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A Continuous Review Inventory System with Retrial Customers and Two-Stage Service

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Abstract: In this paper, we developed a model of a retrial inventory system in which there exists two-stage service. The demanded item is delivered to the customer after a random time of service. The interarrival times of customers, lead times, the retrial times and service times are assumed to have independent exponential distribution. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state case. Some important system performance measures and the long run total expected cost rate are calculated. We numerically demonstrate the model and investigate the impact of different model-parameters on the optimal decisions.

Keywords: Continuous review inventory system, Positive leadtime, Retrial customers, Two-stage serviceContinuous review inventory system, Positive leadtime, Retrial customers, Two-stage service

1 Introduction

Several researchers have been studied inventory system with retrial demands/retrial customers in recent years. The researchers/practitioners have recognized the fact that, the stock level is non-empty the demanded item is directly delivered. But in the case of inventories maintained with service facilities, the demanded item is delivered to the customers only after some service time. This forces the formation of queues in these models. This necessitates the study of both inventory level and queue length joint distributions. Study of such models is beneficial to organizations who

i. provide service to customers by using items from a stock.
ii. maintain stock of items each of which needs service such as assembly or initialization or installation etc., before it is delivered to the customers.

Examples for the first type include firms engaged in servicing consumer products such as cars, Computers and Television etc., and for the second type include firms that supply bikes which need assembly of its parts and that computers which need installation of basic services. Berman et al. [2] have considered an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service times are deterministic and constant, as such queues can form only during stock out periods. They determined optimal order quantity that minimizes the total cost rate.
Berman and Kim [3] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [4,5] studied various concept of queueing - inventory system with service facility. Elango [15] considered a Markovian inventory system with instantaneous supply of orders at a service facility. The service time is assumed to have exponential distribution with parameter depending on the number of waiting customers. Krishnamoorthy, A. et al [12] investigated a survey on inventory models with positive service time.
The concept of inventory system with retrial demands was introduced by Artalejo et al. [1]. The authours analyzed inventory policies with positive lead-time and retrial of customers who could not get service during their earlier attempts to access the service station. Ushakumari [16] considered a retrial inventory system with classical retrial policy. Jeganathan et al. [6] studied a retrial

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inventory system with non-preemptive priority service. Manuel, P. et al [7] studied a perishable inventory system with service facilities and retrial customers.

The author Cohen, J.W. [13,14] consider a queueing system with a two-stage service rule: The server served at a normal service speed when the workload is less, whereas the service speed increases when the workload is high. Bekker and Boxma [9,10] developed in Queues with adaptable service speed and also they have studied an $M/G/1$ Queue with Adaptable Service Speed and some special cases.

In the present paper, we consider the continuous review retrial inventory system with single server served at a two-stage service. If the inventory level is positive and no customer in the system, any arriving primary customer served immediately. When the server is busy or the inventory level is zero and the waiting hall (finite size) is not full, any arriving primary customer waits in the waiting hall. Any arriving primary customer who finds the waiting hall is full, enters into the orbit of finite size. Exponentially distributed two-stage service by the single server. The joint probability distribution of the inventory level, the number of customers in the waiting hall and the number of customers in the orbit is obtained in the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated.

The rest of this paper is organized as follows. In section 2, we describe the mathematical model and the notations used in this paper are defined. Analysis of the model and the steady state solution of the model are proposed in section 3. Some key system performance measures are derived in section 4. In section 5, the total expected cost rate is calculated and we present some numerical examples.

2 Model formulation

In this paper we consider a continuous review inventory system with (s,Q) ordering policy. Under (s,Q) the inventory is refilled to level S whenever its content reaches s, an order for $Q(=S-s)$ units is placed. Primary customers arrive according to a Poisson phenomenon $\lambda(>0)$. If no customers in the waiting hall and the inventory level is positive then the arriving primary customer is immediately taken for service by the server and leaves the system after service of random duration. If the server is busy or inventory level is zero, arriving primary customer joins the waiting hall of finite size M. The single server provided two-stage service based on a waiting hall customer: The service rates μ and $\mu_s(>\mu)$, when the waiting hall customer level is N or below and the customer level exceeds N, respectively, which are exponentially distributed. When the waiting hall is full, any arriving primary customer enters into orbit of finite size H and makes successive repeated attempts until he finds the waiting hall is empty. The inter-retrial times follow an exponential distribution with constant rate α. An arriving primary customer who finds the waiting hall and orbit are full is considered to be lost. The lead time is exponentially distributed with parameter β. We assume that the inter-demand times between primary customer demands, the lead times, retrial customer demand times and service times are mutually independent random variables.

