_\sigma)$ is a $GFSS$ in $GFS(X,E)$, defined as follows: $\forall e \in E, \forall x \in X$.

$(g_{u_2p_2}\ o\ f_{u_1p_1})^{-1}(M_\sigma)(e)(x)$
$= (u_2\ o\ u_1, p_2\ o\ p_1)^{-1}(M_\sigma)(e)(x)$
$= (M(p_2(p_1(e)))(u_2(u_1(x))), \sigma(p_2(p_1(e))))$.

Proposition 3.5. Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping and $F_\mu, H_\nu \in GFS(X,E)$ and $G_\delta, M_\sigma \in GFS(Y,K)$. Then

(1) The generalized fuzzy soft union and generalized fuzzy soft intersection of generalized fuzzy soft images $f_{pu}(F_\mu)$ and $f_{up}(H_\nu)$ in $GFS(Y,K)$ are defined as

$(f_{up}(F_\mu) \sqcup f_{up}(H_\nu))(k)(y) = f_{up}(F_\mu)(k)(y) \bigvee f_{up}(H_\nu)(k)(y),$
$(f_{up}(F_\mu) \sqcap f_{up}(H_\nu))(k)(y)$
$= f_{up}(F_\mu)(k)(y) \bigwedge f_{up}(H_\nu)(k)(y). \forall k \in K, y \in Y.$

(2) The generalized fuzzy soft union and generalized fuzzy soft intersection of generalized fuzzy soft inverse images $f_{up}^{-1}(G_\delta)$ and $f_{up}^{-1}(M_\sigma)$ in $GFS(X,E)$ are defined as

$(f_{up}^{-1}(G_\delta) \sqcup f_{up}^{-1}(M_\sigma))(e)(x) = f_{pu}^{-1}(G_\delta)(e)(x) \bigvee f_{up}^{-1}(M_\sigma)(e)(x),$
$(f_{up}^{-1}(G_\delta) \sqcap f_{up}^{-1}(M_\sigma))(e)(x)$
$= f_{up}^{-1}(G_\delta)(e)(x) \bigwedge f_{up}^{-1}(M_\sigma)(e)(x). \forall e \in E, x \in X.$

Where $\sqcup$ and $\sqcap$ denoted generalized fuzzy soft union and generalized fuzzy soft intersection of generalized fuzzy soft images and generalized fuzzy soft inverse images in $GFS(X,E)$ and $GFS(Y,K)$, respectively.

Theorem 3.6 Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping. For $GFSSs$ F_μ and $H_\nu \in GFS(X,E)$, we have.

(1) $f_{up}(\widetilde{0}_{\theta_X}) = \widetilde{0}_{\theta_Y}$,
(2) $f_{up}(\widetilde{1}_{\triangle_X}) \sqsubseteq \widetilde{1}_{\triangle_Y}$,
(3) If $F_\mu \sqsubseteq H_\nu$, then $f_{up}(F_\mu) \sqsubseteq f_{up}(H_\nu)$,
(4) $f_{up}(F_\mu \sqcup H_\nu) = f_{up}(F_\mu) \sqcup f_{up}(H_\nu)$,
(5) $f_{up}(F_\mu \sqcap H_\nu) \sqsubseteq f_{up}(F_\mu) \sqcap f_{up}(H_\nu)$.

Proof (1) For $k \in K$ and $y \in Y$,
$f_{up}(\widetilde{0}_{\theta_X})(k)(y)$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} \widetilde{0}(e)(x), \bigvee_{e\in p^{-1}(k)} \theta_X(e))$
$= (0,0) = (\widetilde{0}(k)(y), \theta_Y(k)) = \widetilde{0}_{\theta_Y}(k)(y)$.

(2) For $k \in K$ and $y \in Y$,
$f_{up}(\widetilde{1}_{\triangle_X})(k)(y)$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} \widetilde{1}(e)(x), \bigvee_{e\in p^{-1}(k)} \triangle_X(e))$
$\leq (1,1) = (\widetilde{1}(k)(y), \theta(k) = \widetilde{1}_{\triangle_Y}(k)(y)$.

(3) Considering only the non-trival case, for $k \in K$ and $y \in Y$, and since $F_\mu \sqsubseteq H_\nu$, we have

$f_{up}(F_\mu)(k)(y)$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} F(e)(x), \bigvee_{e\in p^{-1}(k)} \mu(e))$
$\leq (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} H(e)(x), \bigvee_{e\in p^{-1}(k)} \nu(e))$
$= f_{up}(H_\nu)(k)(y)$

This give (3).

(4) For $k \in K$ and $y \in Y$, we show that
$f_{up}((F_\mu) \sqcup (H_\nu))(k)(y) = f_{up}(F_\mu)(k)(y) \bigvee f_{up}(H_\nu)(k)(y).$

Consider
$f_{up}(F_\mu \sqcup H_\nu)(k)(y) = f_{up}(M_\sigma)(k)(y) \qquad \text{(say)}$

$$= \begin{cases} (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} M(e)(x), \bigvee_{e\in p^{-1}(k)} \sigma(e)), \\ \qquad if\ u^{-1}(y) \neq \phi, p^{-1}(k) \neq \phi, \\ (0,0), \qquad \text{otherwise,} \end{cases}$$

whwer,
$M(e)(x) = F(e)(x) \bigvee H(e)(x)$ and $\sigma(e) = \mu(e) \bigvee \nu(e)$ for $e \in p^{-1}(k), x \in p^{-1}(y)$.

Considering only the non- trival case, we have
$f_{up}(F_\mu \sqcup H_\nu)(k)(y)$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} [F(e)(x) \bigvee H(e)(x)],$
$\bigvee_{e\in p^{-1}(k)} \mu(e) \bigvee \nu(e))$. (I)

By Proposition (3.5), we have
$(f_{up}(F_\mu) \sqcup f_{up}(H_\nu))(k)(y)$
$= f_{up}(F_\mu)(k)(y) \bigvee f_{up}(H_\nu)(k)(y)$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} F(e)(x), \bigvee_{e\in p^{-1}(k)} \mu(e)) \bigvee$
$(\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} H(e)(x), \bigvee_{e\in p^{-1}(k)} \nu(e))$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} [F(e)(x) \bigvee H(e)(x)],$
$\bigvee_{e\in p^{-1}(k)} \mu(e) \bigvee \nu(e))$. (II)

By (I) and (II) we have (4).

(5) For $k \in K$ and $y \in Y$, using Proposition(3.5) we have

$f_{up}(F_\mu \sqcap H_\nu)(k)(y) = f_{up}(M_\sigma)(k)(y), \qquad \text{(say)}$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} M(e)(x), \bigvee_{e\in p^{-1}(k)} \sigma(e)),$
$= (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} [F(e)(x) \bigwedge H(e)(x)],$
$\bigvee_{e\in p^{-1}(k)} \mu(e) \bigwedge \nu(e))$.
$\leq (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} F(e)(x), \bigvee_{e\in p^{-1}(k)} \mu(e)) \bigwedge$
$(\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} H(e)(x), \bigvee_{e\in p^{-1}(k)} \nu(e))$
$= f_{up}(F_\mu)(k)(y) \bigwedge f_{up}(H_\nu)(k)(y)$.
$= (f_{up}(F_\mu) \sqcap f_{up}(H_\nu))(k)(y)$

This give (5)

In Theorem 3.6, inequalities (2),(5) and implication(3) cannot be reversed in general, as shown in the following.

