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Fractal Integral Inequalities for Harmonic Convex Functions

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Abstract: In this paper, we introduce a new class of convex functions, which is called generalized harmonic convex functions on fractal sets $\mathbb{R}^\alpha$ $(0<\alpha\le 1)$. This new class of convex functions includes generalized convex functions and harmonic convex functions as special cases. We obtain several new Hermite-Hadamard inequalities for generalized harmonic convex functions. Some special cases are also discussed. The ideas and technique of this paper may motivate further research in this field.

Keywords: Generalized convex functions, Harmonic convex functions, Generalized Hermite-Hadamard inequality.
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1 Introduction

Recently, scientists and engineers have given utmost attention to study the fractal calculus. According to Mandlebrot, the set whose Hausdorff dimension strictly exceeds the topological dimension is called a fractal set [14]. The calculus on fractal set is exceedingly practical and comprehensive in science and engineering for their real world models. Many researchers used various approaches to construct fractional calculus on fractal sets. Yang [36] systematically analyzed the local fractional functions on fractal space, which included local fractional calculus. Later on Mo et. al. [17] introduced the generalized convex function on fractal sets and established the generalized Jensen's inequality and generalized Hermite-Hadamard inequality related to generalized convex function. Wei et al. [35] introduced a local fractional integral inequality on fractal space comparable to Anderson's inequality for comprehensive convex functions. The generalized convex function on fractal sets $\mathbb{R}^\alpha$ $(0<\alpha\le 1)$ can be stated as follows:

A function $f: I=[a,b]\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ is said to be generalized convex function, if

$$f\big((1-t)x+ty\big)\le(1-t)^\alpha f(x)+t^\alpha f(y),\qquad \forall x,y\in I, t\in[0,1].$$

In particular, a function $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}^\alpha$ is said to be generalized convex function, if and only if,

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b{}^{(\alpha)}f(x)\le\frac{f(a)+f(b)}{2^\alpha},\qquad(1)$$

which is called generalized Hermite-Hadamard inequality for generalized convex function.
Hermite-Hadamard inequalities are used to find the upper and lower bounds of the mean value. For a novel application of Hermite-Hadamard inequalities in the proof of inequality $e<\left(1+\frac{1}{n}\right)^{n+0.5}$, see Khattri [11]. It is known that the minimum of the differentiable convex functions on the convex sets can be characterized by variational inequalities, see, for example, Noor[19,20,21] and the references therein.

It is well known that the harmonic means have applications in electrical circuits. The total resistance of a set of parallel resistance is obtained by adding up the reciprocals of the individual resistance values and then considering the reciprocal of their total. For example if R_1 and R_2 are the resistances of two parallel resistors, then the total resistance is obtained by the formula $R=\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}=\frac{R_1R_2}{R_1+R_2}$, which is half the harmonic mean. These harmonic means also played a crucial in the developments of parallel algorithms. For the novel

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applications of harmonic means in the stock market, see Al-Azemi and Colin [1]. Noor [21] used the harmonic means to construct a wide class of iterative methods for solving the nonlinear equations. Noor and Noor [22,23] peoved that the optimal conditions of the differentiable harmonic functions can be expressed in term of variational inequalities. Anderson et al. [2] considered the harmonic functions and investigated various applications of harmonic functions. Iscan [29] derived the Hermite-Hadamard type inequality for the harmonic functions. These integral inequalities are used to find the upper and lower bounds. For recent developments and other aspects of harmonic convex functions, see [8,9,14-19] and the references therein.

It is clear that the generalized convex functions and harmonic convex functions are two different classes of convex functions. It is natural to unify these two different classes. Inspired by these investigations, we introduce a new class of generalized convex function, which is called generalized harmonic convex function on fractal sets. This new class of generalized harmonic convex functions include generalized convex functions and harmonic convex functions as special cases. We derive some fractal Hermite-Hadamard inequalities for generalized harmonic convex function. Some special cases are discussed which can be obtained from main results. Results obtained in this paper continue to hold for these cases. Interested readers are encouraged to find the applications of these convex functions in different areas of pure and applied sciences.

2 Preliminaries

Recall the set $\mathbb{R}^\alpha$ of real line numbers and use Gao-Yang-Kangs idea to describe the definitions of the local fractional derivative and local fractional integral, see [36].
For $0<\alpha\leq 1$, we have the following α-type set.

