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Generalized Coordinated Nonconvex Functions and Integral Inequalities

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Abstract: In this paper, we introduce the notion of the generalized convex functions involving 2-variables on coordinates and arbitrary bifunction $\eta(.,.)$ on coordinates. Using the concepts of ordinary and fractional calculus some new refinements of Hermite-Hadamard like integral inequalities are also derived via generalized convex functions involving 2-variables on coordinates.

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1 Introduction and Preliminaries

In this section, we shall briefly introduce some recent studies of the subject. We discuss some previously known concepts and results. These preliminaries help the readers to understand the main results of the paper. Before proceeding let us recall the classical convexity on coordinates, which is also known as two dimensional classical convexity. Dragomir [7] was the first to investigate this extension of classical convexity in connection with integral inequalities. Let us consider a bidimensional interval $\Omega = [a,b]\times[c,d] \subset \mathbb{R}^2$ with $a<b$ and $c<d$. A function $\mathscr{F}:\Omega\to\mathbb{R}$ is said to be convex function on Ω, if the following inequality

$$\mathscr{F}(tx+(1-t)z,ty+(1-t)w) \le t\mathscr{F}(x,y)+(1-t)\mathscr{F}(z,w),$$

holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$.

A function $\mathscr{F}:\Omega\to\mathbb{R}$ is said to be convex on Ω, if the partial functions $\mathscr{F}_y:[a,b]\to\mathbb{R}$, $\mathscr{F}_y(u)=\mathscr{F}(u,y)$ and $\mathscr{F}_x:[c,d]\to\mathbb{R}$, $\mathscr{F}_x(v)=\mathscr{F}(x,v)$ are convex for all $x\in[a,b]$ and $y\in[c,d]$.

Definition 1. *Consider the rectangle $\Omega=[a,b]\times[c,d]\subset\mathbb{R}^2$. A function $\mathscr{F}:\Omega\to\mathbb{R}$ is said to be coordinated convex function on Ω, if*

$$\begin{aligned}&\mathscr{F}(tx+(1-t)y,ru+(1-r)w)\\ \le\ & tr\mathscr{F}(x,u)+t(1-r)\mathscr{F}(x,w)\\ &+r(1-t)\mathscr{F}(y,u)+(1-t)(1-r)\mathscr{F}(y,w),\end{aligned}$$

whenever $x,y\in[a,b]$, $u,w\in[c,d]$ and $t,r\in[0,1]$.

Recently Gordji et al. [11] introduced the class of η-convex function.

Definition 2. *A function $\mathscr{F}:I\subset\mathbb{R}\to\mathbb{R}$ is said to be η-convex function with respect to $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, if*

$$\mathscr{F}(tx+(1-t)y)\le\mathscr{F}(y)+t\eta(\mathscr{F}(x),\mathscr{F}(y)),\quad \forall x,y\in I, t\in[0,1].$$

This class generalizes the class of convex functions. For some recent studies on η-convex functions, interested readers are referred to [4,10].
Riemann-Liouville integrals are defined as follows:

Definition 3([13]). *Let $\mathscr{F}\in L[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+}\mathscr{F}$ and $J^{\alpha}_{b^-}\mathscr{F}$ of order $\alpha>0$ are defined by*

$$J^{\alpha}_{a^+}\mathscr{F}(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\mathscr{F}(t)dt,\quad x>a,$$

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and

$$J^{\alpha}_{b^-}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\mathscr{F}(t)dt, \quad x<b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t}x^{\alpha-1}dx,$$

is the well known Gamma function.

For more details and studies on the concepts and results of fractional calculus, see [13].
Sarikaya [18] gave new extensions of classical Riemann-Liouville integrals as follows:

Definition 4. *Let $\mathscr{F} \in L([a,b]\times[c,d])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$ and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta>0$ with $a,c\geq 0$ are defined by*

$$J^{\alpha,\beta}_{a^+,c^+}\mathscr{F}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}\mathscr{F}(t,s)\mathrm{d}s\mathrm{d}t, \quad x>a, y>c$$

$$J^{\alpha,\beta}_{a^+,d^-}\mathscr{F}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}\mathscr{F}(t,s)\mathrm{d}s\mathrm{d}t, \quad x>a, y<d$$

$$J^{\alpha,\beta}_{b^-,c^+}\mathscr{F}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}\mathscr{F}(t,s)\mathrm{d}s\mathrm{d}t, \quad x<b, y>c$$

$$J^{\alpha,\beta}_{b^-,d^-}\mathscr{F}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}\mathscr{F}(t,s)\mathrm{d}s\mathrm{d}t, \quad x<b, y<d,$$

respectively.

