Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120517

A Survey on the Oscillation of Differential Equations with Several Non-Monotone Arguments

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Received: 2 Jun. 2018, Revised: 12 Jul. 2018, Accepted: 17 Jul. 2018
Published online: 1 Sep. 2018

Abstract: Consider the first-order linear differential equation with several retarded arguments $x'(t)+\sum_{i=1}^{m}p_i(t)x(\tau_i(t))=0,\ t\geq t_0$, where the functions $p_i,\ \tau_i\in C\left([t_0,\infty),\ 211d^{+}\right)$, for every $i=1,2,\ldots,m$, $\tau_i(t)\leq t$ for $t\geq t_0$and $\lim_{t\to\infty}\tau_i(t)=\infty$. A survey of the most interesting oscillation conditions is presented. An example illustrating the results is given.

Keywords: Oscillation, retarded, differential equations, non-monotone arguments.

1 Introduction

Consider the first-order linear differential equation with several non-monotone retarded arguments

$$x'(t)+\sum_{i=1}^{m}p_i(t)x(\tau_i(t))=0,\ t\geq t_0, \tag{1.1}$$

where the functions $p_i,\tau_i\in C([t_0,\infty),\mathbb{R}^{+})$, for every $i=1,2,\ldots,m$, (here $\mathbb{R}^{+}=[0,\infty)$), $\tau_i(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau_i(t)=\infty$.

Let $T_0\in[t_0,+\infty)$, $\tau(t)=\min_{1\leq i\leq m}\{\tau_i(t)\}$ and $\tau_{-1}(t)=\sup\{s:\tau(s)\leq t\}$. By a solution of the equation (1.1) we understand a function $x\in C([T_0,+\infty),\mathbb{R})$, continuously differentiable on $[\tau_{-1}(T_0),+\infty]$ and that satisfies (1.1) for $t\geq\tau_{-1}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *non-oscillatory.*

In the special case where $m=1$ equation (1.1) reduces to the equation

$$x'(t)+p(t)x(\tau(t))=0,\ t\geq t_0, \tag{1.2}$$

where the functions $p,\ \tau\in C([t_0,\infty),\mathbb{R}^{+})$, $\tau(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$.

For the general theory of these equations the reader is referred to [13,16, 18, 19, 32].

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equations (1.1) and (1.2) has been the subject of many investigations. See, for example, [1-40] and the references cited therein.

In the case of monotone arguments, a survey of the most interesting oscillation conditions for Eq.(1.2) can be found in [36]. While in the general case of non-monotone arguments we present the following interesting sufficient oscillation conditions.

In 1994, Koplatadze and Kvinikadze [26] established the following: Assume

$$\sigma(t):=\sup_{s\leq t}\tau(s),\quad t\geq 0. \tag{1.3}$$

Clearly $\sigma(t)$ is non-decreasing and $\tau(t)\leq\sigma(t)$ for all $t\geq 0$. *Let* $k\in\mathbb{N}$ exist such that

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left\{\int_{\sigma(s)}^{\sigma(t)}p(\xi)\psi_k(\xi)d\xi\right\}ds>1-c(\mathfrak{a}), \tag{1.4}$$

where $\mathfrak{a}:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds\leq\frac{1}{e}$,

$$\psi_1(t)=0,\ \psi_k(t)=\exp\left\{\int_{\tau(t)}^{t}p(\xi)\psi_{k-1}(\xi)d\xi\right\},\ k=2,3,\ldots\ \text{for } t\in\ \mathbb{R}^{+}, \tag{1.5}$$

and

$$c(\mathfrak{a})=\begin{cases}0 & \text{if } \mathfrak{a}>\frac{1}{e},\\ \frac{1}{2}\left(1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}\right) & \text{if } 0<\mathfrak{a}\leq\frac{1}{e}.\end{cases} \tag{1.6}$$

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Then all solutions of equation (1.2) oscillate.

In 2011 Braverman and Karpuz [6] derived the following sufficient oscillation condition for Eq.(1.2)

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\}ds>1, \quad (1.7)$$

while in 2014 Stavroulakis [37] improved the above condition as follows:

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\}ds>1-\frac{1}{2}\left(1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}\right) \quad (1.8)$$

In 2018 Chatzarakis, Purnaras and Stavroulakis [9] improved further these conditions as follows: Assume that for some $k\in\mathbb{N}$

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right)ds>1, \quad (1.9)$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right)ds>1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \quad (1.10)$$

where $0<\mathfrak{a}\le\frac{1}{e}$, and

$$P_k(t)=p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right]$$

with $P_0(t)=p(t)$. Then all solutions of Eq. (1.2) oscillate. Concerning the differential equation (1.1) with several non-monotone arguments the following related oscillation results have been recently published.