1.$\mathbb{Z}^\alpha$: the α-type set of the integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.
2.$\mathbb{Q}^\alpha$: the α-type set of the rational numbers is defined as the set $\{m^\alpha = (p/q)^\alpha : p\in\mathbb{Z}, q\neq 0\}$
3.$\mathbb{J}^\alpha$: the α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq (p/q)^\alpha : p\in\mathbb{Z}, q\neq 0\}$.
4.$\mathbb{R}^\alpha$: the α-type set of the real line numbers is defined as the set $\mathbb{R}^\alpha = \mathbb{Q}^\alpha\cup\mathbb{J}^\alpha$.

If a^α, b^α and c^α belong to the set $\mathbb{R}^\alpha$ of real line numbers, then

1.$a^\alpha+b^\alpha$ and $a^\alpha b^\alpha$ belongs to the set $\mathbb{R}^\alpha$
2.$a^\alpha+b^\alpha = b^\alpha+a^\alpha = (a+b)^\alpha = (b+a)^\alpha$
3.$a^\alpha+(b^\alpha+c^\alpha) = (a^\alpha+b^\alpha)+c^\alpha$
4.$a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$
5.$a^\alpha(b^\alpha c^\alpha) = (a^\alpha b^\alpha)c^\alpha$
6.$a^\alpha(b^\alpha+c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$
7.$a^\alpha+0^\alpha = 0^\alpha+a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$

The definition of local fractional derivative and local fractional integral may be given as:

Definition 1.[36]. *The local fractional derivative of $f(x)$ of order α at $x=x_o$ is defined by*

$$f^{(\alpha)}(x_o) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x)\bigg|_{x=x_o} = \lim_{x\to x_o}\frac{\Delta^\alpha(f(x)-f(x_o))}{(x-x_o)^\alpha},$$

where $\Delta^\alpha(f(x)-f(x_o)) = \Gamma(1+\alpha)(f(x)-f(x_o))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x\in I\subseteq\mathbb{R}$, then we denote $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2\dots$

Definition 2.[36]. *The local fractional integral of $f(x)$ is defined by*

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(\mathrm{d}t)^\alpha.$$

Lemma 1.[36].

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}x^{k\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(\mathrm{d}t)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha}-a^{(k+1)\alpha}), \quad k\in\mathbb{R}.$$

Definition 3.[33]. *A set $I\subseteq\mathbb{R}\setminus\{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx+(1-t)y}\in I, \qquad \forall x,y\in I, t\in[0,1].$$

It has been shown that the minimum of a differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of variational inequalities, which is called the harmonic variational inequality. This shows that the harmonic functions have some properties which convex functions have. For recent developments, see [21,22,23,30] and the references therein. We now introduce the new concept of harmonic convex functions.

Definition 4. *A function $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ is said to be generalized harmonic convex function and $(0<\alpha\leq 1)$, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq (1-t)^\alpha f(x) + t^\alpha f(y), \qquad \forall x,y\in I, t\in[0,1]. \tag{2}$$

The function f is said to be generalized harmonic concave function, if -f is generalized harmonic convex function. If $t=\frac{1}{2}$, then 2 reduces to

$$f\left(\frac{2xy}{x+y}\right) \le \frac{f(x)+f(y)}{2^\alpha},$$

which is called the generalized Jensen harmonic convex function.

If $\alpha=1$, then definition 4 reduces to the definition of harmonic convex function, see [10]. For the recent development in harmonic convex functions, see [15, 24, 25, 26, 27]

Definition 5.[15]. *A function $f:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonic symmetric with respect to $\frac{2ab}{a+b}$, if*

$$f(x)=f\left(\frac{abx}{(a+b)x-ab}\right) \quad \forall x\in[a,b].$$

Definition 6.[32]. *Two functions f,g are said to be similarly ordered (f is g-monotone), if and only if,*

$$\langle f(x)-f(y), g(x)-g(y)\rangle \ge 0, \qquad \forall x,y\in\mathbb{R}^n.$$

One can show that the product of two similarly ordered generalized harmonic convex functions is again a generalized harmonic convex function.

3 Main Results

In this section, we derive generalized Hermite-Hadamard type inequalities for generalized harmonic convex functions.