Example 3.7. Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping where

$X = \{a,b,c\}, Y = \{x,y,z\}, E = \{e_1,e_2,e_3,e_4\}$ and $K = \{e_1^{'}, e_2^{'}, e_3^{'}\}$. For (2) we define mappings $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ as

$u(a) = x \qquad u(b) = y \qquad u(c) = x,$
$p(e_1) = e_2^{'} \quad p(e_2) = e_1^{'}, \quad p(e_3) = e_2^{'}, \quad p(e_4) = e_1^{'}.$
$\widetilde{1}_{\triangle_Y} \not\sqsubseteq \{(e_1^{'} = \{\frac{x}{1}, \frac{y}{1}, \frac{z}{0}\}, 1), (e_2^{'} = \{\frac{x}{1}, \frac{y}{1}, \frac{z}{0}\}, 1),$
$(e_3^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0)\} = f_{pu}(\widetilde{1}_{\triangle_X})$.

For (3) and (5), define mapping $u : X \longrightarrow Y$and $p : E \longrightarrow K$ as

$u(a) = y \quad u(b) = y \quad u(c) = y$, $p(e_1) = e_2^{'}$, $p(e_2) = e_1^{'}, p(e_3) = e_2^{'}$, $p(e_4) = e_1^{'}$. Choose two generalized fuzzy soft sets in $GFS(X,E)$ as $F_\mu = \{(e_3 = \{\frac{a}{0.3}, \frac{b}{0.7}, \frac{c}{0.5}\}, 0.2)\}, H_\nu = \{(e_3 = \{\frac{a}{0.5}, \frac{b}{0.1}, \frac{c}{1}\}, 0.3)\}$. Then the calculations give $f_{up}(F_\mu) = \{(e_1^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0), (e_2^{'} = \{\frac{x}{0}, \frac{y}{7}, \frac{z}{0}\}, 0.2), (e_3^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0)\} \sqsubseteq \{(e_1^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0), (e_2^{'} = \{\frac{x}{0}, \frac{y}{1}, \frac{z}{0}\}, 0.3), (e_3^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0)\} = f_{up}(H_\nu)$ but $F_\mu \not\sqsubseteq H_\nu$. Also, we have $f_{up}(F_\mu) \sqcap f_{up}(H_\nu) = \{(e_1^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0), (e_2^{'} = \{\frac{x}{0}, \frac{y}{7}, \frac{z}{0}\}, 0.2), (e_3^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0)\} \not\sqsubseteq \{(e_1^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0), (e_2^{'} = \{\frac{x}{0}, \frac{y}{5}, \frac{z}{0}\}, 0.2), (e_3^{'} = \{\frac{x}{0}, \frac{y}{0}, \frac{z}{0}\}, 0)\} = f_{up}(F_\mu \sqcap H_\nu)$.

Theorem 3.8. Let $F_\mu \in GFS(X,E)$, $\{F_\mu\}_{i\in J} \subset GFS(X,E)$ where J is an index set.

(1) $f_{pu}(\sqcup_{i\in J}(F_\mu)_i) = \sqcup_{i\in J} f_{up}(F_\mu)_i$.

(2) $f_{up}(\sqcap_{i\in J}(F_\mu)_i) = \sqcap_{i\in J} f_{up}(F_\mu)_i$, if f_{up} is injective.

(2) $f_{up}(\widetilde{1}_{\triangle X}) = \widetilde{1}_{\triangle Y}$, if f_{up} is surjective.

Proof The straightforward proof is omitted.

Theorem 3.9. Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping. For $GFSSs$ G_δ, J_σ and $(G_\delta)_i \in GFS(Y,K)$ $\forall i \in J$, where J is an index set, we have.

(1) $f_{up}^{-1}(\widetilde{0}_{\theta_Y}) = \widetilde{0}_{\theta_X}$,

(2) $f_{up}^{-1}(\widetilde{1}_{\triangle_Y}) = \widetilde{1}_{\triangle_X}$,

(3) If $G_\delta \sqsubseteq J_\sigma$. Then $f_{up}^{-1}(G_\delta) \sqsubseteq f_{up}^{-1}(J_\sigma)$,

(4) $f_{up}^{-1}(G_\delta \sqcup J_\sigma) = f_{up}^{-1}(G_\delta) \sqcup f_{up}^{-1}(J_\sigma)$. In general, $f_{up}^{-1}(\sqcup_{i\in J}(G_\delta)_i) = \sqcup_{i\in J} f_{up}^{-1}(G_\delta)_i$,

(5) $f_{up}^{-1}(G_\delta \sqcap J_\sigma) = f_{up}^{-1}(G_\delta) \sqcap f_{up}^{-1}(J_\sigma)$. In general, $f_{up}^{-1}(\sqcap_{i\in J} G_\delta)_i) = \sqcap_{i\in J} f_{up}^{-1}(G_\delta)_i$.

Proof (1) $f_{up}^{-1}(\widetilde{0}_{\theta_Y})(e)(x)$
$= (\widetilde{0}(p(e)(u(x)), \theta_Y(p(e))$
$= (0,0) = \widetilde{0}_{\theta_X}(e)(x)$, $\forall e \in E, x \in X$.

(2) $f_{up}^{-1}(\widetilde{1}_{\triangle_Y}) = \widetilde{1}_{\triangle_X}$,
$= (1,1) = \widetilde{1}_{\triangle X}(e)(x)$, $\forall e \in E, x \in X$.

(3) Since $G_\delta \sqsubseteq J_\sigma$, we have $f_{up}^{-1}(G_\delta)(e)(x)$
$= (G(p(e))(u(x)), \delta(p(e))$
$= (G(k)(u(x)), \delta(k), k \in K$
$\leq (J(k)(u(x)), \sigma(k)$
$= f_{up}^{-1}(J_\sigma)(e)(x)$. (4) For $e \in E$ and $x \in X$, we have
$f_{up}^{-1}(G_\delta \sqcup J_\sigma)(e)(x)$
$= f_{up}^{-1}(N_\psi)(e)(x)$
$= (N(p(e))(u(x)), \psi(p(e))$
$= (N(k)(u(x)), \psi(p(e)), \quad p(e) \in K, u(x) \in Y$
$= (N(k)(u(x)), \psi(k)$, where
$k = p(e) = ((G(k) \bigvee J(k))(u(x)), (\delta \bigvee \sigma)(k)$
$= (G(k)(u(x)) \bigvee J(k)(u(x)), \delta(k) \bigvee \sigma(k)$. (I)

Next, using Proposition (3.5), we get
$[f_{up}^{-1}(G_\delta) \sqcup f_{up}^{-1}(J_\sigma)](e)(x)$
$= f_{up}^{-1}(G_\delta)(e)(x) \bigvee f_{up}^{-1}(J_\sigma)(e)(x)$
$= (G(p(e))(u(x)), \delta(p(e) \bigvee (J(p(e))(u(x)), \sigma(p(e))$
$= (G(k)(u(x)) \bigvee J(k)(u(x)), \delta(k) \bigvee \sigma(k)$. $\quad (II)$

From (I) and (II), we get (4).

(5) For $e \in E$, $x \in X$ and using Proposition (3.5), we have
$f_{up}^{-1}(G_\delta \sqcap J_\sigma)(e)(x)$
$= f_{up}^{-1}(N_\psi)(e)(x)$
$= (N(p(e))(u(x)), \psi(p(e)), \quad p(e) \in K$
$(N(k)(u(x)), \psi(k), k = p(e)$
$= ((G(k) \bigwedge J(k))(u(x)), (\delta \bigwedge \sigma)(k)$
$= (G(k)(u(x)) \bigwedge J(k)(u(x)), \delta(k) \bigwedge \sigma(k)$
$= f_{up}^{-1}(G_\delta)(e)(x) \bigwedge f_{up}^{-1}(J_\sigma)(e)(x)$.
$= (f_{up}^{-1}(G_\delta) \sqcap f_{up}^{-1}(J_\sigma))(e)(x)$

This give (5).

The implication in (3) is not reversible, in general, as can be shown in the following Example.

Example 3.10. Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping where the mappings $u : X \longrightarrow Y$ and $u : E \longrightarrow K$ ard defined by

$u(a) = x \quad u(b) = x \quad u(c) = y$,
$p(e_1) = e_1^{'} \quad p(e_2) = e_3^{'}, \quad p(e_3) = e_3^{'}, \quad p(e_4) = e_1^{'}$.