Note that

$$J^{0,0}_{a^+,c^+}\mathscr{F}(x,y) = J^{0,0}_{a^+,d^-}\mathscr{F}(x,y) = J^{0,0}_{b^-,c^+}\mathscr{F}(x,y) = J^{0,0}_{b^-,d^-}\mathscr{F}(x,y) = \mathscr{F}(x,y),$$

and

$$J^{1,1}_{a^+,c^+}\mathscr{F}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y \mathscr{F}(t,s)\mathrm{d}s\mathrm{d}t.$$

Also,

$$J^{\alpha}_{a^+}\mathscr{F}\left(x,\tfrac{c+d}{2}\right) = \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\mathscr{F}\left(t,\tfrac{c+d}{2}\right)\mathrm{d}t, \quad x>a$$

$$J^{\alpha}_{b^-}\mathscr{F}\left(x,\tfrac{c+d}{2}\right) = \tfrac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\mathscr{F}\left(t,\tfrac{c+d}{2}\right)\mathrm{d}t, \quad x<b$$

$$J^{\alpha}_{c^+}\mathscr{F}\left(\tfrac{a+b}{2},y\right) = \tfrac{1}{\Gamma(\beta)}\int_c^y (y-s)^{\beta-1}\mathscr{F}\left(\tfrac{a+b}{2},s\right)\mathrm{d}s, \quad y>c$$

$$J^{\alpha}_{d^-}\mathscr{F}\left(\tfrac{a+b}{2},y\right) = \tfrac{1}{\Gamma(\beta)}\int_y^d (s-y)^{\beta-1}\mathscr{F}\left(\tfrac{a+b}{2},s\right)\mathrm{d}s, \quad y<d.$$

Following auxiliary results will play significant role in the development of the main results of the paper.

Lemma 1([19]). *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$, then the following equality holds:*

$$\bigwedge_{\mathscr{F}}(a,b,c,d;x,y) = \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2t)(1-2s) \times\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\mathrm{d}t\mathrm{d}s,$$

where

$$\bigwedge_{\mathscr{F}}(a,b,c,d;x,y) = \frac{\mathscr{F}(a,c)+\mathscr{F}(a,d)+\mathscr{F}(b,c)+\mathscr{F}(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \mathscr{F}(x,y)\mathrm{d}y\mathrm{d}x - \frac{1}{2}\left[\frac{1}{b-a}\int_a^b[\mathscr{F}(x,c)+\mathscr{F}(x,d)] + \frac{1}{d-c}\int_c^d[\mathscr{F}(a,y)+\mathscr{F}(b,y)]\right].$$

Lemma 2([20]). *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with*

$a<b$ and $c<d$. If $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$, then the following equality holds:

$$\begin{aligned}&\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\\&=\frac{(b-a)(d-c)}{4}\\&\times\left\{\int_0^1\int_0^1 t^\alpha s^\beta\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\right.\\&-\int_0^1\int_0^1(1-t)^\alpha s^\beta\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\&-\int_0^1\int_0^1 t^\alpha(1-s)^\beta\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1(1-t)^\alpha(1-s)^\beta\\&\qquad\left.\times\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\mathrm{d}s\mathrm{d}t\right\},\end{aligned}$$

where

$$\begin{aligned}&\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\\&=\frac{\mathscr{F}(a,c)+\mathscr{F}(a,d)+\mathscr{F}(b,c)+\mathscr{F}(b,d)}{4}\\&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}\mathscr{F}(b,d)+J^{\alpha,\beta}_{a^+,d^-}\mathscr{F}(b,c)\right.\\&\left.+J^{\alpha,\beta}_{b^-,c^+}\mathscr{F}(a,d)+J^{\alpha,\beta}_{b^-,d^-}\mathscr{F}(a,c)\right]\\&\quad-\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+}\mathscr{F}(a,d)+J^\beta_{c^+}\mathscr{F}(b,d)\right.\\&\qquad\left.+J^\beta_{d^-}\mathscr{F}(a,c)+J^\beta_{d^-}\mathscr{F}(b,c)\right]\\&\quad-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+}\mathscr{F}(b,c)+J^\alpha_{a^+}\mathscr{F}(b,d)\right.\\&\qquad\left.+J^\alpha_{b^-}\mathscr{F}(a,c)+J^\alpha_{b^-}\mathscr{F}(a,d)\right].\end{aligned}$$

For some recent details on different generalizations of classical convexity and integral inequalities of Hermite-Hadamard type, see [1,2,3,5,6,7,9,12,15,16,17].
Inspired by the research work discussed above, we define a new class of convexity which is a joint generalization of coordinated convex functions and η-convex functions. This class is called as generalized convex functions of 2-variables on coordinates. We also derive some Hermite-Hadamard like inequalities via generalized convex functions of 2-variables on coordinates. This is the main motivation of this paper. It is expected that the results obtained in this paper may stimulate further research in this direction.