Assume that there exist non-decreasing functions $\sigma_i\in C([t_0,\infty),\mathbb{R}^+)$ such that

$$\tau_i(t)\le\sigma_i(t)\le t,\ i=1,2,\dots,m. \quad (1.11)$$

In 2015 Infante, Kopladatze and Stavroulakis [21] proved that if

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}\sum_{i=1}^{m}p_i(\xi)\exp\left(\int_{\tau_i(\xi)}^{\xi}\sum_{i=1}^{m}p_i(u)\,du\right)d\xi\right)ds\right]^{1/m}>\frac{1}{m^m}, \quad (1.12)$$

then all solutions of Eq. (1.1) oscillate.
Also in 2015 Kopladatze [27] improved the above condition as follows: Let there exist some $k\in\mathbb{N}$ such that

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(m\int_{\tau_i(s)}^{\sigma_i(t)}\left(\prod_{\ell=1}^{m}p_\ell(\xi)\right)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right)ds\right]^{\frac{1}{m}}>\frac{1}{m^m}\left[1-\prod_{i=1}^{m}c_i(\alpha_i)\right], \quad (1.13)$$

where

$$\psi_1(t)=0,\ \psi_i(t)=\exp\left(\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\left(\prod_{\ell=1}^{m}p_\ell(s)\right)^{\frac{1}{m}}\psi_{i-1}(s)\,ds\right),\ i=2,3,\dots,$$

$$0<\alpha_i:=\liminf_{t\to\infty}\int_{\sigma_i(t)}^{t} p_i(s)\,ds<\frac{1}{e},\ i=1,2,\dots,m, \quad (1.14)$$

and

$$c_i(\alpha_i)=\frac{1-\alpha_i-\sqrt{1-2\alpha_i-\alpha_i^2}}{2},\ i=1,2,\dots,m, \quad (1.15)$$

then all solutions of Eq. (1.1) oscillate.

In 2016 Bravermen, Chatzarakis and Stavroulakis [7] obtained the following iterative sufficient oscillation conditions

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(h(t),\tau_i(u))\,du>1, \quad (1.16)$$

or

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(h(t),\tau_i(u))\,du>1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \quad (1.17)$$

or

$$\liminf_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(h(t),\tau_i(u))\,du>\frac{1}{e} \quad (1.18)$$

where

$$h(t)=\max_{1\le i\le m}h_i(t)\ \text{and}\ h_i(t)=\sup_{t_0\le s\le t}\tau_i(s),\ i=1,2,\dots,m,$$

$$0<\alpha:=\liminf_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(s)\,ds\le\frac{1}{e} \quad (1.19)$$

and

$$a_1(t,s)=\exp\left(\int_{s}^{t}\sum_{i=1}^{m}p_i(u)\,du\right),$$

$$a_{r+1}(t,s)=\exp\left(\int_{s}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(u,\tau_i(u))\,du\right),\ r\in\mathbb{N}.$$

Also, in 2016 Akca, Chatzarakis and Stavroulakis [1] improved that result replacing condition (1.8) by the iterative condition

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(h(u),\tau_i(u))\,du>\frac{1+\ln\lambda_0}{\lambda_0} \quad (1.20)$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$,

$$0 < \alpha := \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)\,ds \le \frac{1}{e}$$

and $\tau(t) = \max\limits_{1\le i\le m} \{\tau_i(t)\}$.

In 2017 Chatzarakis [8] derived the following: Assume that for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} P(s)\exp\left(\int\limits_{\tau(s)}^{h(t)} P_k(u)\,du\right) ds > 1, \qquad (1.21)$$

or

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} P(s)\exp\left(\int\limits_{\tau(s)}^{h(t)} P_k(u)\,du\right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \qquad (1.22)$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_k(u)du\right) ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \qquad (1.23)$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u)du\right) ds > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \qquad (1.24)$$

or

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u)du\right) ds > \frac{1}{e}, \qquad (1.25)$$

where $h(t), \tau(t), \alpha$ are defined as above, λ_1 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$, and

$$P_k(t) = P(t)\left[1 + \int\limits_{\tau(t)}^{t} P(s)\exp\left(\int\limits_{\tau(s)}^{t} P(u)\exp\left(\int\limits_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right) du\right) ds\right]$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$. Then all solutions of Eq. (1.1) oscillate.

