Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120407

Multiplicity Results of Multi-Point Boundary Value Problem of Nonlinear Fractional Differential Equations

*Kamal Shah**, *Salman Zeb and Rahmat Ali Khan*

Department of Mathematics, University of Malakand, Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan

Received: 16 Feb. 2017, Revised: 19 Apr. 2017, Accepted: 20 Apr. 2018
Published online: 1 Jul. 2018

Abstract: This article is concerned to study the existence and multiplicity of positive solutions to a class with multi-point boundary conditions of nonlinear differential equations having fractional order. Where the nonlinear term is a continuous function. Sufficient conditions for multiplicity results of positive solutions to the problem under consideration are obtained by using Leggett - Williams 's fixed point theorem. Further, the generalization of the concerned results are also obtained for more than three positive solutions. For the demonstration of our results, we provide an example.

Keywords: Nonlinear fractional differential equations; multi-point boundary conditions; Multiplicity of solutions; Green's function; Classical fixed point theorems.

1 Introduction

Differential equations of fractional order is one of the fast growing area of research in the field of mathematics. The concerned area has been recently proved to be valuable tools in the modeling of many phenomena in biology, chemistry, physics, networking, dynamics, fluid mechanics, viscoelasticity, electro-chemistry, control theory, movement through porous media, electromagnetic theory, etc [26,24,2,13,15]. The mentioned area of differential equations of fractional order became a candidate to solve problems of complex systems that appear in various fields of sciences, [3]. The application of differential equations of fractional order can also be traced in physics, see[4]. Recently, the fractional differential equations have been applied to model the phenomenon and process of manufacturing of polymers and rheology, see [5]. Fractional derivatives provide a powerful tools for the description of memory and hereditary properties of various materials and processes. Fractional-order derivatives and integrals are proved to be more useful for the formulation of certain electrochemical problems than the classical models, (see for detail [6]). The phenomenon related to chaos, fractals theory and bioengineering can be excellently models with the help of fractional differential equations as compared to classical ones, see [7].

The area devoted to study the existence and uniqueness of solutions to boundary value problems of fractional differential equations has been studied very well and plenty of research work is available on it, (see for example [10,11,12,16,19,29,20,36] and the references therein). Boundary value problems of differential and integral equations arise in various branches of physics because any physical differential equation represents it. Further, boundary value problems of differential equations have significant applications in the mathematical modeling of physical, biological and engineering problems, for detail see [37]. Recently the applications of boundary value problems have been traced in chemical reactor theory and related applications, (see [1]). To make a boundary value problem useful in applications, it should be well posed. For these purposes existence theory is very important aspect of differential equations which tell us about the aforesaid behavior. The concerned study was carried out by using the tools of classical Fixed Point theory such as Banach Fixed Point theorem, Leray-Schauder Fixed Point theorem etc to form conditions for at least one solution.
In [14] the author established appropriate condition for the existence and multiplicity of positive solutions to the

* Corresponding author e-mail: kamalshah408@gmail.com

following problem

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, t\in(0,1), \\ u(0)=u(1)=0, \end{cases}$$

where $1<\alpha\leq 2$, D^{α} is the Riemann-Liouvilli fractional derivative. The aforesaid problems were further investigated in [32] under the following conditions

$$\begin{cases} D^{\alpha}u(t)=f(t,u(t));\ t\in(0,1), \\ u(0)=u(1)=u^{'}(0)=u^{'}(1)=0, \end{cases}$$

where, $3<\alpha\leq 4$, $f:[0,1]\times R\to R$, and D is the standard Riemann-Liouville derivative.
Goodrich [17] studied the existence of at least three solutions for the following boundary value problems