Choose two generalized fuzzy soft sets in $GFS(Y,K)$ as

$$G_\delta = \{(e_2^{'} = \{\frac{x}{0.6}, \frac{y}{0}, \frac{z}{0.5}\}, 0.5)\},$$
$$J_\sigma = \{(e_2^{'} = \{\frac{x}{0.2}, \frac{y}{0.1}, \frac{z}{0.9}\}, 0.3)\}.$$

Then calculations give
$f_{up}^{-1}(G_\delta) = \widetilde{0}_{\theta_X} \sqsubseteq \widetilde{0}_{\theta_X} = f_{pu}^{-1}(J_\sigma)$, but $G_\delta \not\sqsubseteq J_\sigma$.

Theorem 3.11 Let $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping. For $F_\mu \in GFS(X,E)$ and $G_\delta \in GFS(Y,K)$, the following statements are true.

(1) $f_{up}^{-1}(G_\delta)^c = (f_{up}^{-1}(G_\delta))^c$.

(2) $f_{up}(f_{up}^{-1}(G_\delta)) \subseteq G_\delta$, if f_{up} is surjective, the equality holds.

(3) $F_\mu \subseteq f_{up}^{-1}(f_{up}(F_\mu))$, if f_{up} is injective, the equality holds.

Proof

(1)$f_{up}^{-1}((G_\delta)^c)(e)(x) = (G^c(p(e)(u(x)), \delta^c(p(e)))$, if $e \in E, x \in X$. (I)

On other hand, for every $x \in X, e \in E$, we have
$(f_{up}^{-1}(G_\delta))^c(e)(x) = 1 - (f_{up}^{-1}(G_\delta)(e)(x)$, if $e \in E, x \in X$
$= (1 - G(p(e)(u(x)), 1 - \delta(p(e)))$, if $e \in E, x \in X$
$= (G^c(p(e)(u(x)), \delta^c(p(e)))$, if $e \in E, x \in X.(II)By(I)and(II)wehave(1)$. (2) $f_{up}(f_{up}^{-1}(G_\delta))(k)(y)$
$= \bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} f_{up}^{-1}(G_\delta)(e)(x))$
$\leq \bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(k)} (G(p(e)u(x)), \delta(p(e)))$
$= (G(k)(y), \delta(k))$
$= G_\delta(k)(y)$.

Therefore
$f_{up}(f_{up}^{-1}(G_\delta))(k)(y) \leq G_\delta(k)(y), \quad \forall k \in K, \forall y \in Y$.

(3) $f_{up}^{-1}(f_{up}(F_\mu))(e)(x) = f_{up}(F_\mu)(k)(y)$
$= f_{up}(F_\mu)(p(e)(u(y))$
$=$
$(\bigvee_{x\in u^{-1}(u(x))} \bigvee_{e\in p^{-1}(p(e))} F(e)(x), \bigvee_{e\in p^{-1}}(p(e))\mu(e))$

$\geq (F(e)(x), \mu(e)) = F_\mu(e)(x)$, for all $e \in E, \forall x \in X$. This completes the proof.

Theorem 3.12. Let $F_\mu \in GFS(X,E)$,$G_\delta \in GFS(Y,K)$, and $f_{up} : GFS(X,E) \longrightarrow GFS(Y,K)$ be a GFS mapping. Then

(1) $G_\delta \overline{q} f_{up}(F_\mu) \Longrightarrow f_{up}^{-1}(G_\delta)\overline{q}F_\mu$.

(2) $G_\delta q f_{up}(F_\mu) \Longrightarrow f_{up}^{-1}(G_\delta)qF_\mu$.

Proof (1) $G_\delta \overline{q} f_{up}(F_\mu) \Longrightarrow f_{up}(F_\mu) \sqsubseteq (G_\delta)^c$

$\Longrightarrow F_\mu \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu)) \sqsubseteq f_{up}^{-1}(G_\delta^c)$

$\Longrightarrow F_\mu \sqsubseteq (f_{up}^{-1}(G_\delta))^c$

$\Longrightarrow f_{up}^{-1}(G_\delta)\overline{q}F_\mu$.

(2) Let $f_{up}(F_\mu)qG_\delta$ and $F_\mu \overline{q} f_{up}^{-1}(G_\delta)$. Then

$F_\mu \sqsubseteq (f_{up}^{-1}(G_\delta))^c = f_{up}^{-1}(G_\delta^c)$. It follows that $f_{up}(F_\mu) \sqsubseteq f_{up}(f_{up}^{-1}(G_\delta^c)) \sqsubseteq G_\delta^c$. This shows that $f_{pu}(F_\mu)\overline{q}G_\delta$. This is a contradiction.

4 Generalized fuzzy soft continuous mappings

Defintion 4.1. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces, a generalized fuzzy soft mapping $f_{pu} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ is called a generalized fuzzy soft continuous [in short GFS-continuous] if $f_{up}^{-1}(G_\delta) \in T_1$ for all $G_\delta \in T_2$.

Next, we give an example about GFS-continuous.

Example 4.2 Let $X = \{x_1,x_2,x_3\}, Y = \{y_1,y_2,y_3\}$,

$E = \{e_1,e_2\}$ and $K = \{e_1',e_2'\}$.

$T_1 = \{\widetilde{0}_{\theta_X}\widetilde{1}_{\triangle_X},(F_\mu)_1,(F_\mu)_2\}$, where $(F_\mu)_1$ and $(F_\mu)_2$ are two $GFSSs$ over (X,E) defined as follows:

$(F_\mu)_1 = \{(e_1 = \{\frac{x_2}{0.4}\},0.1),(e_2 = \{\frac{x_1}{0.1}\},0.2)\}$,

$(F_\mu)_2 = \{(e_1 = \{\frac{x_2}{0.5},\frac{x_3}{0.6}\},0.7),(e_2 = \{\frac{x_1}{0.7}\frac{x_2}{0.9}\},0.3)\}$.

Then T_1 is a GFS topology over (X,E) and hence (X,T_1,E) is a $GFST$-space over (X,E).

$T_2 = \{\widetilde{0}_{\theta_Y}\widetilde{1}_{\triangle_Y},(G_\delta)_1,(G_\delta)_2\}$, where $(G_\delta)_1$ and $(G_\delta)_2$ are two $GFSSs$ over (Y,K) defined as follows:

$(G_\delta)_1 = \{(e_1' = \{\frac{y_1}{0.4}\},0.1),(e_2' = \{\frac{y_2}{0.1}\},0.2)\}$,

$(G_\delta)_1 = \{(e_1' = \{\frac{y_1}{0.5},\frac{y_3}{0.6}\},0.7),(e_2' = \{\frac{y_1}{0.9}\frac{y_2}{0.7}\},0.3)\}$.

Then T_2 is a GFS topology over (Y,K) and hence (Y,T_2,K) is a $GFST$-space over (Y,K).

If f_{up} is a mapping from X to Y defined as follows: $u(x_1) = y_2$ $u(x_2) = y_1$ $u(x_3) = y_3$,

$p(e_1) = e_1'$ $p(e_2) = e_2'$.

Then it is easy to verify that $f_{up}^{-1}(G_\delta) \in T_1$ for all $G_\delta \in T_2$. Thus f_{up} is a GFS-continuous mapping from (X,T_1,E) to (Y,T_2,K).

Theorem 4.3 F_μ is GFS open if and only if for each $GFSS$ G_δ contained in F_μ, F_μ is a GFS-nbd of G_δ.

Proof. ($\Longrightarrow$). Obvious.

($\Longleftarrow$). Since $F_\mu \sqsubseteq F_\mu$, there exists a $GFSS$ open set H_ν such that $F_\mu \sqsubseteq H_\nu \sqsubseteq F_\mu$. Hence $H_\nu = F_\mu$ and F_μ is $GFSS$ open.

