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Hermite-Hadamard Inequalities for Exponentially Convex Functions

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Abstract: In this paper, we introduce and investigate a new class of convex functions, which is called exponentially convex functions. Several new Hermite-Hadamard type integral inequalities via exponentially convex functions are established. Some special cases are discussed as applications of our results. The ideas and techniques of this paper may be the starting point for further research in this field.

Keywords: Convex functions, exponentially convex functions, Hermite-Hadamard inequality.
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1 Introduction and Preliminaries

In recent years much attention has been given in studying and investigating various aspects of classical concept of convexity. Resultantly this concept has been extended and generalized in different directions. For some useful details, see [1,2,4,5,8,10,11].
A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex in the classical sense, if for all $x, y \in I$ and $t \in [0,1]$, we have

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y).$$

Convexity in connection with integral inequalities is an interesting field of research. As many inequalities are direct consequences of the applications of convex functions. Hermite-Hadamard type inequality is one of the most significant result in convex analysis, which provides a necessary and sufficient condition for a function to be convex. This famous result of Hermite and Hadamard reads as follows:
Let $f: I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function, then we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

For some recent studies on Hermite-Hadamard type inequalities, see [3,7,12].

Motivated by the ongoing research in this important research area, we introduce a new class of convex functions, which is called exponentially convex functions. These exponentially convex functions are nonconvex functions. For the basic properties and other aspects of exponentially convex functions, see Noor [9]. We would like to point out out that every convex function is a exponentially convex function, but the converse is not true. We establish some Hermite-Hadamard type inequalities for exponentially convex functions. We also discuss some special cases, which can be obtained from the main results.
The following auxiliary result will be helpful in obtaining some of our main results.

Lemma 1([6]). *Let $f: I = [a,b] \subset \mathbb{R} \to R$ be a differentiable function on I°, where I° is the interior of I. If $f' \in L_1[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{f(a)+f(b)}{2}$$
$$= \frac{b-a}{2}\left[\int_0^1 (1-2t)f'((1-t)a+tb)\mathrm{d}t\right].$$

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2 Exponential convex functions

We now introduce a new class of convex functions, which is called ”exponentially convex functions.”

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponential convex function, if*

$$f((1-t)x+ty) \leq (1-t)\frac{f(x)}{e^{\alpha x}} + t\frac{f(y)}{e^{\alpha y}}, \tag{1}$$

for all $x,y \in I$, $t \in [0,1]$ and $\alpha \in R$.. If (1) holds in the reversed sense, then f is said to be exponentially concave function.

Note that if $\alpha = 0$, then the class of exponentially convex functions reduce to class of classical convex function. However, the converse is not true.

3 Main Results

In this section, we derive our main results.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an integrable exponentially convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(u)}{e^{\alpha u}}\mathrm{d}u \leq \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}.$$

Proof. Let f be an exponentially convex function. Then

$$2f\left(\frac{x+y}{2}\right) \leq \frac{f(x)}{e^{\alpha x}} + \frac{f(y)}{e^{\alpha y}}.$$

Let $x = (1-t)a+tb$ and $y = ta+(1-t)b$, we have

$$2f\left(\frac{a+b}{2}\right) \leq \frac{f((1-t)a+tb)}{e^{\alpha((1-t)a+tb)}} + \frac{f(ta+(1-t)b)}{e^{\alpha(ta+(1-t)b)}}.$$

Integrating with respect to t on $[0,1]$ and using the change of variable technique, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(u)}{e^{\alpha u}}\mathrm{d}u. \tag{2}$$

Again utilizing the fact that f is an exponentially convex function, we have

$$f((1-t)a+tb) \leq (1-t)\frac{f(a)}{e^{\alpha a}} + t\frac{f(b)}{e^{\alpha b}}.$$

Integrating with respect to t on $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b \frac{f(u)}{e^{\alpha u}}\mathrm{d}u \leq \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}. \tag{3}$$

Summation of inequalities (2) and (3) completes the proof. □

Remark. Note that, if $\alpha = 0$ in Theorem 1, then we recover the classical Hermite-Hadamard inequality.

Our next result is Hermite-Hadamard like inequality via product of two exponentially convex functions.