In 2018 Attia et al [3] established the following oscillation conditions.

Assume that

$$0 < \rho := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \le \frac{1}{\mathrm{e}},$$

and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t)$$
$$\int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{g_k(t)}^{t}\sum_{i=1}^{n} p_i(s)ds + (\lambda(\rho)-\varepsilon)\int_{\tau_k(s)}^{g_k(t)}\sum_{\ell=1}^{n} p_\ell(u)du} ds,$$
$$\varepsilon \in (0, \lambda(\rho)),$$

or

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q_1(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t)$$
$$\int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{g_k(t)}^{t}\sum_{i=1}^{n} p_i(s)ds + \int_{\tau_k(s)}^{g_k(t)}\sum_{\ell=1}^{n}(\lambda(q_\ell)-\varepsilon_\ell)p_\ell(u)du} ds, \quad \varepsilon_\ell \in (0, \lambda(q_\ell)),$$

and

$$q_\ell = \liminf_{t\to\infty} \int_{\tau_\ell(t)}^{t} p_\ell(s)ds, \quad \ell = 1, 2, \dots, m$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{n}\left(\prod_{k=1}^{n}\int_{g_j(t)}^{t} R_k(s)ds\right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n}\mathrm{e}^{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\sum_{\ell=1}^{n} p_\ell(s)ds} > \frac{1}{n^n},$$

where

$$R_k(s) = \mathrm{e}^{\int_{g_k(s)}^{s}\sum_{i=1}^{n} p_i(u)du}$$
$$\sum_{i=1}^{n} p_i(s)\int_{\tau_i(s)}^{s} p_k(u)\mathrm{e}^{(\lambda(\rho)-\varepsilon)\int_{\tau_k(u)}^{g_k(s)}\sum_{\ell=1}^{n} p_\ell(v)dv} du, \quad \varepsilon \in (0, \lambda(\rho)),$$

and

$$0 < \beta_k := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)ds \le \frac{1}{\mathrm{e}}.$$

Then Eq. (1.1) is oscillatory.

Recently Bereketoglu et al [4] improved the above conditions as follows:

Theorem 1. *Assume that there exist non-decreasing functions $\sigma_i \in C([t_0, \infty), \mathbb{R}^+)$ such that (1.11) is satisfied and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int\limits_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int\limits_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right) ds\right)\right]^{1/m} > \frac{1}{m^m}, \qquad (1.26)$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int\limits_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int\limits_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right) ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1 - \prod_{i=1}^{m} c_i(\alpha_i)\right], \qquad (1.27)$$

where

$$P_k(t) = \sum_{j=1}^{m} p_j(t)\left\{1 + m\left[\prod_{i=1}^{m}\int\limits_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int\limits_{\tau_i(s)}^{t} P_{k-1}(u)\,du\right) ds\right]^{1/m}\right\},$$

with

$$P_0(t) = m\left[\prod_{\ell=1}^{m} p_\ell(t)\right]^{1/m},$$

α_i is given by (1.14) and $c_i(\alpha_i)$ by (1.15). Then all solutions of Eq.(1.1) oscillate.

*Remark.*It is clear that the left-hand sides of both conditions (1.26) and (1.27) are identically the same and also the right-hand side of (1.27) reduces to (1.26) when $c_i(\alpha_i) = 0$. So it seems that both conditions are the same when $c_i(\alpha_i) = 0$. One may notice, however, that the condition (1.14) with $0 \le \alpha_i \le 1/e$ is required in (1.27) but not in (1.26).

In the case of monotone arguments we have the following theorem.

Theorem 2. *Let τ_i be non-decreasing functions and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} \left(p_i(s) \exp\left(\int_{\tau_i(s)}^{\tau_i(t)} P_k(u)\,du \right) ds \right) \right]^{1/m} > \begin{cases} \frac{1}{m^m} \\ or \\ \frac{1}{m^m}\left[1 - \prod_{i=1}^{m} c_i(\alpha_i)\right] \end{cases}$$

where

$$P_k(t) = \Sigma_{j=1}^{m} p_j(t) \left\{ 1 + m \left[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{t} P_{k-1}(u)\,du \right) ds \right]^{1/m} \right\},$$

with

$$P_0(t) = m \left[\prod_{\ell=1}^{m} p_\ell(t) \right]^{1/m},$$

$$\alpha_i = \liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)\,ds, \; i = 1, 2, \dots, m,$$

and

$$c_i(\alpha_i) = \begin{cases} 0, & \text{if } \alpha_i > \frac{1}{e} \\ \frac{1 - \alpha_i - \sqrt{1 - 2\alpha_i - \alpha_i^2}}{2}, & \text{if } 0 \le \alpha_i \le \frac{1}{e}. \end{cases} \quad (1)$$

Then all solutions of Eq.(1.1) oscillate.