Theorem 1.*Let $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ be generalized harmonic convex function. If $f(x)\in {}_aI_b{}^{(\alpha)}[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{(ab)^\alpha}{(b-a)^\alpha}\int_a^b \frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^\alpha \le \frac{f(a)+f(b)}{2^\alpha}, \quad x\in[a,b]. \tag{3}$$

*Proof.*Let f be generalized harmonic convex function with $t=\frac{1}{2}$ in the inequality 2, then

$$f\left(\frac{2xy}{x+y}\right) \le \frac{f(x)+f(y)}{2^\alpha}, \qquad \forall x,y\in I.$$

Let $x=\frac{ab}{ta+(1-t)b}$ and $y=\frac{ab}{(1-t)a+tb}$. Then

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2^\alpha}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

Thus, integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{2ab}{a+b}\right)(\mathrm{d}t)^\alpha \\
&\le \frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)(\mathrm{d}t)^\alpha \right.\\
&\quad \left. +\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)(\mathrm{d}t)^\alpha\right] \\
&= \frac{(ab)^\alpha}{(b-a)^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^\alpha.
\end{aligned}$$

This implies

$$f\left(\frac{2ab}{a+b}\right) \le \frac{(ab)^\alpha}{(b-a)^\alpha}\int_a^b \frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^\alpha. \tag{4}$$

Now f is a generalized harmonic convex function, then

$$\begin{aligned}
f\left(\frac{ab}{ta+(1-t)b}\right) &\le (1-t)^\alpha f(a)+t^\alpha f(b) \\
f\left(\frac{ab}{(1-t)a+tb}\right) &\le t^\alpha f(a)+(1-t)^\alpha f(b).
\end{aligned}$$

By adding these inequalities we have

$$\begin{aligned}
&f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right) \\
&\le (1-t)^\alpha f(a)+t^\alpha f(b)+t^\alpha f(a)+(1-t)^\alpha f(b) \\
&= f(a)+f(b).
\end{aligned}$$

Then, integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)(\mathrm{d}t)^\alpha \right.\\
&\qquad \left. +\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)(\mathrm{d}t)^\alpha\right] \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^1 [f(a)+f(b)](\mathrm{d}t)^\alpha.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)(\mathrm{d}t)^\alpha \right.\\
&\qquad \left. +\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)(\mathrm{d}t)^\alpha\right] \\
&= \frac{2^\alpha(ab)^\alpha}{(b-a)^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^\alpha,
\end{aligned}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 [f(a)+f(b)](\mathrm{d}t)^\alpha = \frac{f(a)+f(b)}{\Gamma(1+\alpha)}.$$

So

$$\frac{(ab)^\alpha}{(b-a)^\alpha}\int_a^b \frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^\alpha \le \frac{f(a)+f(b)}{2^\alpha}. \tag{5}$$

Combining 4 and 5, we obtain the required result.

Theorem 2. *Let $f,g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^{\alpha}$ be generalized harmonic convex functions. If $fg \in I_x^{(\alpha)}[a,b]$, then*

$$\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)g(\frac{abx}{(a+b)x-ab})}{x^{2\alpha}}(\mathrm{d}x)^{\alpha} \leq \beta_{\alpha}M(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}N(a,b),$$

where

$$M(a,b) = f(a)g(a)+f(b)g(b), \tag{6}$$

$$N(a,b) = f(a)g(b)+f(b)g(a), \tag{7}$$

$$\begin{aligned}\beta_{\alpha} &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}.\end{aligned} \tag{8}$$

Proof. Let f,g be generalized harmonic convex functions. Then

$$\begin{aligned}&\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)g(\frac{abx}{(a+b)x-ab})}{x^{2\alpha}}(\mathrm{d}x)^{\alpha}\\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)(\mathrm{d}t)^{\alpha}\\ &\leq \frac{1}{\Gamma(1+\alpha)}\int_0^1[(1-t)^{\alpha}f(a)+t^{\alpha}f(b)]\\ &\quad\times[t^{\alpha}g(a)+(1-t)^{\alpha}g(b)](\mathrm{d}t)^{\alpha}\\ &= f(a)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}\\ &\quad+[f(a)g(a)+f(b)g(b)]\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\\ &\quad+f(b)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\\ &= \beta_{\alpha}[f(a)g(a)+f(b)g(b)]\\ &\quad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}[f(a)g(b)+f(b)g(a)]\\ &= \beta_{\alpha}M(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}N(a,b),\end{aligned}$$

which is the required result. □

Theorem 3.*Let $f,g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^{\alpha}$ be generalized harmonic convex functions. If $fg \in I_x^{(\alpha)}[a,b]$, then*

$$\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)g(x)}{x^{2\alpha}}(\mathrm{d}x)^{\alpha} \leq \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}M(a,b)+\beta_{\alpha}N(a,b),$$

where $M(a,b)$, $N(a,b)$ and β_{α} are given by 6, 7 and 8 respectively.