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0,\ t\in(0,1), \\ u^{(i)}(0)=0,\ i=0,1,2...n-2, \\ D^{\gamma}u(1)=0,\ 2\leq\gamma\leq n-2, \end{cases}$$

where $n-1<\alpha\leq n$, and D is the standard Riemann-Liouvilli fractional derivative of order α, and $n>3, n\in\mathbb{N}$, $f:J\times[0,\infty)\to[0,\infty)$ is continuous function.
Similarly in [29], authors developed sufficient conditions for existence and uniqueness of nontrivial solutions by using LeraySchauder Fixed Point theorem of nonlinear alternative, and condensing mapping principle for nonlinear fractional order differential equation given by

$$\begin{cases} D_t^{\alpha}u(t)=f(t,u(t));\quad t\in(0,1), \\ D^{\alpha-2}u(0)=\gamma_0 D^{\alpha-2}u(T), \\ \quad D^{\alpha-1}u(0)=\mu_0 D^{\alpha-1}u(T), \end{cases}$$

where $1<\alpha\leq 2$, $\gamma_0,\mu_0\neq 1$.
In very recent years the concerned area has been explored very well, we refer some fresh work [33,34]. To the best of our knowledge, the area devoted to the study of multiple positive solutions corresponding to multi point boundary value problems of nonlinear fractional order differential equations is rarely studied. In this regard, very few papers can be found in the literature dealing with the existence and multiple results to multi-point boundary value problems for fractional differential equations [27, 32,35].
In this paper, we investigate sufficient conditions for multiplicity of positive solutions to the (BVP) given in (1)

$$\begin{cases} D_{0+}^{q}u(t)+f(t,u(t))=0,\quad t\in(0,1), \\ u^{(i)}(0)=0,\ 0\leq i\leq n-2, \\ \quad u(1)=\sum_{i=1}^{m-2}\delta_i u(\eta_i), \end{cases}\tag{1}$$

where $n-1<q\leq n$ and D_{0+}^{q} is the standard Riemann-Liouville fraction derivative of order $q, n\geq 3$, $\delta_i,\eta_i\in(0,1)$ with $\sum_{i=1}^{m-2}\delta_i\eta_i^{q-1}<1$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. The concerned conditions are obtained by using the classical Fixed Point theorem such as Leggett-Williams 's Fixed Point theorem for triple positive solutions. Moreover, the results are further extended to search out conditions demonstrating multiple positive solutions. For the applicability of our results, we provide an example.

2 Preliminaries

In this section, we review some notation, definitions and preliminary results which are used throughout this paper. The concerned materials can be found in [2,24,26].
Definition 2.1. The fractional integral of order $q>0$ of a function $y:(0,\infty)$ is given by

$$I_{0+}^{q}y(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s)\,ds,$$

provided that the integral converges.
Definition 2.2. The fractional derivative of order $q>0$ of a continuous function $y:(0,\infty)\to R$ is given by

$$D_{0+}^{q}y(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-q-1}y(s)\,ds,$$

where n=[q]+1, provided that the right side is point wise defined on $(0,\infty)$.
Definition 2.3. A mapping θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\theta:P\to[0,\infty)$ is continuous and

$$\theta(tx+(1-t)y)\geq t\theta(x)+(1-t)\theta(y),$$

for all $x,y\in P$ and $0\leq t\leq 1$.
The next two lemmas provide an important base for obtaining the equivalent integral equation of (BVP) (1).
Lemma 2.3[28] If we assume $u\in C(0,1)\cap L(0,1)$, then the fractional differential equation of order $q>0$

$$D_{0+}^{q}u(t)=0,$$

has a unique solution of the form

$$u(t)=C_1t^{q-1}+C_2t^{q-2}+...+C_Nt^{q-N},\quad C_i\in R,\quad i=1,2,...,N.$$

The following law of composition can be easily deduced from above Lemma .
Lemma 2.4. Assume that $u\in C(0,1)\cap L(0,1)$, with a fractional derivative of order q that belongs to $C(0,1)\cap L(0,1)$, then

$$I_{0+}^{q}D_{0+}^{q}u(t)=u(t)+C_1t^{q-1}+C_2t^{q-2}+...+C_nt^{q-n},$$

where $C_i\in R,\quad i=1,2,...,n$.
Lemma 2.5. [23] Let P be a cone in a real Banach space

