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Fuzzy Fixed Point Results Via Rational Type Contractions Involving Control Functions in Complex-Valued Metric Spaces

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Abstract: In this work, we have proved some common fuzzy fixed point results satisfying rational contractive condition in the consideration of different types of control function used as coefficients in contractive condition. The results in this paper generalizes some results already proved in literature. Examples are given in the support of our constructed results.

Keywords: Complex-valued metric spaces, common fixed point, fuzzy mappings, cauchy sequence, contractive condition

1 Introduction

Fixed point theory is an extensive and attractive subject for research in both pure and applied mathematics. This theory deals with the results regarding the existence and uniqueness of fixed point which are very useful to find the solution to systems of differential equations, integral equations and functional equations. With the passage of time fixed point theory. It has been improved by proving results associated to fixed point for self and nonself mappings in metric spaces. In 1922 Banach [1] established a contraction principle which plays a key role to obtain unique fixed point in complete metric spaces. Banach contraction principle is a strong tool for solving linear and non linear differential equations of both classic and fractional calculus. For instance we refer to the articles [24,25]. It is based on iteration, so it can be applied on computers. Due to the simplicity of Banach contraction principle, fixed point theory has gained more attention and importance of researchers. Several researchers have been extended this contraction principle for different type of linear contraction in different spaces, like cone metric spaces, G metric spaces, quasi metric spaces etc (for details see [6,9,15,21]). Particularly in [10] the authors established rational type contraction and obtained fixed point results satisfying rational type contraction in complete metric space. Fixed point results for rational contraction have been further extended by several mathematicians in different concepts of spaces. During research on the generalization of rational contraction researchers realized that rational type contraction is not meaningful in cone metric spaces where vector division occurs.

To overcome the problem to utilize rational type contraction for vectors case Azam *et. al* established a special class of cone metric spaces and derived fixed point results by utilizing rational contraction [5]. This newly established class is known as complex-valued metric space, which became an attractive topic for further research. Several researchers generalized the initial work in complex-valued metric space by changing the contractive condition for self and nonself mappings and obtained common fixed point results in complex-valued metric space, for instance, see [5,16,18]. Afterwards in [17] the aforementioned results of Azam *et. al* was improved by Sintunavarat and Kumam by substituting the constant coefficients in contractive condition by point dependent control functions. Besides of these, fixed point results are also obtained by mathematicians for multivalued mappings involving control functions as coefficients in contractive inequality, for instance we refer to [1,3,14].

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Zadeh improved the mathematical framework by introducing the notion of fuzzy sets [26]. The classical work of Zadhe has a great interest in mathematics and other sciences having mathematical techniques for research. The notion of fuzzy mappings was established by Weiss and Batnario [22] and Hilpern proved fixed point theorem for these mappings in complete metric space[13]. Further the said fixed point theorem for fuzzy mappings were extended by many mathematicians in complete metric spaces for instance [2,4,7,11,12,20,23, 27]. In this paper we have derived common fixed point results for fuzzy mappings by generalizing the work of [17]. An appropriate example is demonstrated to validate our main result. In addition we have obtained multivalued results as application of our results.

2 Preliminaries

Throughout this paper $\mathscr{C}$ will represent the set of complex numbers.

Definition 2.1. [5] For $g, j \in \mathscr{C}$ the partial order $\precsim$ on $\mathscr{C}$ is defined by:

$$g \precsim j \iff Re(g) \leq Re(j) \text{ and } Im(g) \leq Im(j).$$

$$g \prec j \iff Re(g) < Re(j) \text{ and } Im(g) < Im(j).$$

Note that

$i)$ $p_1, p_2 \in \mathbb{R}$ and $p_1 \leq p_2 \Rightarrow p_1 g \precsim p_2 g$, for all $g \in \mathscr{C}$;

$ii)$ $0 \precsim g \precnsim j \Rightarrow |g| < |j|$, for all $g, j \in \mathscr{C}$;

$iii)$ $g \precsim j$ and $j \prec j^* \Rightarrow g \prec j^*$, for all $g, j, j^* \in \mathscr{C}$.