Proof. Let f,g be generalized harmonic convex functions. Then

$$\begin{aligned}&\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{f(x)g(x)}{x^{2\alpha}}(\mathrm{d}x)^{\alpha}\\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)(\mathrm{d}t)^{\alpha}\\ &\leq \frac{1}{\Gamma(1+\alpha)}\int_0^1[(1-t)^{\alpha}f(a)+t^{\alpha}f(b)]\\ &\quad\times[(1-t)^{\alpha}g(a)+t^{\alpha}g(b)](\mathrm{d}t)^{\alpha}\\ &= f(a)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}\\ &\quad+[f(a)g(b)+f(b)g(a)]\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\\ &\quad+f(b)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\\ &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}[f(a)g(a)+f(b)g(b)]\\ &\quad+\beta_{\alpha}[f(a)g(b)+f(b)g(a)]\\ &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}M(a,b)+\beta_{\alpha}N(a,b),\end{aligned}$$

which is the required result. □

Theorem 4. *Let $f,g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^{\alpha}$ be generalized harmonic convex functions. If $fg \in I_x^{(\alpha)}[a,b]$, then*

$$\begin{aligned}&\frac{2^{\alpha}}{\Gamma(1+\alpha)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\\ &-\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{f(x)}{x^{2\alpha}}(\mathrm{d}x)^{\alpha}\\ &\leq \beta_{\alpha}M(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}N(a,b),\end{aligned}$$

where $M(a,b)$, $N(a,b)$ and β_{α} are given by 6, 7 and 8 respectively.

*Proof.*Let f,g be two generalized harmonic convex functions, with $t=\frac{1}{2}$. Then

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2^{\alpha}}, \qquad \forall x,y \in I.$$
$$g\left(\frac{2xy}{x+y}\right) \leq \frac{g(x)+g(y)}{2^{\alpha}}, \qquad \forall x,y \in I.$$

Let $x=\frac{ab}{ta+(1-t)b}$, and $y=\frac{ab}{(1-t)a+tb}$. Then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2^{\alpha}}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right].$$
$$g\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2^{\alpha}}\left[g\left(\frac{ab}{ta+(1-t)b}\right)+g\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

Thus

$$f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)$$

$$
\begin{aligned}
&\leq \frac{1}{4^{\alpha}}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right]\\
&\quad\times\left[g\left(\frac{ab}{ta+(1-t)b}\right)+g\left(\frac{ab}{(1-t)a+tb}\right)\right]\\
&=\frac{1}{4^{\alpha}}\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right.\\
&\quad+f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\\
&\quad+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\\
&\quad\left.+f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right]\\
&\leq\frac{1}{4^{\alpha}}\left\{f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right.\\
&\quad+f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\\
&\quad+\left[(1-t)^{\alpha}f(a)+t^{\alpha}f(b)\right]\left[t^{\alpha}g(a)+(1-t)^{\alpha}g(b)\right]\\
&\quad\left.+\left[t^{\alpha}f(a)+(1-t)^{\alpha}f(b)\right]\left[(1-t)^{\alpha}g(a)+t^{\alpha}g(b)\right]\right\}.
\end{aligned}
$$

Integrating over $[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\\
&\leq\frac{1}{4^{\alpha}}\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\right.\\
&\quad\times g\left(\frac{ab}{ta+(1-t)b}\right)(\mathrm{d}t)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)\\
&\quad\times g\left(\frac{ab}{(1-t)a+tb}\right)(\mathrm{d}t)^{\alpha}\\
&\quad+2^{\alpha}[f(a)g(a)+f(b)g(b)]\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\\
&\quad+[f(a)g(b)+f(b)g(a)]\\
&\quad\left.\times\frac{1}{\Gamma(1+\alpha)}\int_0^1(t^{2\alpha}+(1-t)^{2\alpha})(\mathrm{d}t)^{\alpha}\right\}\\
&=\frac{1}{2^{\alpha}}\left\{\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{f(x)g(x)}{x^{2\alpha}}(\mathrm{d}x)^{\alpha}\right.\\
&\quad+\beta_{\alpha}[f(a)g(a)+f(b)g(b)]\\
&\quad\left.+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}[f(a)g(b)+f(b)g(a)]\right\}\\
&=\frac{1}{2^{\alpha}}\left\{\frac{(ab)^{\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{f(x)g(x)}{x^{2\alpha}}(\mathrm{d}x)^{\alpha}\right.\\
&\quad\left.+\beta_{\alpha}M(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}N(a,b)\right\},
\end{aligned}
$$