Notations

$[A]_{ij}$	:	The element/submatrix at (i,j)th position of A.
$\mathbf{0}$	:	Zero matrix.
$\mathbf{I}$	:	Identity matrix.
$\mathbf{e}$	:	A column vector of $1'$s of appropriate dimension.
δ_{ij}	:	$\begin{cases} 1, \text{ if } i=j, \\ 0, \text{ otherwise.} \end{cases}$
$\bar{\delta}_{ij}$	:	$1-\delta_{ij}$
E_1	:	$\{0,1,2,\ldots,S\}$
E_2	:	$\{0,1,2,\ldots,N,N+1,\ldots,M\}$
E_3	:	$\{0,1,2,\ldots,H\}$
E	:	$E_1\times E_2\times E_3$

3 Analysis

Let $L(t)$, $X(t)$ and $Y(t)$, respectively, denote the on hand inventory level, the number of demands in the waiting hall and the number of demands in the orbit at time t. From the assumptions made on the input and output processes, it can be shown that the triplet $\{(L(t),X(t),Y(t)),\ t\geq 0\}$ is a continuous time Markov chain with the state space given by E.

To determine the infinitesimal generator

$$A=(a((i,k,m)),((j,l,n))),\qquad (i,k,m),(j,l,n)\in E$$

of this process we use the following arguments :

∗ Transitions due to primary arrival of customers :

–$(i,k,m)\rightarrow(i,k+1,m)$: the rate is λ, for $0\leq i\leq S$, $0\leq k\leq M-1$, $0\leq m\leq H$.
–$(i,M,m)\rightarrow(i,M,m+1)$: the rate is λ, for $0\leq i\leq S$, $0\leq m\leq H-1$.

∗ Transitions due to retrial request of customers :

–$(i,k,m)\rightarrow(i,k+1,m-1)$: the rate is α, for $1\leq i\leq S$, $k=0$, $1\leq m\leq H$.

∗ Transitions due to first stage service completion in the system:

–$(i,k,m)\rightarrow(i-1,k-1,m)$: the rate is μ, for $1\leq i\leq S$, $1\leq k\leq N$, $0\leq m\leq H$.

∗ Transitions due to second stage service completion in the system:

–$(i,k,m) \to (i-1,k-1,m)$: the rate is μ_s, for $1 \le i \le S$, $N+1 \le k \le M$, $0 \le m \le H$.

∗ Transitions due to replenishments:

–$(i,k,m) \to (i+Q,k,m)$: the rate is β, for $0 \le i \le s$, $0 \le k \le M$, $0 \le m \le H$.

∗ We observe that no transition other than the above is possible.

Denoting $\mathbf{q}$=$((q,0,0),(q,0,1),\ldots,(q,0,H),(q,1,0),(q,1,1),\ldots,(q,M,0),(q,M,1),\ldots,(q,M,H))$ for $q = 0,1,\ldots,S$. By ordering states lexicographically, the infinitesimal generator A can be conveniently expressed in a block partitioned matrix with entries

$$[A]_{ij} = \begin{cases} A_2, & j=i, \quad i=s+1,s+2,\ldots,S \\ A_1, & j=i, \quad i=1,2,\ldots,s \\ A_0, & j=i, \quad i=0 \\ B, & j=i-1, \quad i=1,2,\ldots,S \\ C, & j=i+Q, \quad i=0,1,\ldots,s \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

where

$$[A_2]_{ij} = \begin{cases} P_1, & j=i+1, \quad i=0 \\ P_2, & j=i+1, \quad i=1,2,\ldots,M-1 \\ P_3, & j=i, \quad i=M \\ P_4, & j=i, \quad i=0 \\ P_5, & j=i, \quad i=1,2,\ldots,N \\ P_6, & j=i, \quad i=N+1,N+2,\ldots,M-1 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