Definition 2.2.[5] Let $\mathbb{X}$ be a non-empty set and let a mapping $d : \mathbb{X} \times \mathbb{X} \to \mathscr{C}$ satisfying the following axioms:

$i)$ $0 \precsim d(g_1, g_2)$, for all $g_1, g_2 \in X$ and $d(g_1, g_2) = 0$ if and only if $g_1 = g_2$;

$ii)$ $d(g_1, g_2) = d(g_2, g_1)$, for all $g_1, g_2 \in \mathbb{X}$;

$iii)$ $d(g_1, g_2) \precsim d(g_1, g_3) + d(g_3, g_2)$, $\forall\, g_1, g_2, g_3 \in \mathbb{X}$.

Then $(\mathbb{X}, d)$ is said to be a complex-valued metric space.

Definition 2.3.[5] If $(\mathbb{X}, d)$ is a complex-valued metric space. A point $u \in \mathbb{X}$ is known as interior point of a set $M \subseteq \mathbb{X}$, if there exists $0 \prec \varepsilon \in \mathscr{C}$ such that

$$\mathscr{B}(u, \varepsilon) = \{v \in \mathbb{X} : d(u, v) \prec \varepsilon\} \subseteq M.$$

A point $u \in \mathbb{X}$ is known to be a limit point of M if for $0 \prec \varepsilon$

$$\mathscr{B}(u, \varepsilon) \cap \big(M \setminus \{u\}\big) \neq \phi, \text{ where } \varepsilon \in \mathscr{C}$$

M is called open if every member of M is an interior point of M. Further, if $\mathscr{B} \subseteq \mathbb{X}$ then $\mathscr{B}$ is known as closed if it contains all its limit points. The family

$$G = \{\mathscr{B}(u, \varepsilon) : u \in \mathbb{X}, 0 \prec \varepsilon\}$$

is a subbasis for a Hausdroff topology H on $\mathbb{X}$.

Definition 2.4.[1] Let $(\mathbb{X}, d)$ be complex-valued metric space. Let us denote the collection of all non-empty closed bounded subsets of complex-valued metric space $(\mathbb{X}, d)$ by $\mathfrak{CB}(\mathbb{X})$ then for $p \in \mathscr{C}$

$$s(p) = \{q \in \mathscr{C} : p \preceq q\}$$

and for $m_1 \in \mathbb{X}$ and $\mathrm{N} \in \mathfrak{CB}(\mathbb{X})$.

$$s\big(m_1, \mathrm{N}\big) = \bigcup_{n_1 \in \mathrm{N}} s\big(d(m_1, n_1)\big) = \bigcup_{n_1 \in \mathrm{N}} \big\{q \in \mathscr{C} : d(m_1, n_1) \preceq q\big\}.$$

For $\mathrm{M}, \mathrm{N} \in \mathfrak{CB}(\mathbb{X})$, we denote

$$s\big(\mathrm{M}, \mathrm{N}\big) = \big(\bigcap_{m_1 \in \mathrm{M}} s\big(m_1, \mathrm{N}\big)\big) \bigcap \big(\bigcap_{n_1 \in \mathrm{N}} s\big(n_1, \mathrm{M}\big)\big).$$

Let $K : \mathbb{X} \to \mathfrak{CB}(\mathbb{X})$ be a multivalued mapping, for $u \in \mathbb{X}$ and $\mathrm{M} \in \mathfrak{CB}(\mathbb{X})$ we define

$$\mathbb{W}_u(\mathrm{M}) = \big\{d(u, m_1) : m_1 \in \mathrm{M}\big\}.$$

Thus for $u, v \in \mathbb{X}$

$$\mathbb{W}_u(Kv) = \big\{d(u, v) : v \in Kv\big\}.$$

Lemma 2.5.[21] Assume that $(\mathbb{X}, d)$ is a complex-valued metric space.

$i)$ Let $w_1, w_2 \in \mathscr{C}$. If $w_1 \preceq w_2$, then $s(w_1) \subset s(w_2)$.

$ii)$ Let $w_1 \in \mathbb{X}$ and $F \in \mathbb{N}(\mathbb{X})$. If $\vartheta \in s(w_1, F)$, then $w_1 \in F$.