which is the required result. □

Theorem 5. *Let $f,g : I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^{\alpha}$ be generalized harmonic convex functions. If $fg\in I_x^{(\alpha)}[a,b]$, then*

$$
\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3}\int_a^b\int_a^b\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)\\
&\times g\left(\frac{xy}{tx+(1-t)y}\right)(\mathrm{d}t)^{\alpha}(\mathrm{d}y)^{\alpha}(\mathrm{d}x)^{\alpha}\\
&\leq\frac{2^{\alpha}\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^{\alpha}}{(\Gamma(1+\alpha))^2}\int_a^b f(x)g(x)(\mathrm{d}x)^{\alpha}\\
&\quad+\frac{(b-a)^{2\alpha}x^{2\alpha}}{2^{\alpha}(\Gamma(1+\alpha))^2(ab)^{2\alpha}}\beta_{\alpha}\left[M(a,b)+N(a,b)\right],
\end{aligned}
$$

where $M(a,b)$, $N(a,b)$ and β_{α} are given by 6, 7 and 8 respectively.

Proof. Let f,g be two generalized harmonic convex functions on I. Then

$$
f\left(\frac{xy}{tx+(1-t)y}\right)\leq(1-t)^{\alpha}f(x)+t^{\alpha}f(y),
$$
$$
g\left(\frac{xy}{tx+(1-t)y}\right)\leq(1-t)^{\alpha}g(x)+t^{\alpha}g(y).
$$

Thus

$$
\begin{aligned}
&f\left(\frac{xy}{tx+(1-t)y}\right)g\left(\frac{xy}{tx+(1-t)y}\right)\\
&\leq\left[(1-t)^{\alpha}f(x)+t^{\alpha}f(y)\right]\left[(1-t)^{\alpha}g(x)+t^{\alpha}g(y)\right]\\
&=(1-t)^{2\alpha}f(x)g(x)+t^{\alpha}(1-t)^{\alpha}\left[f(x)g(y)+g(x)f(y)\right]\\
&\quad+t^{2\alpha}f(y)g(y).
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)g\left(\frac{xy}{tx+(1-t)y}\right)(\mathrm{d}t)^{\alpha}\\
&\leq\frac{1}{\Gamma(1+\alpha)}f(x)g(x)\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}f(y)g(y)\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\left[f(x)g(y)+g(x)f(y)\right]\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\\
&=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[f(x)g(x)+f(y)g(y)\right]+\beta_{\alpha}\left[f(x)g(y)+g(x)f(y)\right]
\end{aligned}
$$

Now, integrating over $[a,b]$, we have

$$
\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3}\int_a^b\int_a^b\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)\\
&\times g\left(\frac{xy}{tx+(1-t)y}\right)(\mathrm{d}t)^{\alpha}(\mathrm{d}y)^{\alpha}(\mathrm{d}x)^{\alpha}\\
&\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{(\Gamma(1+\alpha))^2}\left[\int_a^b\int_a^b f(x)g(x)(\mathrm{d}y)^{\alpha}(\mathrm{d}x)^{\alpha}\right.\\
&\quad\left.+\int_a^b\int_a^b f(y)g(y)(\mathrm{d}y)^{\alpha}(\mathrm{d}x)^{\alpha}\right]
\end{aligned}
$$