E, $P_c = \{u \in P : \|u\| \le c\}$, θ a nonnegative continuous concave function on P such that $\theta(u) \le \|u\|$ for all $u \in \tilde{P}_c$, and $P(\theta,b,d) = \{u \in P : b \le \theta(u), \|u\| \le d\}$. Suppose $T : \tilde{P}_c \to \tilde{P}_c$ is completely continuous operator such that there exist constants $0 < a < b < d \le c$ satisfy

$(i)\{u \in P(\theta,b,d) \mid \theta(u) > b\} \ne \emptyset$, and $\theta(Tu) > b$ for $u \in P(\theta,b,d)$
$(ii)\|Tu\| < a$ for $u \le a$
$(iii)\theta(Tu) > b$ for $u \in P(\theta,b,c)$ with $\|Tu\| > d$,

then T has at least three fixed points u_1,u_2,u_3 with $a > \|u_1\|$, $\theta(u_2) > b$, $\|u_3\| > a$ with $b > \theta(u_3)$.

3 Existence of multiplicity results

In this section, we develop sufficient conditions, under which the (BVP) (1) has at least three solutions. More over the criteria is extended to develop sufficient conditions leading to multiplicity of positive solutions. Use $X = C[0,1]$ for the Banach space of all continuous real-valued functions on $[0,1]$ with norm $\|u\| = \max\limits_{0\le t\le 1}|u(t)|$ and a cone by P such that

$$P = \{u \in X : u(t) \ge 0, t \in [0,1]\}.$$

Define nonnegative continuous concave functional θ on the cone P as given by

$$\theta(u) = \min_{0.25\le t\le 0.75}|u(t)|. \tag{2}$$

Lemma 3.1. For $y(t) \in C[0,1]$, the linear BVP

$$D^q_{0+}u(t) + y(t) = 0; \quad 0 < t < 1, n-1 < q \le n,\ n \ge 3,$$
$$u^{(i)}(0) = 0,\ 0 \le i \le n-2 \quad u(1) = \sum_{i=1}^{m-2}\delta_i u(\eta_i), \tag{3}$$

has a unique solution of the form $u(t) = \int_0^1 G(t,s)y(s)\,ds$, where the Green function

$$G(t,s) = G_1(t,s) + t^{q-1}\sum_{i=1}^{m-2}\delta_i G_2(\eta_i,s) \tag{4}$$

is given by

$$G_1(t,s) = \begin{cases} \dfrac{t^{q-1}(1-s)^{q-1} - (1-\lambda)(t-s)^{q-1}}{(1-\lambda)\Gamma(q)}, \\ 0 \le s \le t \le 1, \\ \dfrac{t^{q-1}(1-s)^{q-1}}{(1-\lambda)\Gamma(q)}, 0 \le t \le s \le 1, \end{cases} \tag{5}$$

$$G_2(\eta,s) = \begin{cases} \dfrac{(1-s)^{q-1} - (\eta_i - s)^{q-1}}{(1-\lambda)\Gamma(q)}, \\ 0 \le s \le \eta_i,\ \text{for } i = 1,2,3,...,m-2, \\ \dfrac{(1-s)^{q-1}}{(1-\lambda)\Gamma(q)}, \\ \eta_i \le s \le 1,\ \text{for } i = 1,2,3,...,m-2. \end{cases} \tag{6}$$

*Proof.*In view of Lemma 2.4, we obtain

$$u(t) = -I^q_{0+}y(t) + C_1t^{q-1} + C_2t^{q-2} + C_3t^{q-3} + \cdots + C_nt^{n-q}, \tag{7}$$

for some $C_i \in R$. The initial condition $u^{(i)}(0) = 0$ implies $C_2 = C_3 = C_4 = \cdots = C_n = 0$ and the boundary condition

$$u(1) = \sum_{i=1}^{m-2}\delta_i u(\eta_i),$$

yields

$$C_1 = \frac{1}{1-\lambda}\left[I^q_{0+}y(1) - \sum_{i=1}^{m-2}\delta_i I^q_{0+}y(\eta_i)\right],$$

where

$$\lambda = \sum_{i=1}^{m-2}\delta_i\eta_i^{q-1} < 1.$$

Hence, (7) takes the form

$$u(t) = -I^q_{0+}y(t) + \frac{t^{q-1}}{1-\lambda}\left[I^q_{0+}y(1) - \sum_{i=1}^{m-2}\delta_i I^q_{0+}y(\eta_i)\right]. \tag{8}$$