$iii)$ Let $w_1 \in \mathscr{C}, \mathrm{M}, \mathrm{N} \in \mathfrak{CB}(\mathbb{X})$ and $m_1 \in \mathrm{M}$. If $w_2 \in s(\mathrm{M}, \mathrm{N})$, then $w_2 \in s(m_1, \mathrm{N})$ for all $m_1 \in \mathrm{M}$ or $w_2 \in s(\mathrm{M}, n_1)$ for all $n_1 \in \mathrm{N}$.

Definition 2.6.[5] Let $\{g_r\}$ be a sequence in complex-valued metric space $(\mathbb{X}, d)$ and $g \in \mathbb{X}$, then

$i)$ g is known as a limit point of $\{g_r\}$ if for every $j \in \mathscr{C}$ with $j \succ 0$ there exists $r_0 \in \mathscr{N}$ such that $d(g_r, g) \preceq j$ for all $r \succ r_0$ and we write $\lim_{r\to\infty} g_r = g$.

$ii)$ $\{g_r\}$ is said to be a Cauchy sequence if for any $j \in \mathscr{C}$ with $j \succ 0$ there exists $r_0 \in \mathscr{N}$ such that $d(g_r, g_{r+s}) \prec j$ for all $r \succ r_0$ where $s \in \mathscr{N}$.

$iii)$ $(\mathbb{X}, d)$ is known as complete complex-valued metric space if every Cauchy sequence is convergent in $(\mathbb{X}, d)$.

Definition 2.7.[2] Let (V, d) be a metric linear space. A function Q in V define by $Q : V \to [0, 1]$ is called membership function, which assigns a grade of membership to each value of V. A set consisting the tuples of elements of V together with their grade of membership is called a fuzzy set. For simplicity we denote Q as a fuzzy set. The α-level set of Q denotes by $[Q]_\alpha$ and it is defined as below,

$$[Q]_\alpha = \{w : Q(w) \geq \alpha\} \; if \quad \alpha \in (0,1],$$

$$[Q]_0 = \{\overline{w : Q(w) > 0}\}.$$

Here $\overline{Q}$ represents the closer of the set Q.
Suppose $\mathcal{F}(\mathscr{Y})$ be the family of all fuzzy sets in a metric space $\mathscr{Y}$. For $\mathrm{T}, \mathrm{U} \in \mathcal{F}(\mathscr{Y}), \mathrm{T} \subset \mathrm{U}$ means $\mathrm{T}(w) \leq \mathrm{U}(w)$ for every $w \in \mathscr{Y}$.
Definition 2.8.[13] Assume that W is an arbitrary set and (Y,d) be complex-valued metric space. The mapping $K : W \to \mathcal{F}(Y)$ is called a fuzzy mapping. A fuzzy mapping K is a fuzzy subset on $W \times Y$ with a membership function $K(w)(y)$. The function $K(w)(y)$ is the grade of membership of y in $K(w)$.
Definition 2.9.[11] Let $(\mathbb{X},d)$ be a complex-valued metric space and $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings. If $w_1 \in [Kw_1]_\alpha$ where $\alpha \in [0,1]$ and $w_1 \in \mathbb{X}$ then w_1 is called a fuzzy fixed point of K and $w_1 \in \mathbb{X}$ is known as common fuzzy fixed point of K,L if $w_1 \in [Kw_1]_\alpha \cap [Lw_1]_\alpha$.
Definition 2.10.[1] Let $(\mathbb{X},d)$ be complex-valued metric space. A fuzzy mapping K from $\mathbb{X}$ into $\mathcal{F}(\mathbb{X})$ is said to have greatest lower bound property on $(\mathbb{X},d)$, if for any $u \in \mathbb{X}$ and $\alpha \in (0,1]$, greatest lower bound of $\mathbb{W}_u([Kv]_\alpha)$ exists in $\mathscr{C}$ for all $u,v \in \mathbb{X}$. Here we mention $d(u,[Kv]_\alpha)$ by the glb of $\mathbb{W}_u([Kv]_\alpha)$, i.e

$$d\big(u,[Kv]_\alpha\big) = \inf\{d(u,v_1) : v_1 \in [Kv]_\alpha\}.$$

3 Main results

In this section, we first prove some lemmas that are required to obtain our main fixed point results and then considering different types of control function as coefficients in rational contractive condition for fuzzy mappings in the context of complex-valued metric space.
Lemma 3.1. Let $(\mathbb{X},d)$ be complex-valued metric space, $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings and $w_0 \in \mathbb{X}$. Define a sequence $\{w_q\}$ by