$$+\frac{1}{(\Gamma(1+\alpha))^2}\beta_\alpha\left[\int_a^b\int_a^b f(x)g(y)(\mathrm{d}y)^\alpha(\mathrm{d}x)^\alpha\right.$$
$$\left.+\int_a^b\int_a^b g(x)f(y)(\mathrm{d}y)^\alpha(\mathrm{d}x)^\alpha\right]$$
$$=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{(\Gamma(1+\alpha))^2}\left[\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\right.$$
$$\left.+\int_a^b f(y)g(y)(\mathrm{d}y)^\alpha\right]$$
$$+\frac{1}{(\Gamma(1+\alpha))^2}\beta_\alpha\left[\int_a^b f(x)(\mathrm{d}x)^\alpha\int_a^b g(y)(\mathrm{d}y)^\alpha\right.$$
$$\left.+\int_a^b g(x)(\mathrm{d}x)^\alpha\int_a^b f(y)(\mathrm{d}y)^\alpha\right]$$
$$=\frac{2^\alpha\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{(\Gamma(1+\alpha))^2}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha$$
$$+\frac{(b-a)^{2\alpha}x^{2\alpha}}{2^\alpha(\Gamma(1+\alpha))^2(ab)^{2\alpha}}\beta_\alpha\left[(f(a)+f(b)(g(a)+g(b)\right]$$
$$=\frac{2^\alpha\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{(\Gamma(1+\alpha))^2}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha$$
$$+\frac{(b-a)^{2\alpha}x^{2\alpha}}{2^\alpha(\Gamma(1+\alpha))^2(ab)^{2\alpha}}\beta_\alpha\left[M(a,b)+N(a,b)\right],$$

which is the required result. □

Theorem 6. *Let $f,g : I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ be generalized harmonic convex functions. If $fg\in I_x^{(\alpha)}[a,b]$, then*

$$\frac{1}{\Gamma(1+\alpha)^2}\int_a^b\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)$$
$$\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha(\mathrm{d}x)^\alpha$$
$$\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha$$
$$+\frac{\Gamma(1+2\alpha)}{4^\alpha\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}[M(a,b)+N(a,b)]$$
$$+\frac{(b-a)^\alpha x^{2\alpha}}{2^\alpha\Gamma(1+\alpha)(ab)^\alpha}\beta_\alpha\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$, $N(a,b)$ and β_α are given by 6, 7 and 8 respectively.

Proof. Let f,g be two generalized harmonic convex functions on I. Then

$$f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\leq(1-t)^\alpha f(x)+t^\alpha f\left(\frac{2ab}{a+b}\right),$$
$$\forall x,\frac{2ab}{a+b}\in I,t\in[0,1].$$
$$g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\leq(1-t)^\alpha g(x)+t^\alpha g\left(\frac{2ab}{a+b}\right),$$
$$\forall x,\frac{2ab}{a+b}\in I,t\in[0,1].$$

Thus

$$f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)$$
$$\leq\left[(1-t)^\alpha f(x)+t^\alpha f\left(\frac{2ab}{a+b}\right)\right]$$
$$\times\left[(1-t)^\alpha g(x)+t^\alpha g\left(\frac{2ab}{a+b}\right)\right]$$
$$=(1-t)^{2\alpha}f(x)g(x)$$
$$+t^\alpha(1-t)^\alpha\left[f(x)g\left(\frac{2ab}{a+b}\right)+g(x)f\left(\frac{2ab}{a+b}\right)\right]$$
$$+t^{2\alpha}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right).$$

Integrating over $[0,1]$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)$$
$$\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha$$
$$\leq\frac{1}{\Gamma(1+\alpha)}f(x)g(x)\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^\alpha$$
$$+\frac{1}{\Gamma(1+\alpha)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\int_0^1 t^{2\alpha}(\mathrm{d}t)^\alpha$$
$$+\frac{1}{\Gamma(1+\alpha)}\left[f(x)g\left(\frac{2ab}{a+b}\right)\right.$$
$$\left.+\frac{1}{\Gamma(1+\alpha)}g(x)f\left(\frac{2ab}{a+b}\right)\right]\int_0^1 t^\alpha(1-t)^\alpha(\mathrm{d}t)^\alpha$$
$$=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[f(x)g(x)+f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\right]$$
$$+\beta_\alpha\left[f(x)g\left(\frac{2ab}{a+b}\right)+g(x)f\left(\frac{2ab}{a+b}\right)\right].$$