we write (8) as

$$\begin{aligned} u(t) &= \frac{-1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + \frac{t^{q-1}}{(1-\lambda)\Gamma(q)}\int_0^1 (1-s)^{q-1}y(s)ds \\ &- \frac{t^{q-1}}{(1-\lambda)\Gamma(q)}\sum_{i=1}^{m-2}\delta_i\int_0^{\eta_i}(\eta_i - s)^{q-1}y(s)ds \\ &= \frac{-1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds \\ &+ \frac{1}{(1-\lambda)\Gamma(q)}\left[\int_0^t [t(1-s)]^{q-1}y(s)ds + \int_t^1 [t(1-s)]^{q-1}y(s)ds\right] \\ &+ \frac{t^{q-1}}{(1-\lambda)\Gamma(q)}\sum_{i=1}^{m-2}\delta_i\int_0^{\eta_i}\left[(1-s)^{q-1} - (\eta_i - s)^{q-1}\right]y(s)ds \\ &+ \frac{t^{q-1}}{(1-\lambda)\Gamma(q)}\sum_{i=1}^{m-2}\delta_i\int_{\eta_i}^1 (1-s)^{q-1}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

Hence in view of this Lemma (1) can be written as

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 3.2. The Green's function defined by (4) satisfies the following conditions:

$(i)G(t,s) \in C([0,1]\times[0,1])$ and $G(t,s) > 0$, for $t,s \in (0,1)$;
(ii)There exists a positive function $\gamma(s) \in C((0,1),(0,\infty))$ such that

$$\min_{0.25\le t\le 0.75} G(t,s) \ge \gamma(s)L(s),\ \text{for } 0 < s < 1, \tag{9}$$

where $L(s) = G_1(s,s) + \frac{\sum_{i=1}^{m-2}\delta_i}{(1-\lambda)\Gamma(q)}G_2(\eta_i,s),\ s \in (0,s)$.

Proof.(i) The expression for $G(t,s)$ in (4) clearly shows that $G(t,s) > 0$, for $s,t \in (0,1)$. Moreover continuity of $G(t,s)$ is obvious. (ii) Let us denote

$$g_1(t,s) = \frac{t^{q-1}(1-s)^{q-1} - (1-\lambda)(t-s)^{q-1}}{(1-\lambda)\Gamma(q)},$$
$$g_2(t,s) = \frac{t^{q-1}(1-s)^{q-1}}{(1-\lambda)\Gamma(q)},$$

then, $\frac{\partial g_1(t,s)}{\partial t} < 0$ for $s \le t$, which implies that $g_1(t,s)$ is decreasing function. While $\frac{\partial g_2(t,s)}{\partial t} > 0$ for $s \le t$ yields that $g_2(t,s)$ is increasing function. It follows that $G_1(t,s)$ is decreasing with respect to t for $s \le t$ and increasing with respect to t for $t \le s$. Consequently,

$$\min_{0.25\le t\le 0.75} G_1(t,s) = \begin{cases} g_1(0.75,s), \ s \in (0,0.25], \\ \min\{g_1(0.75,s), g_2(0.25,s)\}, \\ s \in [0.25,0.75], \\ g_2(0.25,s), \ s \in [0.75,1), \end{cases}$$
$$= \begin{cases} g_1(0.75,s), \ s \in (0,\varepsilon], \\ g_2(0.25,s), \ s \in [\varepsilon,1), \end{cases}$$
$$= \begin{cases} \dfrac{[0.75(1-s)]^{q-1} - (1-\lambda)(0.75-s)^{q-1}}{(1-\lambda)\Gamma(q)}, \ s \in (0,\varepsilon], \\ \dfrac{[0.25(1-s)]^{q-1}}{(1-\lambda)\Gamma(q)}, \ s \in [\varepsilon,1), \end{cases}$$

where ε is the unique solutions obtained from $g_1(0.75,\varepsilon) = g_2(0.25,\varepsilon)$. Further

$$\max_{t\in[0,1]} G_1(t,s) = G_1(s,s) = \frac{s^{q-1}(1-s)^{q-1}}{(1-\lambda)\Gamma(q)} > 0, \ s \in (0,1).$$