Theorem 2. *Let $f,g : I \subset \mathbb{R} \to \mathbb{R}$ be two integrable exponentially convex functions, then*

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq \frac{2}{b-a}\int_a^b \frac{f(u)g(u)}{e^{2\alpha u}}\mathrm{d}u \\
&\quad + \frac{1}{e^{\alpha(a+b)}}\left[\frac{1}{6}M(a,b;e)+\frac{1}{3}N(a,b;e)\right] \\
&\leq \frac{1}{3}M(a,b;e)+\frac{1}{6}N(a,b;e),
\end{aligned}$$

where

$$M(a,b;e) := \frac{f(a)g(a)}{e^{\alpha a}} + \frac{f(b)g(b)}{e^{\alpha b}}, \tag{4}$$

and

$$N(a,b;e) := \frac{1}{e^{\alpha(a+b)}}\left[f(b)g(a)+f(a)g(b)\right], \tag{5}$$

respectively.

Proof. Since f and g are exponentially convex functions, we have

$$\begin{aligned}
&4f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \\
&\leq \left\{\left(\frac{f(x)}{e^{\alpha x}}+\frac{f(y)}{e^{\alpha y}}\right)\right\}\left\{\left(\frac{g(x)}{e^{\alpha x}}+\frac{g(y)}{e^{\alpha y}}\right)\right\}.
\end{aligned}$$

Let $x = (1-t)a+tb$ and $y = ta+(1-t)b$, then we have

$$\begin{aligned}
&4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq \left\{\left(\frac{f((1-t)a+tb)}{e^{\alpha((1-t)a+tb)}}+\frac{f(ta+(1-t)b)}{e^{\alpha(ta+(1-t)b))}}\right)\right\} \\
&\quad \times\left\{\left(\frac{g((1-t)a+tb)}{e^{\alpha((1-t)a+tb)}}+\frac{g(ta+(1-t)b)}{e^{\alpha(ta+(1-t)b))}}\right)\right\}.
\end{aligned}$$

Integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}
&4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq \frac{2}{b-a}\int_a^b \frac{f(u)g(u)}{e^{2\alpha u}}\mathrm{d}u \\
&\quad + \frac{1}{e^{\alpha(a+b)}}\int_0^1\left[\left\{(1-t)\frac{f(a)}{e^{\alpha a}}+t\frac{f(b)}{e^{\alpha b}}\right\}\right. \\
&\qquad \times\left\{t\frac{g(a)}{e^{\alpha a}}+(1-t)\frac{g(b)}{e^{\alpha b}}\right\}+\left\{t\frac{f(a)}{e^{\alpha a}}+(1-t)\frac{f(b)}{e^{\alpha b}}\right\} \\
&\qquad \left.\times\left\{(1-t)\frac{g(a)}{e^{\alpha a}}+t\frac{g(b)}{e^{\alpha b}}\right\}\right]\mathrm{d}t.
\end{aligned}$$

This implies

$$4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq \frac{2}{b-a}\int_a^b \frac{f(u)g(u)}{e^{2\alpha u}}du$$
$$+\frac{1}{e^{\alpha(a+b)}}\int_0^1\left[2t(1-t)\left\{\frac{f(a)g(a)}{e^{2\alpha a}}+\frac{f(b)g(b)}{e^{2\alpha b}}\right\}\right.$$
$$\left.\times\frac{[t^2+(1-t)^2]}{e^{\alpha(a+b)}}\{f(b)g(a)+f(a)g(b)\}\right]dt.$$

This implies

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq \frac{2}{b-a}\int_a^b \frac{f(u)g(u)}{e^{2\alpha u}}du$$
$$+\frac{1}{e^{\alpha(a+b)}}\left[\frac{1}{6}M(a,b;e)+\frac{1}{3}N(a,b;e)\right]. \tag{6}$$

We now prove second part of the inequality. Since f and g are exponentially convex functions, we have

$$f((1-t)a+tb)g((1-t)a+tb)$$
$$\leq\left[(1-t)\frac{f(a)}{e^{\alpha a}}+t\frac{f(b)}{e^{\alpha b}}\right]\left[(1-t)\frac{f(a)}{e^{\alpha a}}+t\frac{f(b)}{e^{\alpha b}}\right].$$

Integrating with respect to t on $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b \frac{f(u)g(u)}{e^{2\alpha u}}du\leq\frac{1}{3}M(a,b;e)+\frac{1}{6}N(a,b;e). \tag{7}$$

Summation of inequalities (6) and (7) completes the proof. □

Note that if $\alpha=0$, Theorem 2 reduces to a result for classical convex functions.
We now derive some results for differentiable exponentially convex functions.