Now, integrating over $[a,b]$, we have

$$\frac{1}{\Gamma(1+\alpha)^2}\int_a^b\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)$$
$$\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha(\mathrm{d}x)^\alpha$$
$$\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha$$
$$+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)(\mathrm{d}x)^\alpha$$
$$+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\left[g\left(\frac{2ab}{a+b}\right)\int_a^b f(x)(\mathrm{d}x)^\alpha\right.$$
$$\left.+f\left(\frac{2ab}{a+b}\right)\int_a^b g(x)(\mathrm{d}x)^\alpha\right]$$
$$\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha$$

$$
\begin{aligned}
&+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\\
&+\frac{(b-a)^\alpha x^{2\alpha}}{\Gamma(1+\alpha)(ab)^\alpha}\beta_\alpha\left[\left(\frac{g(a)+g(b)}{2^\alpha}\right)\left(\frac{f(a)+f(b)}{2^\alpha}\right)\right.\\
&\left.+\left(\frac{f(a)+f(b)}{2^\alpha}\right)\left(\frac{g(a)+g(b)}{2^\alpha}\right)\right]\\
\leq&\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\left[\left(\frac{f(a)+f(b)}{2^\alpha}\right)\left(\frac{g(a)+g(b)}{2^\alpha}\right)\right]\\
&+\frac{2^\alpha(b-a)^\alpha x^{2\alpha}}{\Gamma(1+\alpha)(ab)^\alpha}\beta_\alpha\left[\left(\frac{g(a)+g(b)}{2^\alpha}\right)\left(\frac{f(a)+f(b)}{2^\alpha}\right)\right]\\
=&\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{\Gamma(1+2\alpha)}{4^\alpha\Gamma(1+3\alpha)}\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}[M(a,b)+N(a,b)]\\
&+\frac{(b-a)^\alpha x^{2\alpha}}{2^\alpha\Gamma(1+\alpha)(ab)^\alpha}\beta_\alpha[M(a,b)+N(a,b)],
\end{aligned}
$$

which is the required result. □

Theorem 7. *Let $f,g: I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ be similarly ordered generalized harmonic convex functions. If $fg\in I_x^{(\alpha)}[a,b]$, then*

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)^2}\int_a^b\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\\
&\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha(\mathrm{d}x)^\alpha\\
\leq&\frac{1}{\Gamma(1+2\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+2\alpha)}[M(a,b)+N(a,b)],
\end{aligned}
$$

where $M(a,b)$, $N(a,b)$ and β_α are given by 6, 7 and 8 respectively.

Proof. Let f,g be two similarly ordered generalized harmonic convex functions on I. Then

$$
\begin{aligned}
&f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\leq(1-t)^\alpha f(x)+t^\alpha f\left(\frac{2ab}{a+b}\right),\\
&\forall x,\frac{2ab}{a+b}\in I, t\in[0,1].\\
&g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\leq(1-t)^\alpha g(x)+t^\alpha g\left(\frac{2ab}{a+b}\right),\\
&\forall x,\frac{2ab}{a+b}\in I, t\in[0,1].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\\
&\leq(1-t)^\alpha f(x)g(x)+t^\alpha f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right).
\end{aligned}
$$

Integrating over $[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\\
&\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha\\
\leq&\frac{1}{\Gamma(1+\alpha)}f(x)g(x)\int_0^1(1-t)^\alpha(\mathrm{d}t)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\int_0^1 t^\alpha(\mathrm{d}t)^\alpha\\
=&\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(x)g(x)+f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\right].
\end{aligned}
$$

Now, integrating over $[a,b]$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)^2}\int_a^b\int_0^1 f\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)\\
&\times g\left(\frac{x\left[\frac{2ab}{a+b}\right]}{tx+(1-t)\left[\frac{2ab}{a+b}\right]}\right)(\mathrm{d}t)^\alpha(\mathrm{d}x)^\alpha\\
\leq&\frac{1}{\Gamma(1+2\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{1}{\Gamma(1+2\alpha)}\int_a^b f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)(\mathrm{d}x)^\alpha\\
\leq&\frac{1}{\Gamma(1+2\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+2\alpha)}[f(a)+f(b)][g(a)+g(b)]\\
=&\frac{1}{\Gamma(1+2\alpha)}\int_a^b f(x)g(x)(\mathrm{d}x)^\alpha\\
&+\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+2\alpha)}[M(a,b)+N(a,b)],
\end{aligned}
$$

which is the required result. □

4 Conclusion.

In this paper, we have introduced and investigated a new class of convex functions, which is called the general harmonic convex functions. Several new Hermite-Hadamard type inequalities are derived. Results obtained in this paper represent significant contribution in this field. It is expected that results obtained in this paper may stimulate further research.

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