with

$$[P_1]_{kl} = \begin{cases} \lambda, & l=k, \quad k=0,1,\ldots,H \\ \alpha, & l=k-1, \quad k=1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[P_2]_{kl} = \begin{cases} \lambda, & l=k, \quad k=0,1,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[P_3]_{kl} = \begin{cases} \lambda, & l=k+1, \quad k=0,1,\ldots,H-1 \\ -(\lambda+\mu_s), & l=k, \quad k=0,1,\ldots,H-1 \\ -\mu_s, & l=k, \quad k=H \\ 0, & \text{otherwise} \end{cases}$$

$$[P_4]_{kl} = \begin{cases} -\lambda, & l=k, \quad k=0 \\ -(\lambda+\alpha), & l=k, \quad k=1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[P_5]_{kl} = \begin{cases} -(\lambda+\mu), & l=k, \quad k=0,1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[P_6]_{kl} = \begin{cases} -(\lambda+\mu_s), & l=k, \quad k=0,1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} P_1, & j=i+1, \quad i=0 \\ P_2, & j=i+1, \quad i=1,2,\ldots,M-1 \\ D_3, & j=i, \quad i=M \\ D_4, & j=i, \quad i=0 \\ D_5, & j=i, \quad i=1,2,\ldots,N \\ D_6, & j=i, \quad i=N+1,N+2,\ldots,M-1 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

with

$$[D_3]_{kl} = \begin{cases} \lambda, & l=k+1, \quad k=0,1,\ldots,H-1 \\ -(\lambda+\mu_s+\beta), & l=k, \quad k=0,1,\ldots,H-1 \\ -(\mu_s+\beta), & l=k, \quad k=H \\ 0, & \text{otherwise} \end{cases}$$

$$[D_4]_{kl} = \begin{cases} -(\lambda+\beta), & l=k, \quad k=0 \\ -(\lambda+\alpha+\beta), & l=k, \quad k=1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[D_5]_{kl} = \begin{cases} -(\lambda+\mu+\beta), & l=k, \quad k=0,1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[D_6]_{kl} = \begin{cases} -(\lambda+\mu_s+\beta), & l=k, \quad k=0,1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} P_2, & j = i+1, \quad i = 0,1,2,\ldots,M-1 \\ D_4, & j = i, \quad i = 0 \\ J_3, & j = i, \quad i = M \\ J_4, & j = i, \quad i = 0,1,\ldots,M-1 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

with

$$[J_3]_{kl} = \begin{cases} \lambda, & l = k+1, \quad k = 0,1,\ldots,H-1 \\ -(\lambda+\beta), & l = k, \quad k = 0,1,\ldots,H-1 \\ -\beta, & l = k, \quad k = H \\ 0, & \text{otherwise} \end{cases}$$

$$[J_4]_{kl} = \begin{cases} -(\lambda+\beta), & l = k, \quad k = 0,1,2,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[B]_{ij} = \begin{cases} L_1, & j = i-1, \quad i = 1,2,\ldots,N \\ L_2, & j = i-1, \quad i = N+1,N+2,\ldots,M \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

with

$$[L_1]_{kl} = \begin{cases} \mu, & l = k, \quad k = 0,1,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[L_2]_{kl} = \begin{cases} \mu_s, & l = k, \quad k = 0,1,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

$$[C]_{ij} = \begin{cases} C_1, & j = i, \quad i = 0,1,2,\ldots,M \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

$$[C_1]_{kl} = \begin{cases} \beta, & l = k, \quad k = 0,1,\ldots,H \\ 0, & \text{otherwise} \end{cases}$$

It can be noted that the matrices A_2, A_1, A_0, B and C are square matrices of order $(M+1)(H+1)$ and $P_1, P_2, P_3, P_4, P_5, P_6, D_3, D_4, D_5, D_6, J_3, J_4, L_1, L_2$ and C_1 are square matrices of order $(H+1)$.