Theorem 4.4. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces. For a GFS mapping $f_{up} : (X,T_1,E) \longrightarrow (Y,T_2,K)$, the following statements are equivalent:

(1) f_{up} is GFS-continuous;

(2) for $GFSS$ F_μ in $GFS(X,E)$, the inverse image of every GFS-nbd of $f_{up}(F_\mu)$ is a GFS-nbd of F_μ;

(3) for each $GFSS$ F_μ in $GFS(X,E)$ and each GFS-nbd M_σ of $f_{up}(F_\mu)$, there is a GFS-nbd H_ν of F_μ such that $f_{up}(H_\nu) \sqsubseteq M_\sigma$.

Proof (1)$\Longrightarrow$(2). Let f_{up} be GFS-continuous, if M_σ is a GFS-nbd of $f_{up}(F_\mu)$, then M_σ contains an open GFS-nbd K_γ of $f_{up}(F_\mu)$. Since $f_{up}(F_\mu) \sqsubseteq M_\sigma$, $f_{up}^{-1}(f_{up}(F_\mu)) \sqsubseteq f_{up}^{-1}(K_\gamma) \sqsubseteq f_{up}^{-1}(M_\sigma)$. But $F_\mu \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))$ and $f_{up}^{-1}(K_\gamma)$ is a GFS open. Consequntly, $f_{up}^{-1}(M_\sigma)$ is a GFS-nbd of F_μ.

(2) $\Longrightarrow$ (1). We use Theorem (4.3). We prove that if $G_\delta \in T_2$ then $f_{up}^{-1}(G_\delta) \in T_1$. Let F_μ be any GFS sub set of $f_{up}^{-1}(G_\delta)$. Then G_δ is an open GFS-nbd of $f_{up}(F_\mu)$, and by (2) $f_{up}^{-1}(G_\delta)$ is a GFS-nbd of F_μ. This shows that $f_{up}^{-1}(G_\delta)$ is a GFS open set.

(2) $\Longrightarrow$ (3) Let F_μ be any $GFSS$ over (X,E) and let M_σ be any GFS-nbd of $f_{up}(F_\mu)$. By (2), $f_{up}^{-1}(M_\sigma)$ is a GFS-nbd of F_μ. Then there exists a GFS open set H_ν in (X,T_1,E) such that $F_\mu \sqsubseteq H_\nu \sqsubseteq f_{up}^{-1}(M_\sigma)$. Thus, we have an open GFS-nbd H_ν of F_μ such that $f_{up}(F_\mu) \sqsubseteq f_{up}(H_\nu) \sqsubseteq M_\sigma$.

(3) $\Longrightarrow$ (2) Let M_σ be any GFS-nbd of $f_{up}(F_\mu)$. There is a GFS-nbd H_ν of F_μ such that $f_{up}(H_\nu) \sqsubseteq M_\sigma$. Hence $f_{up}^{-1}(f_{up}(H_\nu)) \sqsubseteq f_{up}^{-1}(M_\sigma)$. Furthermore, since $H_\nu \sqsubseteq f_{up}^{-1}(f_{up}(H_\nu))$, $f_{up}^{-1}(M_\sigma)$ is a GFS-nbd of F_μ.

Theorem 4.5. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces and $f_{up} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ be a GFS mapping. Then the followings are equivalent:

(1) f_{up} is GFS-continuous;

(2) $f_{up}^{-1}(G_\delta) \in T_1', \forall G_\delta \in T_2'$;

(3) $\overline{f_{up}^{-1}(G_\delta)} \sqsubseteq f_{up}^{-1}(\overline{G_\delta}), \forall G_\delta \in GFS(Y,K)$.

Proof (1) $\Longrightarrow$ (2) Let G_δ be a GFS-closed set over (Y,K). Then, $G_\delta^c \in T_2$ and by (1) $f_{up}^{-1}(G_\delta^c) \in T_1$.

Since $f_{up}^{-1}(G_\delta^c) = (f_{up}^{-1}(G_\delta))^c$,

we have $f_{up}^{-1}(G_\delta)$ is GFS closed over (X,E).

(2) $\Longrightarrow$ (3) Let $G_\delta \in GFS(Y,K)$, $\overline{G_\delta} \in T_2'$ by (1) $f_{up}^{-1}(\overline{G_\delta}) \in T_1'$. Then

$$\overline{f_{up}^{-1}(G_\delta)} \sqsubseteq \overline{f_{up}^{-1}(\overline{G_\delta})} = f_{up}^{-1}(\overline{G_\delta}).$$

(3) $\Longrightarrow$ (1) Let $G_\delta \in T_2$. Then $G_\delta^c = \overline{G_\delta^c}$. From the hypothesis,

$\overline{f_{up}^{-1}(G_\delta^c)} \sqsubseteq f_{up}^{-1}(\overline{G_\delta^c}) = f_{up}^{-1}(G_\delta^c)$. Then $f_{up}^{-1}(G_\delta^c)$ is GFS closed.

Since $f_{up}^{-1}(G_\delta^c) = (f_{up}^{-1}(G_\delta))^c$ by Theorem (3.11), we have $f_{up}^{-1}(G_\delta)$ is GFS open over (X,E).

Theorem 4.6. Let If $f_{u_1p_1} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ and $g_{u_2p_2} : (Y,T_2,K) \longrightarrow (Z,T_3,D)$ are GFS-continuous mappings, then $g_{u_2p_2} o f_{u_1p_1} : (X,T_1,E) \longrightarrow (Z,T_3,D)$ is also GFS-continuous.

Proof. For a $GFSS$ $G_\delta \in GFS(Z,D) (g_{u_2p_2} \ o \ f_{u_1p_1})^{-1}(G_\delta)(e)(x) = (u_2 \ o \ u_1, p_2 \ o \ p_1)^{-1}(G_\delta(e)(x) = e(G(p_2(p_1(e)))(u_2(u_1(x))), \delta(p_2(p_1(e)))) = u_1^{-1}(u_2)^{-1}(G(p_2(p_1(e)))(x), \delta(p_2(p_1(e)))) = (u_1,p_1)^{-1}((u_2,p_2)^{-1}(G_\delta))(e)(x)$. Hence $(g_{u_2p_2} \ o \ f_{u_1p_1})^{-1}(G_\delta) = (u_1,p_1)^{-1}((u_2,p_2)^{-1}(G_\delta))$, $(u_2,p_2)^{-1}(G_\delta) \in T_2$ since $g_{u_2p_2}$ is GFS continuous, and so $(g_{u_2p_2} \ o \ f_{u_1p_1})^{-1}(G_\delta) = f_{u_1p_1}^{-1}(g_{u_2p_2}^{-1}(G_\delta)) \in T_1$ since $f_{u_1p_1}$ GFS continuous.

Defintion 4.7. A GFS mapping $f_{up} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ is called GFS constant mapping if $u : X \longrightarrow Y$ and $u : E \longrightarrow K$ are constant.

Remark 4.8. In grneral topology spaces the constant mapping is always continuous, but in $GFST$-spaces it is not true in general.

Example 4.9. Let $X = Y = \{x_1,x_2,x_3\}$, $E = K = \{e_1,e_2,e_3\}$ and $f_{up} : (X,T^0,E) \longrightarrow (Y,T^1,K)$ a constant mapping, where $T^0 = \{\widetilde{0}_{\theta_X}, \widetilde{1}_{\triangle_X}\}$ and $T^1 = GFS(Y,K)$.