Theorem 3. *Let $f:I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, where I° is the interior of I. If $|f'|$ is a exponentially convex function, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{b-a}{8}\left[\left|\frac{f'(a)}{e^{\alpha a}}\right|+\left|\frac{f'(b)}{e^{\alpha b}}\right|\right].$$

Proof. Using Lemma 1, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$=\left|\frac{b-a}{2}\int_0^1(1-2t)f'((1-t)a+tb)dt\right|$$
$$\leq\frac{b-a}{2}\int_0^1|1-2t||f'((1-t)a+tb)|dt.$$

Since $|f'|$ is exponential convex function, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{b-a}{2}\int_0^1|1-2t|\left[(1-t)\left|\frac{f'(a)}{e^{\alpha a}}\right|+t\left|\frac{f'(b)}{e^{\alpha b}}\right|\right]dt$$
$$=\frac{b-a}{8}\left[\left|\frac{f'(a)}{e^{\alpha a}}\right|+\left|\frac{f'(b)}{e^{\alpha b}}\right|\right].$$

This completes the proof. □

Theorem 4. *Let $f:I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, where I° is the interior of I. If $|f'|^q$ is an exponentially convex function, where $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{1}{2}\left\{\left|\frac{f'(a)}{e^{\alpha a}}\right|^{\frac{p}{p-1}}+\left|\frac{f'(b)}{e^{\alpha b}}\right|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}.$$

Proof. Using Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$=\left|\frac{b-a}{2}\int_0^1(1-2t)f'((1-t)a+tb)dt\right|$$
$$\leq\frac{b-a}{2}\int_0^1|1-2t||f'((1-t)a+tb)|dt.$$

Using Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)du\right|$$
$$\leq\frac{b-a}{2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1|f'((1-t)a+tb)|^q dt\right)^{\frac{1}{q}}, \tag{8}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now using the fact that $|f'|^q$ is an exponentially convex function, we have

$$\int_0^1 |f'((1-t)a+tb)|^q \leq \int_0^1 \left[(1-t)\left|\frac{f'(a)}{e^{\alpha a}}\right|^q + t\left|\frac{f'(b)}{e^{\alpha b}}\right|^q\right] dt = \frac{1}{2}\left[\left|\frac{f'(a)}{e^{\alpha a}}\right|^q + \left|\frac{f'(b)}{e^{\alpha b}}\right|^q\right], \tag{9}$$

and

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1}. \tag{10}$$

Using (8), (9) and (10), we have the required result. □

Theorem 5. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, where is the interior of I. If $|f'|^q$ is an exponentially convex function, where $q\geq 1$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{1}{4}\left\{\left|\frac{f'(a)}{e^{\alpha a}}\right|^q + \left|\frac{f'(b)}{e^{\alpha b}}\right|^q\right\}\right]^{\frac{1}{q}}.$$

Proof. Using Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| = \left|\frac{b-a}{2}\int_0^1 (1-2t)f'((1-t)a+tb)\mathrm{d}t\right| \leq \frac{b-a}{2}\int_0^1 |1-2t||f'((1-t)a+tb)|\mathrm{d}t.$$

Using power-mean inequality, we have

$$\int_0^1 |1-2t||f'((1-t)a+tb)|\mathrm{d}t \leq \left(\int_0^1 |1-2t|\mathrm{d}t\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |1-2t||f'((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}. \tag{11}$$

Using the exponentially convexity of $|f'|^q$, we have

$$\int_0^1 |1-2t||f'((1-t)a+tb)|^q\mathrm{d}t \leq \int_0^1 |1-2t|[(1-t)\left|\frac{f'(a)}{e^{\alpha a}}\right|^q + t\left|\frac{f'(b)}{e^{\alpha b}}\right|^q]\mathrm{d}t = \frac{1}{4}\left\{\left|\frac{f'(a)}{e^{\alpha a}}\right|^q + \left|\frac{f'(b)}{e^{\alpha b}}\right|^q\right\}, \tag{12}$$

where

$$\int_0^1 |1-2t|\mathrm{d}t = \frac{1}{2} \tag{13}$$

Using (11), (12) and (13), we have the required result. □

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