$$w_{2q+1} \in [Kw_{2q}]_\alpha, \quad w_{2q+2} \in [Lw_{2q+1}]_\alpha, \;\; \forall\, q = 0,1,2,...,$$

where $[Kw]_\alpha, [Lw]_\alpha$ are non-empty closed and bounded subsets of $\mathbb{X}$. Assume that there exists a mapping $\varphi : \mathbb{X} \to [0,1)$ such that $\varphi(u) \preceq \varphi(w)$ for all $u \in [Kw]_\alpha$ with $\varphi(v) \preceq \varphi(w) \;\; \forall \;\; v \in [Lw]_\alpha$. Then $\varphi(w_{2q}) \preceq \varphi(w_0)$ and $\varphi(w_{2q+1}) \preceq \varphi(w_1)$.
Proof. Let $w \in \mathbb{X}$ and $q = 0,1,2,\cdots$ Then we have,

$$\begin{aligned}\varphi(w_{2q}) &\preceq \varphi(w_{2q-2}) \quad for \;\; w_{2q-1} \in [Kw_{2q-2}]_\alpha,\\ &\preceq \varphi(w_{2q-4}) \quad for \;\; w_{2q-2} \in [Kw_{2q-4}]_\alpha,\\ &\preceq \cdots \preceq \varphi(w_0).\end{aligned}$$

Similarly we have
$\varphi(w_{2q+1}) \preceq \varphi(w_1)$.
Hence proved.
Example 3.2 Let $\mathbb{X} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}$. Let $d : \mathbb{X} \times \mathbb{X} \to \mathscr{C}$ be defined by
$d(w,x) = i|w-x|$,
then clearly $(\mathbb{X},d)$ is a complex-valued metric space. Assume that fuzzy mappings K,L are defined by

$$K(w)(a) = \begin{cases} \alpha \;\; if \; 0 \leq a \leq \dfrac{1}{b+2} \\ \dfrac{\alpha}{2} \;\; if \; \dfrac{1}{b+2} < a \leq 1. \end{cases}$$

$$L(w)(a) = \begin{cases} \alpha \;\; if \; 0 \leq a \leq \dfrac{1}{b+2} \\ \dfrac{\alpha}{3} \;\; if \; \dfrac{1}{b+2} < t \leq 1. \end{cases}$$

Then $[Kw]_\alpha = [Lw]_\alpha = [0, \frac{1}{b+2}]$.

Setting the sequence $\{w_m\}$ as $\{w_m\} = \frac{1}{m+1}, m = 0,1,2,\cdots$. Then $w_0 = 1 \in \mathbb{X}$.

Clearly $w_{2m+1} \in [Kw_{2m}]_\alpha$ and $w_{2m+2} \in [Lw_{2m+1}]_\alpha$.

Consider a mapping $\beta : \mathbb{X} \to [0,1)$ defined by $\beta(w) = \frac{w}{9}$, for all $w \in \mathbb{X}$.

Obviously $\beta(u) \leq \beta(w)$ and $\beta(v) \leq \beta(w), \forall \;\; w \in \mathbb{X}, u \in [Kw]_\alpha, v \in [Lw]_\alpha$ for all $w,x \in \mathbb{X}$.
Consider

$$\beta(w_{2m}) = \frac{1}{9(2m+1)} \leq \frac{1}{9} = \beta(w_0),$$

that is $\beta(w_{2m}) \leq \beta(w_0), m = 0,1,2\cdots$. Also consider

$$\beta(w_{2m+1}) = \frac{1}{9(2m+2)} \leq \frac{\frac{1}{2}}{9} = \beta(w_1),$$

that is $\beta(w_{2m+1}) \leq \beta(w_1), m = 0,1,2\cdots$. Hence Lemma 3.1 is verified.
Lemma 3.3 Let $(\mathbb{X},d)$ be complex-valued metric space, $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings and $w_0 \in \mathbb{X}$. Define a sequence $\{w_q\}$ by

$$w_{2q+1} \in [Kw_{2q}]_\alpha, \quad w_{2q+2} \in [Lw_{2q+1}]_\alpha, \;\; \forall\, q = 0,1,2,..., \tag{1}$$