Consider $u(x) = x_1, \forall x \in X$ and $p(e) = e_1, \forall e \in E$, if we take

$G_\delta = \{(e_1 = \{\frac{x_1}{0.5}, \frac{x_2}{0.4}, \frac{x_3}{0}\}, 0.2), (e_2 = \{\frac{x_1}{0.7}, \frac{x_2}{0}, \frac{x_3}{0}\}, 0.6)\}, (e_3 = \{\frac{x_1}{0}, \frac{x_2}{0}, \frac{x_3}{0}\}, 0)\}$, then

$f_{up}^{-1}(G_\delta)(e_1)(x_1) = (G_\delta(p(e_1))(u(x_1)), \delta(p(e_1))) = (G(e_1)(x_1), \delta(e_1)) = (0.5, 0.2)$

and similary,

$f_{up}^{-1}(G_\delta)(e_1)(x_2) = (G(e_1)(x_1), \delta(e_1)) = (0.5, 0.2)$

$f_{up}^{-1}(G_\delta)(e_1)(x_3) = (G(e_1)(x_1), \delta(e_1)) = (0.5, 0.2)$

$f_{up}^{-1}(G_\delta)(e_2)(x_1) = (G_\delta(p(e_2))(u(x_1)), \delta(p(e_2)))$
$= (G(e_1)(x_1), \delta(e_1)) = (0.5, 0.2)$

and similary,

$f_{up}^{-1}(G_\delta)(e_2)(x_2) = f_{up}^{-1}(G_\delta)(e_2)(x_3) = (0.5, 0.2)$,

$f_{up}^{-1}(G_\delta)(e_3)(x_1) = (G(p(e_3))(u(x_1)), \delta(p(e_3)))$
$= (G(e_1)(x_1), \delta(e_1)) = (0.5, 0.2)$,

and similary,

$f_{up}^{-1}(G_\delta)(e_3)(x_2) = f_{up}^{-1}(G_\delta)(e_3)(x_3) = (0.5, 0.2)$.

Hence $f_{up}^{-1}(G_\delta) \notin T^0$, which $G_\delta \in T^1$.

Definition 4.10. Let (X,T,E) be a $GFST$-space. A $GFSS$ F_μ in $GFS(X,E)$ is called $Q-$generalized fuzzy soft neighborhood (briefly, $Q-GFS$ neighborhood) of H_ν if and only if there exists a GFS open set J_σ such that $H_\nu q J_\sigma$ and $J_\sigma \sqsubseteq F_\mu$.

Definition 4.11. A $GFSS$ F_μ in $GFS(X.E)$ is called $Q-GFS$ neighborhood of a generalized fuzzy soft point $(x_\alpha,e_\lambda)\widetilde{\in}\widetilde{1}_{\triangle_X}$ if and only if there exists a GFS open set J_σ such that $(x_\alpha,e_\lambda) q J_\sigma$ and $J_\sigma \sqsubseteq F_\mu$.

Remark 4.12. If F_μ is GFS open set, the F_μ is a $Q-GFS$ neighborhood if and only if $F_\mu q J_\sigma$.

Theorem 4.13 Let $F_\mu \in GFS(X.E)$ and $(x_\alpha e_\lambda) \in \widetilde{1}_{\triangle_X}$ Then $(x_\alpha,e_\lambda)\widetilde{\in}\overline{F_\mu}$ if and only if each open $Q-GFS$ neighborhood of (x_α,e_λ) is generalized soft quasi-coincident with F_μ.

Proof. Let $(x_\alpha,e_\lambda)\widetilde{\in}\overline{F_\mu}$. For every GFS closed set H_ν which F_μ, $(x_\alpha,e_\lambda) \in H_\nu$. Suppose that M_σ is an open $Q-GFS$ neighborhood of (x_α,e_λ) and $M_\sigma \bar{q} F_\mu$. Then $F_\mu \sqsubseteq (M_\sigma)^c$. Since M_σ is $Q-GFS$ neighborhood of (x_α,e_λ), by theorem 2.19(3), (x_α,e_λ) does not belong to $(M_\sigma)^c$. Therefore, we have that (x_α,e_λ) does not belong to $\overline{F_\mu}$. This is a contradiction.

Conversely, let each open $Q-GFS$ neighborhood of (x_α,e_λ) be generalized soft quasi-coincident with F_μ. Suppose that (x_α,e_λ) does not belong to $\overline{F_\mu}$. Then there exists a GFS closed set H_ν which is contains F_μ such that (x_α,e_λ) does not belong to H_ν. By Theorem 2.19(3), we have $(x_\alpha,e_\lambda) q (H_\nu)^c$. Then $(H_\nu)^c$ is open $Q-GFS$ neighborhood of (x_α,e_λ) and by Theorem 2.19(1), $F_\mu \bar{q} (H_\nu)^c$, a contradiction.

Theorem 4.14. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces and $f_{up} : (X,T_1,E) \longrightarrow (Y,T_2,K)$ be a GFS mapping. Then the followings are equivalent:

(1) f_{up} is GFS-continuous;

(2) $f_{up}^{-1}(G_\delta) \sqsubseteq (f_{up}^{-1}(G_\delta))^0, \forall G_\delta \in T_2$;

(3) $f_{up}(\overline{F_\mu}) \sqsubseteq \overline{f_{up}(F_\mu)}, \forall F_\mu \in GFS(X,E)$;

(4) $\overline{f_{up}^{-1}(G_\delta)} \sqsubseteq f_{up}^{-1}(\overline{G_\delta}), \forall G_\delta \in GFS(Y,K)$;

(5) $f_{up}^{-1}(G_\delta)^0 \sqsubseteq (f_{up}^{-1}(G_\delta))^0, \forall F_\mu \in GFS(X,E)$;

(6) $(f_{up}^{-1}(G_\delta))^b \sqsubseteq f_{up}^{-1}(G_\delta)^b, \forall G_\delta \in GFS(Y,K)$;

(7) $f_{pu}(F_\mu)^b \sqsubseteq (f_{pu}(F_\mu))^b, \forall F_\mu \in GFS(X,E)$.

Proof (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). Let $F_\mu \in GFS(X,E)$ and $f_{up}(x_\alpha,e_\lambda)$ be not GFS subset of $\overline{f_{up}(F_\mu)}$. Then there exists an open $Q-GFS$ neighborhood of G_δ of $f_{up}(x_\alpha,e_\lambda)$ such that $G_\delta \bar{q} f_{up}(F_\mu)$ and hence $f_{up}^{-1}(G_\delta)\bar{q}(F_\mu)$ which implies $(f_{up}^{-1}(G_\delta))^0 \bar{q} F_\mu$. Since $(x_\alpha,e_\lambda) q f_{up}^{-1}(G_\delta)$, by (2), $(x_\alpha,e_\lambda) q (f_{up}^{-1}(G_\delta))^0$. Put $M_\sigma = (f_{up}^{-1}(G_\delta))^0$. Then M_σ is an open $Q-GFS$ neighborhood of (x_α,e_λ) and $M_\sigma \bar{q} F_\mu$. This shows that (x_α,e_λ) is not GFS subset of $\overline{F_\mu}$ which implies that $f_{up}(x_\alpha,e_\lambda)$ is not GFS subset of $f_{up}(\overline{F_\mu})$. Thus $f_{up}(\overline{F_\mu}) \sqsubseteq \overline{f_{up}(F_\mu)}$.

(3)$\Longrightarrow$ (4). Let $G_\delta \in GFS(Y,K)$. Since $f_{up}(f_{up}^{-1}(G_\delta)) \sqsubseteq G_\delta$, we have

$\overline{f_{up}(f_{up}^{-1}(G_\delta))} \sqsubseteq \overline{G_\delta}$. By (3), we obtain $f_{up}(\overline{f_{up}^{-1}(G_\delta)}) \sqsubseteq \overline{G_\delta}$. Thus we have $\overline{f_{up}^{-1}(G_\delta)} \sqsubseteq f_{up}^{-1}(\overline{G_\delta})$.

(4) $\Longleftrightarrow$ (5). These follow from Theorems 3.11(3) and 2.20.