Setting

$$\gamma(s) = \begin{cases} \dfrac{[0.75(1-s)]^{q-1} - (1-\lambda)(0.75-s)^{q-1}}{[s(1-s)]^{q-1}}, \ s \in (0,r], \\ \left(\dfrac{0.25}{s}\right)^{q-1}, s \in [r,1). \end{cases}$$

$$\min_{0.25\le t\le 0.75} G_1(t,s) = \gamma(s) G_1(s,s), \ \ s \in (0,s).$$

Thus

$$\min_{0.25\le t\le 0.75} G(t,s) \ge \min_{0.25\le t\le 0.75} G(t,s)$$
$$+ \min_{0.25\le t\le 0.75} t^{q-1} \sum_{i=1}^{m-2} \delta_1 G_2(\eta_i,s)$$
$$\ge \gamma(s) G_1(s,s) + (0.25)^{q-1} \sum_{i=1}^{m-2} \delta_1 G_2(\eta_i,s) = \chi(s), \ s \in (0,1),$$
$$\max_{t\in[0,1]} G(t,s) \le \max_{t\in[0,1]} G_1(t,s)$$
$$+ \max_{0.25\le t\le 0.75} t^{q-1} \sum_{i=1}^{m-2} \delta_1 G_2(\eta_i,s)$$
$$= G_1(s,s) + \sum_{i=1}^{m-2} \delta_1 G_2(\eta_i,s) = L(s), \ s \in (0,1).$$

Hence $\gamma_1(s) = \dfrac{\chi(s)}{L(s)}$

$$= \frac{\gamma(s)G_1(s,s) + (0.25)^{q-1}\sum_{i=1}^{m-2}\delta_1 G_2(\eta_i,s)}{G_1(s,s) + \sum_{i=1}^{m-2}\delta_1 G_2(\eta_i,s)}.$$

Clearly $\gamma_1 : (0,1) \to (0,\infty)$ is continues.

proof is completed.

In view of Lemma 3.1, the BVP (1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s)) ds \tag{10}$$

and by a solution of the BVP (1), we mean a solution of the integral equation (10) is a fixed point of the operator $T : P \to P$ defined by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s)) ds. \tag{11}$$

Onward, we use these notations

$$M = \left(\int_0^1 G(s,s) ds\right)^{-1}, N = \left(\int_{0.25}^{0.75} \gamma(s) G(s,s) ds\right)^{-1}$$

and

$$\mathbb{Q} = \frac{1}{(1-\lambda)\Gamma(q+1)} + \sum_{i=1}^{m-2} \delta_i \int_0^1 G_2(\eta_i,s) ds. \tag{12}$$

Lemma 3.3. Assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Then the operator $T : P \to P$ defined in (11) is completely continuous.

*Proof.*Due to nonnegativity and continuity of $G(t,s)$ and $f(t,s)$, the operator T is continuous. For each $u \in \Omega = \{u \in P : \|u\| \le \mathscr{R}, \mathscr{R} > 0\}$, we have

$$K = \max_{(t,u)\in[0,1]\times[0,\mathscr{R}]} |f(t,u(t))| + 1. \tag{13}$$

Therefore, we consider

$$|Tu(t)| = \left|\int_0^1 G(t,s) f(s,u(s)) ds\right|$$
$$\le K \int_0^1 G(s,s) ds$$
$$\le K \int_0^1 L(s) ds,$$

which implies that $T(\Omega)$ is bounded.
For equi-continuity of $T : P \to P$, take $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ with $t_2 - t_1 < \delta$ and taking $\varepsilon > 0$ and $n-1 < q$,

such that $\delta = \frac{1}{n}(\frac{\varepsilon}{K\mathbb{Q}})^{q-1}$. Then for $u \in \Omega$, we claim that $|Tu(t_2) - Tu(t_1)| < \varepsilon$. Thus we have