2 Corollaries and Examples

For the case $m = 2$, Eq. (1.1) reduces to the equation

$$x'(t) + p_1(t)x(\tau_1(t)) + p_2(t)x(\tau_2(t)) = 0. \quad (2.1)$$

From Theorem 1 the following corollary is immediate.

Corollary 1. *Assume that there exist non-decreasing functions $\sigma_i \in C([t_0, \infty), \mathbb{R}^+)$ such that (1.11) is satisfied for $i = 1, 2$, and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty} \prod_{j=1}^{2} \left[\prod_{i=1}^{2} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du \right) ds \right]^{1/2} > \begin{cases} \frac{1}{4} \\ or \\ \frac{1}{4}\left[1 - \prod_{i=1}^{2} c_i(\alpha_i)\right] \end{cases}$$

where

$$P_k(t) = \Sigma_{j=1}^{2} p_j(t) \left\{ 1 + 2 \left[\prod_{i=1}^{2} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{t} P_{k-1}(u)\,du \right) ds \right]^{1/2} \right\},$$

with

$$P_0(t) = 2\sqrt{p_1(t)p_2(t)},$$

and for $i = 1, 2$,

$$\alpha_i = \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i\,ds,$$

$$c_i(\alpha_i) = \begin{cases} 0, & \text{if } \alpha_i > \frac{1}{e} \\ \frac{1 - \alpha_i - \sqrt{1 - 2\alpha_i - \alpha_i^2}}{2}, & \text{if } 0 \le \alpha_i \le \frac{1}{e}. \end{cases} \quad (2.2)$$

Then all solutions of Eq.(2.1) oscillate.

Moreover, in the case of the equation (1.2)

$$x'(t) + p(t)x(\tau(t)) = 0,$$

we have the following corollary.

Corollary 2. *Assume that there exists a non-decreasing function $\sigma(t)$ such that $\tau(t) \le \sigma(t) \le t$ and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du \right) ds > \begin{cases} 1 \\ or \\ 1 - c(\alpha) \end{cases} \quad (2.3)$$

where

$$P_k(t) = p(t)\left\{ 1 + \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_{k-1}(u)\,du \right) ds \right\}, \; P_0(t) = p(t),$$

$$\alpha = \liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\,ds$$

and

$$c(\alpha) = \begin{cases} 0, & \text{if } \alpha > \frac{1}{e} \\ \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, & \text{if } 0 \le \alpha \le \frac{1}{e}. \end{cases} \quad (2.4)$$

Then all solutions of Eq.(1.2) oscillate.

The following example (cf. [6],[21]) is given to illustrate our results. It is to be pointed out that in this example it is shown that our conditions essentially improve related known conditions in the literature.

Example 1.(Cf. [6],[21]) Consider the equation

$$x'(t) + px(\tau(t)) = 0, \; t \ge 0, \; p > 0, \quad (2.5)$$

with the retarded argument

$$\tau(t)=\begin{cases} t-1, & t\in[3n,\ 3n+1],\\ -3t+(12n+3), & t\in[3n+1,\ 3n+2],\\ 5t-(12n+13), & t\in[3n+2,\ 3n+3].\end{cases}$$

For this equation, as in [6,21], one may choose the funtion

$$\sigma(t)=\begin{cases} t-1, & t\in[3n,\ 3n+1],\\ 3n, & t\in[3n+1,\ 3n+2.6],\\ 5t-(12n+13), & t\in[3n+2.6,\ 3n+3].\end{cases}$$

If we choose $t_n=3n+3$, then for $k=1$, we find

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right)ds \geq \lim_{n\to\infty}\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+2} P_1(u)\,du\right)ds,$$

where

$$\begin{aligned} P_1(t) &= p\left\{1+\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{t} p du\right)ds\right\}\\ &= p\left\{1+\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+3} p du\right)ds\right\}\\ &= p\left(1+\frac{e^{6p}-e^{p}}{5}\right).\end{aligned}$$

Therefore

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right)ds \geq \frac{p}{5P_1}\left(e^{5P_1}-1\right).$$

For $p=0.27$, $P_1\approx 0.472129$, and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)\approx 1.09775>1.$$

Thus the condition (2.3) is satisfied and therefore all solutions of Eq.(2.5) oscillate.