3.1 Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{L(t), X(t), Y(t) : t \geq 0\}$ on the finite state space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

Table 1: Total expected cost rate as a function of S and s

S \ s	11	12	13	14	15	16
72	7.43067	7.42069	7.41634	7.41713	7.42262	7.43241
73	7.42859	7.41862	7.41421	7.41486	7.42014	7.42965
74	7.42721	7.41725	7.41279	7.41332	7.41841	7.42767
75	7.42647	7.41655	7.41205	7.41248	7.41740	7.42643
76	**7.42636**	**7.41648**	**7.41196**	**7.41230**	**7.41707**	**7.42589**
77	7.42685	7.41702	7.41249	7.41275	7.41739	7.42601
78	7.42791	7.41814	7.41360	7.41381	7.41833	7.42677
79	7.42952	7.41982	7.41529	7.41544	7.41986	7.42814
80	7.43164	7.42202	7.41751	7.41763	7.42195	7.43008

Table 2: Total expected cost rate as a function of S and M

S \ M	14	15	16	17	18
62	6.86026	6.85860	6.85872	6.86131	6.86588
63	6.85890	6.85647	6.85600	6.85811	6.86230
64	**6.85837**	6.85522	6.85419	6.85586	6.85967
65	6.85865	**6.85480**	6.85323	6.85448	6.85795
66	6.85967	6.85516	**6.85308**	**6.85393**	6.85707
67	6.86141	6.85625	6.85369	6.85416	**6.85698**
68	6.86381	6.85805	6.85502	6.85513	6.85766

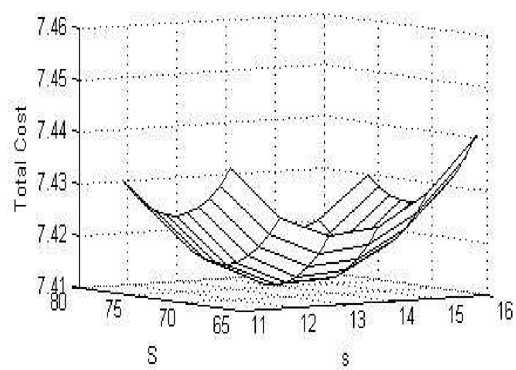

Fig. 1: Convexity of the total cost for various combinations of S and M. $\lambda = 9, \alpha = 1, \mu = 10, \mu_s = 14, \beta = 1.3, c_h = 0.04, c_s = 3.8, c_b = 0.004, c_w = 1, c_o = 0.2, M = 12, H = 5$

$$\pi^{(i,j,k)} = \lim_{t\to\infty} Pr[L(t) = i, X(t) = j, Y(t) = k | L(0), X(0), Y(0)]$$

exists. Let

$$\Pi = (\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(S)})$$

which is partitioned as follows:

$$\Pi^{(i)} = (\Pi^{(i,0)}, \Pi^{(i,1)}, \Pi^{(i,2)}, \ldots, \Pi^{(i,N)}, \Pi^{(i,N+1)}, \ldots, \Pi^{(i,M)}), \quad i = 0,1,\ldots,S$$

Further the above vectors are also partitioned as follows:

$$\Pi^{(i,j)} = (\Pi^{(i,j,0)}, \Pi^{(i,j,1)}, \ldots, \Pi^{(i,j,H)}),$$
$$i = 0,1,2,\ldots,S, \qquad j = 0,1,2,\ldots,M$$

The vector of limiting probabilities Π then satisfies

$$\Pi A = \mathbf{0} \text{ and } \Pi \mathbf{e} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

$$\Pi^{(i)}A_0 + \Pi^{(i+1)}B = \mathbf{0},\ i = 0,$$
$$\Pi^{(i)}A_1 + \Pi^{(i+1)}B = \mathbf{0},\ i = 1,2,\ldots,s,$$
$$\Pi^{(i)}A_2 + \Pi^{(i+1)}B = \mathbf{0},\ i = s+1,\ldots,Q-1,$$
$$\Pi^{(0)}C + \Pi^{(i)}A_2 + \Pi^{(i+1)}B = \mathbf{0},\ i = Q,$$
$$\Pi^{(i-Q)}C + \Pi^{(i)}A_2 + \Pi^{(i+1)}C = \mathbf{0},\ i = Q+1,\ldots,S-1,$$
$$\Pi^{(i-Q)}C + \Pi^{(i)}A_2 = \mathbf{0},\ i = S$$