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&= \left| \int_0^1 [G(t_2,s)f(s,u(s)) - G(t_1,s)f(s,u(s))]\, ds \right| \\
&\leq K \int_0^1 |G(t_2,s) - G(t_1,s)| ds \\
&\leq K \left[\int_0^1 |G_1(t_2,s) - G_1(t_1,s)| ds \right] \\
&+ K \left[(t_2^{q-1} - t_1^{q-1}) \sum_{i=1}^{m-2} \delta_i \int_0^1 G_2(\eta_i,s) ds \right] \\
&\leq K \left[\int_0^{t_1} |G_1(t_2,s) - G_1(t_1,s)| ds \right] \\
&+ K \left[\int_{t_2}^1 |G_1(t_2,s) - G_1(t_1,s)| ds \right] \\
&+ K \left[\int_{t_1}^{t_2} |G_1(t_2,s) - G_1(t_1,s)| ds \right] \\
&+ K \left[(t_2^{q-1} - t_1^{q-1}) \sum_{i=1}^{m-2} \delta_i \int_0^1 G_2(\eta_i,s) ds \right],
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&\leq K \left[\frac{(t_2^{q-1} - t_1^{q-1})}{(1-\lambda)\Gamma(q)} \int_0^1 (1-s)^{q-1} ds \right] \\
&+ K \left[(t_2^{q-1} - t_1^{q-1}) \sum_{i=1}^{m-2} \delta_i \int_0^1 G_2(\eta_i,s) ds \right] \\
&\leq K \left[\frac{1}{(1-\lambda)\Gamma(q+1)} + \sum_{i=1}^{m-2} \delta_i \int_0^1 G_2(\eta_i,s) ds \right] \times \\
&(t_2^{q-1} - t_1^{q-1}) = K\mathbb{Q}(t_2^{q-1} - t_1^{q-1}).
\end{aligned}$$

Further, we explain the above process as

Case I. $\delta \leq t_1 < t_2 < 1$ and using Mean value theorem on $|t_2^{q-1} - t_1^{q-1}|$, we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq K\mathbb{Q}(t_2^{q-1} - t_1^{q-1}) \\
&< K\mathbb{Q}(q-1)\delta^{q-2}(t_2 - t_1) \\
&< K\mathbb{Q}n\delta^{q-1} < \varepsilon.
\end{aligned}$$

Case II. $0 \leq t_1 < \delta, t_2 < n\delta$, we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq K\mathbb{Q}(t_2^{q-1} - t_1^{q-1}) \\
&< K\mathbb{Q}(q-1)(t_2^{q-2} \\
&< K\mathbb{Q}(n\delta)^{q-1} = \varepsilon.
\end{aligned}$$

Hence $T : P \to P$ is equicontinuous. By Arzela-Ascoli theorem, we conclude that the operator $T : P \to P$ is completely continuous.

Now, we show existence of at least three solutions of the BVP (1).

Theorem 3.4. Assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and there exists positive constants $0 < a < b < c$ such that

$(A_1)\, f(t,u) < Ma,$ for $(t,u) \in [0,1] \times [0,a]$
$(A_2)\, f(t,u) \geq Nb,$ for $(t,u) \in [0.25, 0.75] \times [b,c]$
$(A_3)\, f(t,u) \leq Mc,$ for $(t,u) \in [0,1] \times [0,c],$

then the BVP (1) has at least three positive solutions u_1, u_2, and u_3 such that

$$\begin{aligned}
&\max_{0\leq t\leq 1} |u_1(t)| < a, \\
&b < \min_{0.25\leq t\leq 0.75} |u_2(t)| \leq c, \\
&a < \max_{0\leq t\leq 1} |u_3(t)| < c, \\
&\min_{0.25\leq t\leq 0.75} |u_3(t)| < b.
\end{aligned} \tag{14}$$