Observe, however, that when we consider the conditions stated in [21], [27], [7], [1], [6] and [37] for the above equation (2.5), we obtain the following:

1. The condition (1.12) reduces to

$$\lim\sup_{t\to+\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} p\exp\left(\int_{\tau(\xi)}^{\xi} pdu\right)d\xi\right)ds>1, \tag{2.6}$$

and the choice of $t_n=3n+3$, as in [21, Example 4.2], leads to the inequality

$$\frac{(e^{5pe^{p}}-1)}{5e^{p}}>1. \tag{2.7}$$

Observe, however, that for $p=0.27$,

$$\frac{(e^{5pe^{p}}-1)}{5e^{p}}\approx 0.742275<1.$$

Therefore the condition (2.7) is not satisfied.

2. The condition (1.13), for $k=2$, reduces to

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} p\ \psi_2(\xi)\,d\xi\right)ds>1-c(\alpha), \tag{2.8}$$

where $\psi_2(\xi)=1$, that is, to the condition

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right)ds>1-c(\alpha).$$

Observe that, for $t_n=3n+3$,

$$\int_{\sigma(3n+3)}^{3n+3} p\exp\left(\int_{\tau(s)}^{\sigma(3n+3)} pd\xi\right)ds=\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+2} pd\xi\right)ds=\frac{e^{5p}-1}{5},$$

while

$$\alpha=\liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\,ds=p$$

and

$$c(\alpha)=c(p)=\frac{1-p-\sqrt{1-2p-p^2}}{2}.$$

For $p=0.27$, we find

$$\frac{e^{5p}-1}{5}\approx 0.571485,$$

while the right-hand side

$$1-c(p)\approx 0.946086.$$

Therefore the condition (2.8) is not satisfied.

3. The condition (1.16) for $r=1$ reduces to

$$\limsup_{t\to\infty}\int_{h(t)}^{t} pa_1(h(t),\tau(s))\,ds>1, \tag{2.9}$$

where

$$h(t)=\sigma(t)\ \text{and}\ a_1(t,s)=\exp\left(\int_{s}^{t} pdu\right).$$

That is, to the condition

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right)ds>1, \tag{2.10}$$

As before, for $t_n = 3n+3$ and $p = 0.27$,

$$\frac{e^{5p}-1}{5} \approx 0.571485 \qquad (2.11)$$

Therefore the condition (2.9) is not satisfied.

4. Condition (1.20) for $r = 1$ reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} p d\xi\right) ds > \frac{1+\ln\lambda_0}{\lambda_0}, \qquad (2.12)$$

where λ_0 is the smaller root of the equation $\lambda = e^{p\lambda}$. As before, for $t_n = 3n+3$ and $p = 0.27$,

$$\frac{e^{5p}-1}{5} \approx 0.571485,$$

while

$$\frac{1+\ln\lambda_0}{\lambda_0} \approx 0.937188.$$

Therefore the condition (2.12) is not satisfied.

5. Similarly, for $t_n = 3n+3$ and $p = 0.27$, we have

$$\frac{e^{5p}-1}{5} \approx 0.571485 < 1,$$

and therefore the condition (1.7) is not satisfied.

6. Finally, the condition (1.8) reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} p d\xi\right) ds > 1 - \frac{1-p-\sqrt{1-2p-p^2}}{2}$$

and for $t_n = 3n+3$ and $p = 0.27$, as before, we have

$$\frac{e^{5p}-1}{5} \approx 0.571485$$

while

$$1 - \frac{1-p-\sqrt{1-2p-p^2}}{2} \approx 0.946086.$$

Therefore this condition is not satisfied.

We conclude, therefore, that for $p = 0.27$ no one of the conditions (2.6), (2.8) for $k = 2$, (2.9) and (2.12) for $r = 1$, (1.16) and (1.20) is satisfied.

It should be also mentioned that not only for this value of $p = 0.27$ but for all values of $p \in [0.27, 0.3]$

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right) > 1$$

and therefore all solutions of (2.5) oscillate. Observe, however, that for $p = 0.3$

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} \approx 0.974101 < 1,$$

and

$$\frac{e^{5p}-1}{5} \approx 0.696337 < 1$$

$$\frac{e^{5p}-1}{5} \approx 0.696337 < 0.912993 \approx \frac{1+\ln\lambda_0}{\lambda_0}$$

and therefore for $p \in [0.27, 0.3]$ no one of the conditions (2.6), (2.8) for $k = 2$, (2.9) and (2.12) for $r = 1$, (1.16) and (1.20) is satisfied.

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