2 Generalized coordinated convexity

Now we are in a position to define the class of so called Generalized convex functions of 2-variables on coordinates.

Definition 5. *Consider the rectangle*

$$\Omega=[a,b]\times[c,d]\subset\mathbb{R}^2.$$

A function $\mathscr{F}:\Omega\to\mathbb{R}$ is said to be Generalized convex function of 2-variables on Ω with respect to bifunction $\eta(.,.)$, if

$$\begin{aligned}&\mathscr{F}(ta+(1-t)b,rc+(1-r)d)\\&\leq\mathscr{F}(a,c)+r(1-t)\eta(\mathscr{F}(b,d),\mathscr{F}(a,c))\\&\quad+t(1-r)\eta(\mathscr{F}(a,d),\mathscr{F}(a,c))\\&\quad+(1-t)(1-r)\eta(\mathscr{F}(b,d),\mathscr{F}(a,c)).\end{aligned}$$

Note that if we take $\eta(\beta,\alpha)=\beta-\alpha$, then we have

$$\begin{aligned}&\mathscr{F}(ta+(1-t)b,rc+(1-r)d)\leq tr\mathscr{F}(a,c)\\&+r(1-t)\mathscr{F}(b,c)+t(1-r)\mathscr{F}(a,d)\\&+(1-t)(1-r)\mathscr{F}(b,d).\end{aligned}$$

3 Results in connection with ordinary calculus

In this section, we derive some Hermite-Hadamard like inequalities via generalized convex functions on coordinates using the concepts of ordinary calculus.

Theorem 1. *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$. If $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|$ is generalized convex function of 2-variables on coordinates, then the following inequality holds:*

$$\begin{aligned}&\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right|\\&\leq\frac{(b-a)(d-c)}{16}\\&\times\left[\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right.\\&\quad+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\\&\quad\left.+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right].\end{aligned}$$

Proof. Using Lemma 1 and the property of modulus, we have

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| \le \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 |1-2t||1-2s| \times\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}t\mathrm{d}s.$$

Since it is given that $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|$ is generalized convex function of 2-variables on coordinates, so we have

$$\begin{aligned}\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| &\le \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 |1-2t||1-2s| \\ &\times\Bigg[\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+r(1-t)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\ &+t(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\ &+(1-t)(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\Bigg]\mathrm{d}r\mathrm{d}t \\ &=\frac{(b-a)(d-c)}{4} \\ &\times\Bigg[\frac{1}{4}\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\frac{1}{16}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\ &+\frac{1}{16}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\ &+\frac{1}{16}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\Bigg].\end{aligned}$$

This completes the proof. □

Theorem 2. *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$. If $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|^q$ is generalized convex function of 2-variables on coordinates, then for $q>1$, we have*

$$\begin{aligned}\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| &\le \frac{(b-a)(d-c)}{16} \\ &\times\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right) \\ &+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right) \\ &+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 1 and the property of modulus, we have

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| \le \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 |1-2t||1-2s| \times\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}t\mathrm{d}s.$$

Using power mean inequality, we have

$$\begin{aligned}\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| &\le \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1 |1-2t||1-2s|\mathrm{d}t\mathrm{d}s\right)^{1-\frac{1}{q}} \\ &\times\left(\int_0^1\int_0^1 |1-2t||1-2s| \times\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{q}}.\end{aligned}$$

Since it is given that $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|^q$ is generalized convex function of 2-variables on coordinates, so we have

$$\begin{aligned}\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right| &\le \frac{(b-a)(d-c)}{4}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\ &\times\Bigg[\int_0^1\int_0^1 |1-2t||1-2s| \\ &\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+r(1-t)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right) \\ &+t(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right) \\ &+(1-t)(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}\mathrm{d}r\mathrm{d}t\Bigg]^{\frac{1}{q}} \\ &=\frac{(b-a)(d-c)}{16} \\ &\times\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\end{aligned}$$

$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 3. *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$. If $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|^q$ is generalized convex function of 2-variables on coordinates, then for $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}$$
$$\times\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}^{\frac{1}{q}}.$$

Proof. Using Lemma 1 and the property of modulus, we have

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1|1-2t||1-2s|$$
$$\times\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}t\mathrm{d}s.$$

Using Hölder's inequality, we have

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{q}}.$$

Since it is given that $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|^q$ is generalized convex function of 2-variables on coordinates, so we have

$$\left|\bigwedge_{\mathscr{F}}(a,b,c,d;x,y)\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}$$
$$\times\Bigg[\int_0^1\int_0^1$$
$$\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+r(1-t)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+t(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+(1-t)(1-r)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}\mathrm{d}r\mathrm{d}t\Bigg]^{\frac{1}{q}}$$
$$=\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}$$
$$\times\Bigg\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|^q,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|^q\right)\Bigg\}^{\frac{1}{q}}.$$

This completes the proof. □

4 Results in connection with fractional calculus

In this section, we derive some fractional estimates of Hermite-Hadamard like inequalities via generalized convex functions on coordinates using the concepts of fractional calculus.