*Proof.*From Lemma 3.3, the operator $T : P \to P$ is completely continuous. Any u is the solution of BVP(1) if and only if u is the solution of the operator equation $u = Tu$. Now, we show that all the conditions of Lemma 2.5 are satisfied. Let $u \in \overline{P_c}$, then $||u|| \leq c$ and from(A_3), we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s) f(s,u(s)) ds \right| \\
&\leq \int_0^1 G(s,s) f(s,u)s)) ds \\
&\leq \int_0^1 L(s) Mc ds = c.
\end{aligned}$$

Hence $T : \overline{P_c} \to \overline{P_c}$. Choose $u(t) = \frac{b+c}{2}, 0 \leq t \leq 1$. Then using (2), we have

$$u(t) = \frac{b+c}{2} \in P(\theta,b,c), \theta(u) = \theta(\frac{(b+c)}{2}) > b.$$

From which we have

$$\{u \in P(\theta,b,c)\ |\theta(u) > b\} \neq \emptyset.$$

Hence, if $u \in P(\theta,b,c)$, then $b \leq u(t) \leq c$ for $0.25 \leq t \leq 0.75$. Also, from assumption (A_2), we have $f(t,u(t)) \geq Nb$, for $0.25 \leq t \leq 0.75$ and

$$\begin{aligned}
\theta(Tu) &= \min_{0.25\leq t\leq 0.75} |(T(u))| \\
&\geq \int_0^1 \gamma(s) G(s,s) f(s,u(s)) ds \\
&> \int_{0.25}^{0.75} \gamma(s) L(s)) Nb ds = b,
\end{aligned}$$

which implies that

$$\theta(Tu) > b, \text{ for all } u \in P(\theta,b,c).$$

Next, let $u \in \overline{P_a}$, then $||u|| \leq a$. From (A_1) for $t \in [0,1]$

$$\|Tu\| = \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s))ds\right|$$
$$\leq \int_0^1 G(s,s)f(s,u)s))ds$$
$$\leq \int_0^1 L(s)Mads = a.$$

Hence $T : \overline{P_a} \to \overline{P_a}$. Hence all the conditions of Lemma 2.5 are satisfied, so the BVP (1) has at least three positive solutions u_1, u_2,and u_3 satisfying

$$\max_{0\leq t\leq 1}|u_1(t)| < a,$$
$$b < \min_{0.25\leq t\leq 0.75}|u_2(t)| \leq c,$$
$$a < \max_{0\leq t\leq 1}|u_3(t)| < c,$$
$$\min_{0.25\leq t\leq 0.75}|u_3(t)| < b.$$

Proof is completed.

Theorem 3.5. Assume that $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous and there exists positive constants

$$0 < a < b_1 < c_1 < b_2 < c_2 ... b_{k-1} < c_{k-1}, n = 1,2,3,...,$$

such that

$(A_4)\, f(t,u) < Ma$, for $(t,u) \in [0,1]\times[0,a]$
$(A_5)\, f(t,u) \geq Nb_i$, for $(t,u) \in [0.25,0.75]\times[b_i,c_i], 1 \leq i \leq k-1$;
$(A_6)\, f(t,u) \leq Mc_i$, for $(t,u) \in [0,1]\times[0,c_i]$, $1 \leq i \leq k-1$.
then the BVP (1) has at least $2k-1$ positive solutions.

*Proof.*By mathematical induction, when $k = 1$, then from (A_4), the operator T has at least one Fixed Point which is the corresponding solution of BVP(1) by using Schauder Fixed Point theorem. For $k = 2$, the theorem reduces to Theorem 3.4, whose proof has already done. For $k = n$, the statement holds and the BVP(1) has at least $2n-1$ positive solutions satisfying $\max_{t\in[0,1]}|u_i(t)| \leq c_{n-1}; i = 1,2,...,2n-1$. Again to derive result for $k = n+1$ applying Theorem 3.4 to

$$f(t,u) < Mc_{n-1}, \text{ for } (t,u) \in [0,1]\times[0,c_{n-1}];$$
$$f(t,u) \geq Nb_n, \text{ for } (t,u) \in [0.25,0.75]\times[b_n,c_n];$$
$$f(t,u) \leq Mc_n, \text{ for } (t,u) \in [0,1]\times[0,c_n].$$