After lengthy simplifications, the above equations, (except (*)), yield

$$\Pi^{(i)} = \Pi^{(Q)}\Omega_i,\ i = 0,1,\ldots,S.$$

where

$$\Omega_i = \begin{cases} (-1)^{Q-i}(BA_2^{-1})^{(Q-(s+1))}(BA_1^{-1})^{s}(BA_0^{-1}), & i = 0, \\ (-1)^{Q-i}(BA_2^{-1})^{(Q-(s+1))}(BA_1^{-1})^{((s+1)-i)}, & i = 1,\ldots,s, \\ (-1)^{Q-i}(BA_2^{-1})^{(Q-i)}, & i = s+1,\ldots,Q-1, \\ I, & i = Q, \\ \sum_{j=0}^{S-i}(-1)^{((2Q+1)-i)}(BA_2^{-1})^{((S+s)-(i+j+1))} & \\ \times(BA_1^{-1})^{(j+1)}(CA_2^{-1}), & i = Q+1,\ldots,S. \end{cases}$$

$\Pi^{(Q)}$ can be obtained by solving equation $(*)$ and $\Pi\mathbf{e} = 1$. That is,

$$\Pi^{(Q)}\Big((-1)^{Q}(BA_2^{-1})^{(Q-(s+1))}(BA_1^{-1})^{s}(BA_0^{-1})C + A_2 + \sum_{j=0}^{s-1}(-1)^{Q}(BA_2^{-1})^{(2(s-1)-j)}(BA_1^{-1})^{(j+1)}(CA_2^{-1})B\Big) = \mathbf{0},$$

and

4 System Performance Measures

In this section some performance measures of the system under consideration in the steady state are derived.

4.1 Expected inventory level

Let E_i denote the mean inventory level in the steady state. Then

$$E_i = \sum_{i=1}^{S}\sum_{j=0}^{M}\sum_{k=0}^{H} i\left[\pi^{(i,j,k)}\right]$$

4.2 Expected reorder rate

Let E_r denote the expected reorder rate in the steady state. Then

$$E_r = \sum_{k=0}^{H}\left[\sum_{j=1}^{N}\mu\left[\pi^{(s+1,j,k)}\right] + \sum_{j=N(+1)}^{M}\mu_s\left[\pi^{(s+1,j,k)}\right]\right]$$

4.3 Expected blocked customers

Let E_b denote the expected blocked customers in the steady state. Then

$$E_b = \sum_{i=0}^{S}\sum_{k=0}^{H-1}\lambda\left[\pi^{(i,M,k)}\right]$$

Table 3: Effect of λ and μ on optimal values

μ / λ	9.9	10.0	10.1	10.2	10.3
8.9	75 13 7.36182	75 13 7.24200	76 14 7.12423	76 14 7.00902	76 14 6.89661
9.0	75 13 7.53310	76 14 7.41196	76 14 7.29276	76 14 7.17561	76 14 7.06100
9.1	76 13 7.70359	76 13 7.58168	76 13 7.46158	76 14 7.34313	76 14 7.22675
9.2	76 13 7.87276	76 13 7.75058	76 13 7.62988	77 14 7.51084	77 14 7.39304
9.3	76 13 8.04002	76 13 7.91803	76 13 7.79718	77 13 7.67788	77 14 7.55938

4.4 Expected number of customers in the waiting hall

Let E_w denote the expected number of customers in the waiting hall in the steady state. Then

$$E_w = \sum_{i=0}^{S}\sum_{j=1}^{M}\sum_{k=0}^{H} j\left[\pi^{(i,j,k)}\right]$$

Table 4: Effect of λ and μ_s on optimal values

μ_s / λ	13.5	13.6	13.7	13.8	13.9
8.9	75 13 7.29498	75 13 7.28408	75 13 7.27334	75 13 7.26275	75 13 7.25230
9.0	75 13 7.46649	76 13 7.45528	76 13 7.44423	76 13 7.43333	76 13 7.42257
9.1	76 13 7.63741	76 13 7.62597	76 13 7.61468	76 13 7.60354	76 13 7.59254
9.2	76 13 7.80276	76 13 7.79563	76 13 7.78415	76 13 7.77281	76 13 7.76163
9.3	76 13 7.97545	76 13 7.96367	76 13 7.95204	77 13 7.94056	77 13 7.92922