(5) $\Longrightarrow$ (1). Let $G_\delta \in T_2$. By (5), $f_{up}^{-1}(G_\delta) = f_{up}^{-1}(G_\delta)^0 \sqsubseteq (f_{up}^{-1}(G_\delta))^0$ and so $f_{up}^{-1}(G_\delta) \in T_1$.

(4) $\Longrightarrow$ (6). Let G_δ be a $GFSS$ over (Y,K). By (4), Theorem 3.9(5) and Theorem 3.11(1), $(f_{up}^{-1}(G_\delta))^b = \overline{f_{up}^{-1}(G_\delta)} \sqcap \overline{f_{up}^{-1}(G_\delta))^c} \sqsubseteq f_{up}^{-1}(\overline{G_\delta}) \sqcap f_{up}^{-1}(\overline{G_\delta^c}) = f_{up}^{-1}(\overline{G_\delta} \sqcap \overline{G_\delta^c}) = f_{up}^{-1}(G_\delta)^b$ and hence we have $(f_{up}^{-1}(G_\delta))^b \sqsubseteq f_{up}^{-1}(G_\delta)^b$.

(6) $\Longrightarrow$ (1). Let G_δ be a GFS closed set over (Y,K). Then $(G_\delta)^b \sqsubseteq G_\delta$ and $f_{up}^{-1}(G_\delta)^b \sqsubseteq f_{up}^{-1}(G_\delta)$. By (6) we

have $(f_{up}^{-1}(G_\delta))^b \sqsubseteq f_{up}^{-1}(G_\delta)$. This shows that $f_{up}^{-1}(G_\delta)$ is GFS closed set over (X,E). Thus, by Theorem 4.5, f_{up} is GFS-continuous.
(6) $\Longrightarrow$ (7). Let $F_\mu$be a $GFSS$ over (X,E). Then $f_{up}(F_\mu) \in GFS(Y,K)$, by (6), $(f_{up}^{-1}(f_{up}(F_\mu)))^b \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))^b$ and so $(F_\mu)^b \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))^b$. Therefore, we have $f_{pu}(F_\mu)^b \sqsubseteq (f_{pu}(F_\mu))^b$.
(7) $\Longrightarrow$ (6). Let G_δ be a $GFSS$ over (Y,K). Then for $f_{up}^{-1}(G_\delta) \in GFS(X,E)$, by (7) $f_{pu}(f_{up}^{-1}(G_\delta))^b \sqsubseteq (f_{pu}(f_{up}^{-1}(G_\delta)))^b$ and so $f_{pu}(f_{up}^{-1}(G_\delta))^b \sqsubseteq G_\delta^b$. Therefore, we have $(f_{up}^{-1}(G_\delta))^b \sqsubseteq f_{up}^{-1}(G_\delta)^b$.

5 Generalized fuzzy soft open, closed and homeomorphism mappings

Definition 5.1. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces. A GFS mapping $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ is called a generalized fuzzy soft open [GFS-open in short] if $f_{up}(F_\mu)\in T_2$ for each $F_\mu \in T_1$.

Theorem 5.2. Let $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ be a GFS mapping. Then the following statements are equivalent:

(1) f_{up} is GFS-open;
(2) $f_{up}(F_\mu)^0 \sqsubseteq (f_{up}(F_\mu))^0$, $\forall F_\mu \in GFS(X,E)$;
(3) $(f_{up}^{-1}(G_\delta))^0 \sqsubseteq f_{up}^{-1}(G_\delta)^0, \forall G_\delta \in GFS(Y,K)$.
(4) $f_{up}^{-1}(G_\delta)^b \sqsubseteq (f_{up}^{-1}(G_\delta))^b$, $\forall G_\delta \in GFS(Y,K)$;
(5) $f_{up}^{-1}(\overline{G_\delta}) \sqsubseteq \overline{f_{up}^{-1}(G_\delta)}, \forall G_\delta \in GFS(Y,K)$.

Proof (1) $\Longrightarrow$ (2). Let (F_μ) be a $GFSS$ over $GFS(X,E)$. Then $(F_\mu)^0 \sqsubseteq F_\mu$. By using (1), we have $f_{up}(F_\mu)^0 \sqsubseteq (f_{up}(F_\mu))^0$.

(2) $\Longrightarrow$ (3). Let G_δ be a $GFSS$ over (Y,K). Then $f_{up}^{-1}(G_\delta)$ is a $GFSS$ over (X,E). By (2), $f_{up}(f_{up}^{-1}(G_\delta))^0 \sqsubseteq (f_{up}(f_{up}^{-1}(G_\delta))^0 \sqsubseteq (G_\delta)^0$. Therefore, we have $(f_{up}^{-1}(G_\delta))^0 \sqsubseteq f_{up}^{-1}(G_\delta)^0$. (3) $\Longrightarrow$ (4). Let G_δ be a $GFSS$ over (Y,K). Then By using (3), and Theorem 2.22(1), $((f_{up}^{-1}(G_\delta)^b)^c = (f_{up}^{-1}(G_\delta))^0 \sqcup (f_{up}^{-1}(G_\delta)^c)^0 \sqsubseteq f_{up}^{-1}(G_\delta)^0 \sqcup f_{up}^{-1}((G_\delta)^c)^0 = f_{up}^{-1}(G^0 \sqcup (G_\delta^c)^0) = f_{up}^{-1}((G_\delta)^b)^c = (f_{up}^{-1}(G_\delta)^b)^c$ and we have $f_{up}^{-1}(G_\delta)^b \sqsubseteq (f_{up}^{-1}(G_\delta))^b$.

(4) $\Longrightarrow$ (5). Let G_δ be a $GFSS$ over (Y,K). Then By (4), and theorem 2.22(2), $f_{up}^{-1}(\overline{G_\delta}) = f_{up}^{-1}(G_\delta \sqcup G_\delta^b) = f_{up}^{-1}(G_\delta) \sqcup f_{up}^{-1}(G_\delta^b) \sqsubseteq f_{up}^{-1}(G_\delta) \sqcup (f_{up}^{-1}(G_\delta))^b = \overline{f_{up}^{-1}(G_\delta)}$.

(5) $\Longrightarrow$ (3).This follows from Theorem 2.20(1) and Theorem 3.11(1).

(3) $\Longrightarrow$ (1). Let (F_μ) be a $GFSS$ open set in X. Then for $f_{up}(F_\mu) \in GFSS(Y,K)$. By (3), $(f_{up}^{-1}(f_{up}(F_\mu)))^0 \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))^0$. Again since $F_\mu = F_\mu^0 \sqsubseteq (f_{up}^{-1}(f_{up}(F_\mu)))^0 \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))^0$. This shows that f_{up} is GFS-open.

Theorem 5.3. Let $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ be a GFS bijection. Then f_{up} is continuous if and only if $(f_{up}(F_\mu))^0 \sqsubseteq f_{up}(F_\mu)^0$, for every $F_\mu \in GFS(X,E)$.

Proof ($\Longrightarrow$) Let $F_\mu \in GFSS(X,E)$. Then for $f_{up}(F_\mu) \in GFSS(Y,K), (f_{up}(F_\mu))^0 \sqsubseteq f_{up}(F_\mu)$ and so $f_{up}^{-1}(f_{up}(F_\mu))^0 \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu))$. Since f_{up} is bijection and GFS- continuous, $f_{up}^{-1}(f_{up}(F_\mu))^0 \sqsubseteq F_\mu^0$. Again Since f_{up} is surjictiv, $(f_{up}(F_\mu))^0 \sqsubseteq f_{up}(F_\mu)^0$ as claimed.

($\Longrightarrow$) Let G_δ be a GFS open set in Y. Then since f_{up} is surjictiv, $G_\delta = G_\delta^0 = (f_{up}(f_{up}^{-1}(G_\delta)))^0$. By using hypothesis, $G_\delta \sqsubseteq f_{up}(f_{up}^{-1}(G_\delta))^0$. Since f_{up} is injectiv, $f_{up}^{-1}(G_\delta) \sqsubseteq (f_{up}^{-1}(G_\delta))^0$. This shwo that $f_{up}^{-1}(G_\delta)$ is $GFSS$ open set in X.