We get three positive solutions u_0, u_{2n}, u_{2n+1} with

$$\max_{0\leq t\leq 1}|u_0(t)| < c_{n-1},$$
$$b_n < \min_{0.25\leq t\leq 0.75}|u_{2n}(t)| < \max_{0\leq t\leq 1}|u_{2k}(t)|$$
$$\leq c_n, c_{n-1} < \max_{0\leq t\leq 1}|u_{n+1}(t)| < c_n,$$
$$\min_{0.25\leq t\leq 0.75}|u_{n+1}(t)| < b_n.$$

Clearly u_{2n}, u_{2n+1} are different from $u_1, u_2, ..., u_{2n-1}$. Hence BVP(1) has at least $2k+1$ positive solutions. Proof is completed.

4 Example

Example 4.1. For the problem taking $m = 5, \delta_1 = \eta_1 = 0.5,\ \delta_2 = \eta_2 = 0.25, \delta_3 = \eta_3 = 0.125$

$$D^{\frac{5}{2}}_{0+}u(t) + f(t,u) = 0, 0 < t < 1,$$
$$u(0) = u'(0) = 0, \quad u(1) = \sum_{i=1}^{3}\delta_i u(\eta_i), \tag{15}$$

where

$$f(t,u) = \begin{cases} \frac{e^{-2t}}{100} + \frac{u^3}{1000}; & u \leq 1, \\ 4 + \frac{\sin t}{100} + u; & u > 1, \end{cases}$$

we find that $M \approx 2.623438$ and $N = 6.099814$. Choosing $a = 0.04,\ b = 0.75$ and $\ c = 2$, we have

$$f(t,u) = \frac{e^{-2t}}{100} + u^3 \leq 0.011 < Ma \approx 0.10493752,$$
$$for\ (t,u) \in [0,1]\times[0,0.04],$$

$$f(t,u) = 4 + \frac{\sin t}{100} + u \leq 5.01 \geq Nb \approx 4.5748605,$$
$$for\ (t,u) \in [0.25,0.75]\times[1,2],$$

$$f(t,u) = 4 + \frac{\sin t}{100} + u \leq 5.01 \leq Mc \approx 5.246876,$$
$$for\ (t,u) \in [0,1]\times[0,2].$$

Hence, by Lemma 2.5, the BVP (15) has at least three positive solutions $u_1, u_2,$ and u_3 with

$$\max_{0\leq t\leq 1}|u_1(t)| < 0.25,$$
$$1 < \min_{0.25\leq t\leq 0.75}|u_2(t)| \leq 2,$$
$$0.25 < \max_{0\leq t\leq 1}|u_3(t)| < 2,$$
$$\min_{0.25\leq t\leq 0.75}|u_3(t)| < 0.75.$$

5 Conclusion

Thanks to classical Fixed Point theorem due to Leggett-Williams, we have established adequate results for the existence of triple solutions. Also the criteria has been extended to multiplicity results for the considered problem. The established theoretical results have been demonstrated by a proper example. Hence a conclusion, we state that Fixed Point theory provide a powerful tools to treat boundary value problems of FDEs as well as classical differential equations.

Acknowledgement

We are really thankful to the reviewers for theirs useful comments which improved the quality of this paper.

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Kamal Shah is an Assistant Professor of Mathematics in the Department of Mathematics University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan. He received the PhD degree in Applied Mathematics from the University of Malakand in 2016. His research interests lies in the areas of applied mathematics including existence theory and numerical solutions of fractional order differential equations, fixed point theory and applications, numerical solutions of biological and physical models. He serves as a referee of many reputed journals of Mathematical sciences.

Salman Zeb is a Lecturer of Mathematics in the Department of Mathematics University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan. Recently he got his PhD in applied mathematics and his research interests lies in Applied Numerical analysis, Existence theory of Fractional differential equations and Numerical linear algebra.

Rahmat Ali Khan is a professor and Chairman Department of Mathematics University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan. He got his PhD degree in Applied Mathematics from the University of Glasgow UK, in 2006. His research interests lies in the area of Applied mathematics, Fluid Mechanics, Optic and dynamics. He is the reviewer and member of editorial boards of many national and international journal.