Table 7: Effect of μ and μ_s on optimal values

μ_s / μ	13.5	13.6	13.7	13.8	13.9
9.9	75 13 7.58836	75 13 7.57701	75 13 7.56582	75 13 7.55477	75 13 7.54367
10.0	75 13 7.46649	76 13 7.45528	76 13 7.44423	76 13 7.43333	76 13 7.42257
10.1	76 13 7.34653	76 13 7.33551	76 14 7.32464	76 14 7.31387	76 14 7.30324
10.2	76 14 7.22839	76 14 7.21754	76 14 7.20683	76 14 7.19628	76 14 7.18587
10.3	76 14 7.12259	76 14 7.10197	76 14 7.09150	76 14 7.08119	76 14 7.07102

Table 5: Effect of α and μ on optimal values

μ / α	9.9	10.0	10.1	10.2	10.3
0.8	75 13 7.58614	76 13 7.46536	76 13 7.34632	76 14 7.22874	76 14 7.11343
0.9	75 13 7.55789	76 13 7.43682	76 14 7.31757	76 14 7.20013	76 14 7.08513
1.0	75 13 7.53310	76 13 7.41196	76 14 7.29276	76 14 7.17561	76 14 7.06100
1.1	76 13 7.51125	76 13 7.39019	76 14 7.27120	76 14 7.15441	76 14 7.04025
1.2	76 13 7.49190	76 13 7.37101	76 14 7.25235	76 14 7.13595	76 14 7.02224

Table 8: Effect of μ and β on optimal values

β / μ	1.1	1.2	1.3	1.4	1.5
9.9	79 15 7.63405	77 14 7.58066	75 13 7.53310	74 13 7.49019	73 12 7.45041
10.0	79 15 7.51412	77 14 7.46007	76 13 7.41196	74 13 7.436813	73 12 7.32806
10.1	79 15 7.39629	77 14 7.34160	76 14 7.29276	74 13 7.24828	73 12 7.20794
10.2	80 16 7.28061	78 15 7.22527	76 14 7.17561	74 13 7.13083	73 12 7.09022
10.3	80 16 7.16693	78 15 7.11105	76 14 7.06100	75 13 7.01588	73 12 6.97507

Table 6: Effect of α and μ_s on optimal values

μ_s / α	13.5	13.6	13.7	13.8	13.9
0.8	75 13 7.51893	75 13 7.50795	75 13 7.49710	75 13 7.48639	76 13 7.47582
0.9	75 13 7.49098	75 13 7.47987	76 13 7.46891	76 13 7.45807	76 13 7.44738
1.0	75 13 7.46649	76 13 7.45528	76 13 7.44423	76 13 7.43333	76 13 7.42257
1.1	76 13 7.44491	76 13 7.43366	76 13 7.42256	76 13 7.41162	76 13 7.40083
1.2	76 13 7.42582	76 13 7.41455	76 13 7.40343	76 13 7.39247	76 13 7.38167

Table 9: Effect of μ_s and β on optimal values

β / μ_s	1.1	1.2	1.3	1.4	1.5
13.5	79 15 7.56777	77 14 7.51417	76 13 7.46649	74 13 7.42322	73 12 7.38340
13.6	79 15 7.55674	77 14 7.50306	76 13 7.45528	74 13 7.41191	73 12 7.37204
13.7	79 15 7.54587	77 14 7.49209	76 13 7.44423	74 13 7.40074	73 12 7.36083
13.8	79 15 7.53514	77 14 7.48127	76 13 7.43333	74 13 7.38973	73 12 7.34976
13.9	79 15 7.52456	77 14 7.47060	76 13 7.42257	74 13 7.37886	73 12 7.33884