Definition 5.4. Let (X,T_1,E) and (Y,T_2,K) be two $GFST$-spaces. A GFS mapping $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ is called a generalized fuzzy soft closed [GFS-closed in short] if $f_{up}(F_\mu)\in T_2'$ for each $F_\mu \in T_1'$.

Theorem 5.5. A GFS mapping $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ is closed if and only if $\overline{f_{up}(F_\mu)} \sqsubseteq f_{up}\overline{(F_\mu)}, \forall F_\mu \in GFS(X,E)$.

Proof. It can be proved directly.

Theorem 5.5. Let $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ be a GFS bijection. Then f_{up} closed if and only if $f_{up}^{-1}(\overline{G_\delta}) \sqsubseteq \overline{f_{up}^{-1}(G_\delta)}, \forall G_\delta \in GFS(Y,K)$.

Proof. It is similar to that of theorem 5.3.

The concepts of GFS-coninuous, GFS-open, GFS-closed mappings are all independent of each other.

Example 5.7. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2\}$, $E=\{e_1,e_2\}$, $K=\{e_1',e_2'\}$, we define the GFS mapping $f_{up}:(X,T_1,E)\longrightarrow(Y,T_2,K)$ as

$$u(x_1)=y_1, \quad u(x_2)=y_1,$$
$$p(e_1)=e_1', \quad p(e_2)=e_2'.$$

The collection
$T_1 = \{\widetilde{0}_{\theta_X}\widetilde{1}_{\triangle_X},(F_\mu)_1,(F_\mu)_2,(F_\mu)_3,(F_\mu)_4\}$ is GFS topology over (X,E). Where

$$(F_\mu)_1=\{(e_1=\{\tfrac{x_1}{\frac{1}{3}},\tfrac{x_2}{\frac{2}{5}}\},\tfrac{2}{5}),(e_2=\{\tfrac{x_1}{\frac{2}{7}}\tfrac{x_2}{\frac{3}{4}}\},\tfrac{3}{4})\},$$
$$(F_\mu)_2=\{(e_1=\{\tfrac{x_1}{\frac{1}{3}},\tfrac{x_2}{\frac{1}{5}}\},\tfrac{1}{3}),(e_2=\{\tfrac{x_1}{\frac{4}{5}}\tfrac{x_2}{\frac{1}{2}}\},\tfrac{4}{5})\},$$
$$(F_\mu)_3=\{(e_1=\{\tfrac{x_1}{\frac{1}{3}},\tfrac{x_2}{\frac{1}{5}}\},\tfrac{1}{3}),(e_2=\{\tfrac{x_1}{\frac{2}{7}}\tfrac{x_2}{\frac{1}{2}}\},\tfrac{3}{4})\},$$
$$(F_\mu)_4=\{(e_1=\{\tfrac{x_1}{\frac{1}{3}},\tfrac{x_2}{\frac{2}{5}}\},\tfrac{2}{5}),(e_2=\{\tfrac{x_1}{\frac{4}{5}}\tfrac{x_2}{\frac{3}{4}}\},\tfrac{4}{5})\}.$$

Also the collection
$T_2 = \{\widetilde{0}_{\theta_Y}\widetilde{1}_{\triangle_Y},(G_\delta)_1,(G_\delta)_2,(G_\delta)_3,(G_\delta)_4,(G_\delta)_5,(G_\delta)_6,\}$ is GFS topology over (Y,K). Where

$$(G_\delta)_1=\{(e_1'=\{\tfrac{y_1}{\frac{2}{5}},\tfrac{y_2}{0}\},\tfrac{2}{5}),(e_2'=\{\tfrac{y_1}{\frac{3}{4}}\tfrac{y_2}{0}\},\tfrac{3}{4})\},$$
$$(G_\delta)_2=\{(e_1'=\{\tfrac{y_1}{\frac{1}{3}},\tfrac{y_2}{0}\},\tfrac{1}{3}),(e_2'=\{\tfrac{y_1}{\frac{4}{5}}\tfrac{y_2}{0}\},\tfrac{4}{5})\},$$
$$(G_\delta)_3=\{(e_1'=\{\tfrac{y_1}{\frac{1}{4}},\tfrac{y_2}{0}\},\tfrac{1}{4}),(e_2'=\{\tfrac{y_1}{\frac{1}{2}}\tfrac{y_2}{0}\},\tfrac{1}{2})\},$$
$$(G_\delta)_4=\{(e_1'=\{\tfrac{y_1}{\frac{2}{5}},\tfrac{y_2}{0}\},\tfrac{2}{5}),(e_2'=\{\tfrac{y_1}{\frac{4}{5}}\tfrac{y_2}{0}\},\tfrac{4}{5})\},$$
$$(G_\delta)_5=\{(e_1'=\{\tfrac{y_1}{\frac{1}{3}},\tfrac{y_2}{0}\},\tfrac{1}{3}),(e_2'=\{\tfrac{y_1}{\frac{3}{4}}\tfrac{y_2}{0}\},\tfrac{3}{4})\},$$

$(G_\delta)_6 = \{(e_1^{'} = \{\frac{y_1}{\frac{1}{3}}, \frac{y_2}{0}\}, \frac{1}{3}), (e_2^{'} = \{\frac{y_1}{\frac{1}{2}} \frac{y_2}{0}\}, \frac{1}{2})\}$, $f_{up}^{-1}(G_\delta)_5(e_1)(x_1) = (G_5(p(e_1))(u(x_1)), \delta(p(e_1))) = (G_5(e_1^{'})(y_1), \delta(e_1^{'})) = (\frac{1}{3}, \frac{1}{3})$ $f_{up}^{-1}(G_\delta)_5(e_1)(x_2) = (G_5(p(e_1))(u(x_2)), \delta(p(e_1))) = (\frac{1}{3}, \frac{1}{3})$ $f_{up}^{-1}(G_\delta)_5(e_2)(x_1) = (G_5(p(e_2))(u(x_1)), \delta(p(e_2))) = (G_5(e_2^{'})(y_1), \delta(e_2^{'})) = (\frac{3}{4}, \frac{3}{4})$ $f_{up}^{-1}(G_\delta)_5(e_2)(x_2) = (\frac{3}{4}, \frac{3}{4})$. Then $f_{up}^{-1}(G_\delta)_5 = \{(e_1 = \{\frac{x_1}{\frac{1}{3}}, \frac{x_2}{\frac{1}{3}}\}, \frac{1}{3}), (e_2 = \{\frac{x_1}{\frac{3}{4}} \frac{x_2}{\frac{3}{4}}\}, \frac{3}{4})\}$ and $(F_\mu)_3^c = \{(e_1 = \{\frac{x_1}{\frac{2}{3}}, \frac{x_2}{\frac{4}{5}}\}, \frac{2}{3}), (e_2 = \{\frac{x_1}{\frac{5}{7}} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{4})\}$. Put $H_\nu = (F)_3^c$. Then, by calculation we have $f_{up}(H_\nu)(e_1^{'})(y_1) = (\bigvee_{s \in u^{-1}(y_1)} \bigvee_{e \in p^{-1}(e_1^{'})} H(e)(s), \bigvee_{e \in p^{-1}(e_1^{'})} \nu(e)) = (\bigvee_{s \in \{x_1, x_2\}} \{x_1, x_2\}(s), \frac{2}{3}) = (\frac{4}{5}, \frac{2}{3})$, $f_{up}(H_\nu)(e_1^{'})(y_2) = (0, \frac{2}{3})(asu^{-1}(y_2) = \phi)$. By similar calculation consequntly, we have $f_{up}(H_\nu) = f_{up}(F_\mu)_3^c = \{(e_1^{'} = \{\frac{y_1}{\frac{4}{5}}, \frac{y_2}{0}\}, \frac{1}{3}), (e_2^{'} = \{\frac{y_1}{\frac{4}{5}} \frac{y_2}{0}\}, \frac{1}{4})\}$. Here $f_{up}^{-1}(G_\delta)_5 \notin T_1$ and $f_{up}(F_\mu)_3^c$ is not GFS closed set. Thus the GFS mapping is not GFS-continuous and not GFS-closed. But it is GFS-open [as $f_{up}(F_\mu)_1 = (G_\delta)_1, f_{up}(F_\mu)_2 = (G_\delta)_2, f_{up}(F_\mu)_3 = (G_\delta)_6, f_{up}(F_\mu)_4 = (G_\delta)_4$].