Theorem 4. *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$. If $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|$ is generalized convex function of 2-variables on coordinates, then the following inequality holds:*

$$\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)$$
$$\leq\frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)}$$
$$\times\Bigg\{4\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)$$
$$+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)$$
$$+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\Bigg\}\Bigg\}.$$

Proof. Using Lemma 2 and property of modulus, we have

$$\begin{aligned}
&\left|\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\right| \\
&=\frac{(b-a)(d-c)}{4} \\
&\times\left\{\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t\right. \\
&-\int_0^1\int_0^1(1-t)^{\alpha}s^{\beta}\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t \\
&-\int_0^1\int_0^1 t^{\alpha}(1-s)^{\beta}\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t \\
&+\int_0^1\int_0^1(1-t)^{\alpha}(1-s)^{\beta} \\
&\qquad\left.\times\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t\right\}.
\end{aligned}$$

Since it is given that $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|$ is generalized convex function of 2-variables on coordinates, then

$$\begin{aligned}
&\left|\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\right| \\
&\leq\frac{(b-a)(d-c)}{4} \\
&\times\left\{\int_0^1\int_0^1\Big[t^{\alpha}s^{\beta}+(1-t)^{\alpha}s^{\beta}\right. \\
&\qquad+t^{\alpha}(1-s)^{\beta}+(1-t)^{\alpha}(1-s)^{\beta}\Big] \\
&\times\left\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right. \\
&\qquad+s(1-t)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\
&+t(1-s)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\
&\left.\left.+(1-t)(1-s)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right\}\mathrm{d}s\mathrm{d}t\right\} \\
&=\frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)} \\
&\times\left\{4\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right. \\
&\quad+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\
&\quad\left.+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right\}\Bigg\}.
\end{aligned}$$

This completes the proof. □

Theorem 5. *Let $\mathscr{F}:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable functions on $\Omega:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $\frac{\partial^2\mathscr{F}}{\partial t\partial s}\in\mathscr{L}(\Omega)$. If $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|^q$ is generalized convex function of 2-variables on coordinates, then for $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\begin{aligned}
&\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma) \\
&\leq\frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)} \\
&\times\left\{4\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right. \\
&\quad+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right) \\
&\quad\left.+\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right\}\Bigg\}.
\end{aligned}$$

Proof. Using Lemma 2 and property of modulus and the Hölder's inequality, we have

$$\begin{aligned}
&\left|\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\right| \\
&\leq\frac{(b-a)(d-c)}{4} \\
&\times\left\{\left(\int_0^1\int_0^1 t^{p\alpha}s^{p\beta}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1(1-t)^{p\alpha}s^{p\beta}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\right. \\
&+\left(\int_0^1\int_0^1 t^{p\alpha}(1-s)^{p\beta}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left.+\left(\int_0^1\int_0^1(1-t)^{p\alpha}(1-s)^{p\beta}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\right\} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since it is given that $\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}\right|$ is generalized convex function of 2-variables on coordinates, then

$$\begin{aligned}
&\left|\bigvee_{\mathscr{F}}(a,b,c,d;x,y;\alpha,\beta;\Gamma)\right| \\
&\leq\frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{\frac{1}{p}}} \\
&\times\left\{\int_0^1\int_0^1\left\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right.\right.
\end{aligned}$$

$$+s(1-t)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)$$
$$+t(1-s)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)$$
$$+(1-t)(1-s)\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\Big\}\mathrm{d}s\mathrm{d}t\Big\}$$
$$=\frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{\frac{1}{p}}}$$
$$\times\left\{\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right.$$
$$+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)$$
$$\left.+\frac{1}{4}\eta\left(\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(b,d)\right|,\left|\frac{\partial^2\mathscr{F}}{\partial t\partial s}(a,c)\right|\right)\right\}.$$

This completes the proof. □

5 Conclusion

A new extension of convexity called as generalized convex functions of 2-variables on coordinates has been introduced and investigated in connection with integral inequalities of Hermite-Hadamard type using both the concepts of ordinary and fractional calculus. It is worth to mention here that the results obtained in this paper reduce to the results for classical convexity by taking $\eta(\beta,\alpha)=\beta-\alpha$. Thus these results are quite unifying one. Interested readers are encouraged to further explore and investigate the class of generalized convex functions of 2-variables on coordinates in perspective of applications.

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