4.5 Expected number of customers in the orbit

Let E_o denote the expected number of customers in the orbit in the steady state. Then

$$E_o = \sum_{i=0}^{S}\sum_{j=0}^{M}\sum_{k=1}^{H} k\left[\pi^{(i,j,k)}\right]$$

4.6 Probability of arriving customer is lost

Let E_l denote the probability of arriving customer is lost in the steady state. Then

$$E_l = \sum_{i=0}^{S}\left[\pi^{(i,M,H)}\right]$$

Table 10: Effect of c_s and c_h on optimal values

c_s \ c_h	7.78	7.79	7.80	7.81	7.82
0.038	77 14 7.33149	77 14 7.33272	77 14 7.33395	77 14 7.33518	77 14 7.33641
0.039	77 14 7.37089	77 14 7.37212	77 14 7.37335	77 14 7.37458	77 14 7.37581
0.040	76 13 7.40950	76 13 7.41073	76 13 7.41196	76 13 7.41319	76 13 7.41442
0.041	76 13 7.44750	75 13 7.44875	75 13 7.45000	75 13 7.45125	75 13 7.45250
0.042	74 13 7.48515	74 13 7.48642	74 13 7.48769	74 13 7.48896	74 13 7.49023

Table 11: Effect of c_b and c_w on optimal values

c_b \ c_w	0.98	0.99	1.00	1.01	1.02
0.003	76 13 7.32369	76 13 7.36778	76 13 7.41187	76 13 7.45597	76 14 7.49992
0.004	76 13 7.32377	76 13 7.36786	76 13 7.41196	76 13 7.45605	76 14 7.50001
0.005	76 13 7.32385	76 13 7.36795	76 13 7.41204	76 13 7.45613	76 14 7.50009
0.006	76 13 7.32394	76 13 7.36803	76 13 7.41212	76 13 7.45622	76 14 7.50017
0.007	76 13 7.32402	76 13 7.36811	76 13 7.41221	76 13 7.45630	76 14 7.50025

Table 12: Effect of c_b and c_o on optimal values

c_b \ c_o	0.18	0.19	0.20	0.21	0.22
0.003	75 13 7.36131	76 13 7.38661	76 13 7.41187	76 14 7.43698	76 14 7.46174
0.004	75 13 7.36139	76 13 7.38670	76 13 7.41196	76 14 7.43706	76 14 7.46182
0.005	75 13 7.36147	76 13 7.38678	76 13 7.41204	76 14 7.43714	76 14 7.46191
0.006	75 13 7.36156	76 13 7.38086	76 13 7.41212	76 14 7.43723	76 14 7.46199
0.007	75 13 7.36164	76 13 7.38694	76 13 7.41221	76 14 7.43731	76 14 7.46207

4.7 Overall rate of retrials

Let E_{or} denote the overall rate of retrials in the steady state. Then

$$E_{or} = \sum_{i=0}^{S}\sum_{j=0}^{M}\sum_{k=1}^{H} \alpha \left[\pi^{(i,j,k)}\right]$$

Table 13: Effect of c_w and c_o on optimal values

c_w \ c_o	0.18	0.19	0.20	0.21	0.22
0.98	75 13 7.27313	75 13 7.29846	76 13 7.32377	76 13 7.34903	76 14 7.37412
0.99	75 13 7.31726	76 13 7.34259	76 13 7.36786	76 13 7.39313	76 14 7.41797
1.00	76 13 7.36139	76 13 7.38670	76 13 7.41196	76 14 7.43706	76 14 7.46182
1.01	76 13 7.40552	76 13 7.43079	76 13 7.45605	76 14 7.48092	76 14 7.50568
1.02	76 13 7.44962	76 13 7.47488	76 14 7.50001	76 14 7.52477	76 14 7.54953

4.8 The successful retrial rate

Let E_{sr} denote the successful retrial rate in the steady state. Then

$$E_{sr} = \sum_{i=1}^{S}\sum_{k=1}^{H} \alpha \left[\pi^{(i,0,k)}\right]$$

4.9 The fraction of successful rate of retrial

Let E_{fr} denote the fraction of successful retrial rate in the steady state. Then

$$E_{fr} = \frac{E_{sr}}{E_{or}}$$

5 Cost Analysis

To compute the total expected cost per unit time (total expected cost rate), the following costs, are considered.