Example 5.8. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $E = \{e_1, e_2\}$, $K = \{e_1^{'}, e_2^{'}\}$, we define the GFS mapping $f_{up} : (X, T_1, E) \longrightarrow (Y, T_2, K)$ as

$u(x_1) = u(x_2) = y_2$ and $p(e_1) = e_1^{'}$, $p(e_2) = e_2^{'}$.

Here the $GFSSs$ are defined as follows:

$(F_\mu)_1 = \{(e_1 = \{\frac{x_1}{\frac{2}{5}}, \frac{x_2}{\frac{3}{5}}\}, \frac{3}{5}), (e_2 = \{\frac{x_1}{1} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(F_\mu)_2 = \{(e_1 = \{\frac{x_1}{\frac{1}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{\frac{1}{3}} \frac{x_2}{\frac{1}{4}}\}, \frac{1}{4})\}$,

$(F_\mu)_3 = \{(e_1 = \{\frac{x_1}{\frac{3}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{0} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(F_\mu)_4 = \{(e_1 = \{\frac{x_1}{\frac{1}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{0} \frac{x_2}{\frac{3}{4}}\}, \frac{1}{4})\}$,

$(F_\mu)_5 = \{(e_1 = \{\frac{x_1}{\frac{3}{5}}, \frac{x_2}{\frac{3}{5}}\}, \frac{3}{5}), (e_2 = \{\frac{x_1}{1} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(F_\mu)_6 = \{(e_1 = \{\frac{x_1}{\frac{3}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{\frac{2}{3}} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(F_\mu)_7 = \{(e_1 = \{\frac{x_1}{\frac{2}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{0} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(F_\mu)_8 = \{(e_1 = \{\frac{x_1}{\frac{2}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{\frac{4}{5}} \frac{x_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(G_\delta)_1 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{1}{5}}\}, \frac{1}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{\frac{1}{4}}\}, \frac{1}{2})\}$,

$(G_\delta)_2 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{1}{5}}\}, \frac{1}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{\frac{1}{4}}\}, \frac{1}{4})\}$,

$(G_\delta)_3 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{0}\}, \frac{1}{2})\}$,

$(G_\delta)_4 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{1}{5}}\}, \frac{1}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{0}\}, \frac{1}{4})\}$,

$(G_\delta)_5 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{3}{5}}\}, \frac{3}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{\frac{1}{2}}\}, \frac{1}{2})\}$,

$(G_\delta)_6 = \{(e_1^{'} = \{\frac{y_1}{1}, \frac{y_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2^{'} = \{\frac{y_1}{1} \frac{y_2}{\frac{1}{2}}\}, \frac{1}{2})\}$.

Then

$\underset{\sim}{T_1} = \{\widetilde{0}_{\theta_X} \widetilde{1}_{\triangle_X}, (F_\mu)_1, (F_\mu)_2, (F_\mu)_3, (F_\mu)_4, (F_\mu)_5, (F_\mu)_6\}$ and

$\underset{\sim}{T_2} = \{\widetilde{0}_{\theta_Y} \widetilde{1}_{\triangle_Y}, (G_\delta)_1, (G_\delta)_2, (G_\delta)_3, (G_\delta)_4, (G_\delta)_5, (G_\delta)_6\}$

are GFS topologies over (X, E) and (Y, K) respectively. The mapping f_{up} is GFS-closed, but not GFS-open and not GFS-continuous.

Here

$f_{up}(F_\mu)_1 = \{(e_1^{'} = \{\frac{y_1}{0}, \frac{y_2}{\frac{3}{5}}\}, \frac{3}{5}), (e_2^{'} = \{\frac{y_1}{0} \frac{y_2}{1}\}, \frac{1}{2})\}$ is not a GFS open set and

$f_{up}^{-1}(G_\delta)_1 = \{(e_1 = \{\frac{x_1}{\frac{2}{5}}, \frac{x_2}{\frac{2}{5}}\}, \frac{2}{5}), (e_2 = \{\frac{x_1}{\frac{1}{4}} \frac{x_2}{\frac{1}{4}}\}, \frac{1}{2})\} \notin T_1$.

Note:

$f_{up}(F_\mu)_1^c = (G_\delta)_6^c, f_{up}(F_\mu)_2^c = (G_\delta)_2^c, f_{up}(F_\mu)_3^c = (G_\delta)_3^c, f_{up}(F_\mu)_4^c = (G_\delta)_4^c, f_{up}(F_\mu)_5^c = (G_\delta)_5^c, f_{up}(F_\mu)_6^c = (G_\delta)_1^c, f_{up}(F_\mu)_7^c = (G_\delta)_3^c, f_{up}(F_\mu)_8^c = (G_\delta)_6^c$.

Defintion 5.9. Let (X, T_1, E) and (Y, T_2, K) be two $GFST$-spaces. A GFS mapping f_{up} from (X, T_1, E) to (Y, T_2, K) is called a generalized fuzzy soft homeomorphsim [GFS-homeomorphsim in short] if f_{up} is GFS bijective, GFS-continuous, and GFS-open.

When some GFS-homeomorphsim exists, we say that X is generalized fuzzy soft homeomorphic to Y. **Theorem 5.10.** Let (X, T_1, E) and (Y, T_2, K) be two $GFST$-spaces and $f_{up} : (X, T_1, E) \longrightarrow (Y, T_2, K)$ be a GFS bijective mapping. Then the following conditions are equivalent: (1) f_{up} is GFS-homeomorphsim;

(2) f_{up} is GFS-continuous and GFS-closed mapping;

(3) f_{up} is GFS-continuous and GFS-open mapping.

Proof. It is easily obtained.

By Theorem 4.14, 5.2 ,5.3 and 5.5 we can formulate the following theorem:

Theorem 5.11. Let $f_{up} : (X, T_1, E) \longrightarrow (Y, T_2, K)$ be a GFS mapping. Then the following statements are equivalent:

(1) f_{up} is GFS-homeomorphsim;

(2) $f_{up}(F_\mu)^0 = (f_{up}(F_\mu))^0$, $\forall F_\mu \in GFS(X, E)$;

(3) $(f_{up}^{-1}(G_\delta))^0 = f_{up}^{-1}(G_\delta)^0, \forall G_\delta \in GFS(Y, K)$.

(4) $f_{up}^{-1}(G_\delta)^b = (f_{up}^{-1}(G_\delta))^b$, $\forall G_\delta \in GFS(Y, K)$;

(5) $f_{up}^{-1}(\overline{G_\delta}) = \overline{f_{up}^{-1}(G_\delta)}$, $\forall G_\delta \in GFS(Y, K)$.

(6) $f_{up}(\overline{F_\mu}) = \overline{f_{up}(F_\mu)}$, $\forall F_\mu \in GFS(X, E)$.

6 Perspective

In this paper, we have defined the notion of mappings on the families of GFSSs. We have studied the properties of GFS images and GFS inverse images which have been supported by examples and counterexamples. The notions GFS-continuous, $Q - GFS$ neighborhood, GFS-open (closed) mappings and GFS-homeomorphism for generalized fuzzy soft topological spaces are introduced, and some interesting results that may be of value for further research are obtained.

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