c_h : The inventory holding cost per unit item per unit time

c_s : Setup cost per order

c_b : Cost per blocking customer

c_w : Waiting time cost of a customer per unit time

c_o : Waiting time cost of a orbiting customer per unit time

The long run total expected cost rate is given by

$$TC(S,s,M,H) = c_hE_i + c_sE_r + c_bE_b + c_wE_w + c_oE_o$$

Substituting E's into the above equation, we obtain

$$TC(S,s,M,H) = c_h \sum_{i=1}^{S}\sum_{j=0}^{M}\sum_{k=0}^{H} i\left[\pi^{(i,j,k)}\right] +$$
$$c_s \sum_{k=0}^{H}\left[\sum_{j=1}^{N}\mu\left[\pi^{(s+1,j,k)}\right] + \sum_{j=N+1}^{M}\mu_s\left[\pi^{(s+1,j,k)}\right]\right] +$$
$$c_b \sum_{i=0}^{S}\sum_{k=0}^{H-1}\lambda\left[\pi^{(i,M,k)}\right] + c_w \sum_{i=0}^{S}\sum_{j=1}^{M}\sum_{k=0}^{H} j\left[\pi^{(i,j,k)}\right] +$$
$$c_o \sum_{i=0}^{S}\sum_{j=0}^{M}\sum_{k=1}^{H} k\left[\pi^{(i,j,k)}\right]$$

5.1 Numerical Examples

In this section, we discuss some numerical examples that reveal the possible convexity of the total expected cost rate. The convexity of the function $TC(S,s,M,H)$ we have explored the behaviour of this function by considering it as functions of any two variables by fixing the others are constants.

Table 1 gives the total expected cost rate for various combinations of S and s. It can be seen that the minimum expected cost per unit time of 7.41196 is achieved at $TC(S^*,s^*,12,5)$ here $S^* = 76$ and $s^* = 13$ by fixing parameters and costs as $\lambda = 9$, $\alpha = 1$, $\mu = 10$, $\mu_s = 14$, $\beta = 1.3$ and $c_s = 0.04$, $c_h = 7.8$, $c_b = 0.004$, $c_w = 1$, $c_o = 0.2$ respectively. Convexity of the total cost for various combinations of S and s is given in figure 1.

Table 2 gives the total expected cost rate for various combinations of S and M. It can be seen that the minimum expected cost per unit time of 6.85308 is achieved at $TC(S^*,13,M^*,5)$ here $S^* = 66$ and $M^* = 16$ by fixing parameters and costs as $\lambda = 9$, $\alpha = 1$, $\mu = 10$, $\mu_s = 14$, $\beta = 1.3$ and $c_s = 0.04$, $c_h = 3.8$, $c_b = 0.004$, $c_w = 1$, $c_o = 0.2$ respectively. In tables 1 and 2, the minimum total expected cost rate for each row is underlined and that for each column is shown in bold.

Example 1: Now we perform a sensitivity investigation to the optimal values (S^*,s^*) and the total expected cost rate based on changes in the system parameters. The numerical results are shown in tables 3-9 for various system parameters λ, α, μ, μ_s, β (by considering the cost values $c_s = 0.04$, $c_h = 7.8$, $c_b = 0.004$, $c_w = 1$, $c_o = 0.2$ respectively). We observe the following monotonic behavior of (S^*,s^*)

1. The total expected cost rate increases when λ increases and the total expected cost rate decreases when α,μ,μ_s and β increase.
2. The optimal values S^* monotonically increases when λ,α,μ and μ_s increase. We also note that S^* monotonically decreases when β increases.
3. The optimal values s^* monotonically increases when μ and μ_s increase. We also note that s^* monotonically decreases when λ and β increase.

Example 2: In this example, numerical results are shown in tables 10-13 for various cost values c_s, c_h, c_b, c_w, c_o (by considering the parameters $\lambda = 9$, $\alpha = 1$, $\mu = 10$, $\mu_s = 14$, $\beta = 1.3$ respectively). We observe the following monotonic behavior of (S^*,s^*)

1. The total expected cost rate increases when each of c_s, c_h, c_b, c_w and c_o increase.
2. As is to be expected, the optimal values S^* and s^* monotonically increase when c_w and c_o increase. Also, we notice that as S^* and s^* monotonically decrease when c_s increases.

6 Conclusion

In this article, we analyzed a continuous review stochastic retrial queueing-inventory system with Poisson inputs, (s,Q) replenishment policy and two stage service. The interarrival times of customers, lead times, the retrial times and service times are assumed to have independent exponential distribution. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state case. Various system performance measures and the long-run total expected cost rate are derived. The authors are working in the direction of MAP (Markovian arrival process) arrivals and service times follow PH-distributions.

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