where $[Kw]_\alpha, [Lw]_\alpha$ are non-empty closed and bounded subsets of $\mathbb{X}$. Suppose there is a mapping $\varphi : \mathbb{X} \times \mathbb{X} \to [0,1)$ such that $\varphi(u,x) \preceq \varphi(w,x)$ for all $u \in [Kw]_\alpha$ and $\varphi(w,v) \preceq \varphi(w,x)$ for all $v \in [Lx]_\alpha$. Then $\varphi(w_{2q},x) \preceq \varphi(w_0,x)$ and $\varphi(w,w_{2q+1}) \preceq \varphi(w,w_1)$.
Proof. Let $w \in \mathbb{X}$ and $q = 0,1,2,\cdots$ Then we have,

$$\begin{aligned}\varphi(w_{2q},x) &\preceq \varphi(w_{2q-2},x) \quad for \;\; w_{2q-1} \in [Kw_{2q-2}]_\alpha,\\ &\preceq \varphi(w_{2q-4},x) \quad for \;\; w_{2q-2} \in [Kw_{2q-4}]_\alpha,\\ &\preceq \cdots \preceq \varphi(w_0,x).\end{aligned}$$

Similarly, we have

$$\varphi(w, w_{2q+1}) \preceq \varphi(w, w_1).$$

Hence Lemma 3.3 is proved.

Example 3.4. In Example 3, let the mapping $\beta : \mathbb{X} \times \mathbb{X} \to [0,1)$ be defined by $\beta(w,x) = \frac{w}{9} + \frac{x}{11}$, for all $w,x \in \mathbb{X}$. Obviously $\beta(u,x) \leq \beta(w,x)$ and $\beta(w,v) \leq \beta(w,x), \forall w \in \mathbb{X}, u \in [Kw]_\alpha, v \in [Lx]_\alpha$ for all $w,x \in \mathbb{X}$. Consider

$$\beta(w_{2m},y) = \frac{1}{9(2m+1)} + \frac{x}{11} \leq \frac{1}{9} + \frac{x}{11} = \beta(w_0,x),$$

that is $\beta(w_{2m},x) \leq \beta(w_0,x), m = 0,1,2\cdots$. Also consider

$$\beta(w, w_{2m+1}) = \frac{w}{9} + \frac{1}{11(2m+2)} \leq \frac{w}{9} + \frac{\frac{1}{2}}{11} = \beta(w, w_1),$$

that is $\beta(w,w_{2m+1}) \leq \beta(w,w_1), m = 0,1,2\cdots$. Hence Lemma 3.3 is verified.

Theorem 3.5. Let $(\mathbb{X},d)$ be complete complex-valued metric space and $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings satisfying greatest lower bound property, such that for each $w \in \mathbb{X}$ there exists some $\alpha \in (0,1], [Kw]_\alpha, [Lw]_\alpha$ are non-empty closed and bounded subsets of $\mathbb{X}$. If there exists $\varphi_i : \mathbb{X} \to [0,1)$ $i = 1,2,3,4,5$ such that

i) $\varphi_i(u) \preceq \varphi_i(w)$ $i = 1\cdots5$ for all $u \in [Kw]_\alpha$, $\forall\, w \in \mathbb{X}$;

ii) $\varphi_i(v) \preceq \varphi_i(w)$ $i = 1\cdots5$ for all $v \in [Lw]_\alpha$, $\forall\, w \in \mathbb{X}$;

iii) $\Sigma_{i=1}^5 \varphi_i(w) < 1$ $i = 1\cdots5$;

iv)

$$\varphi_1(w)d(w,x) + \varphi_2(w)\frac{d\big(w,[Kw]_\alpha\big)d\big(x,[Lx]_\alpha\big)}{1+d(w,x)} + \frac{\varphi_3(w)d\big(w,[Kw]_\alpha\big)d\big(x,[Lx]_\alpha\big) + \varphi_4(w)d\big(x,[Kw]_\alpha\big)d\big(w,[Lx]_\alpha\big)}{1+d(w,x)} + \varphi_5(w)\frac{d\big(w,[Kw]_\alpha\big)}{1+d(w,x)} \in s\big([Kw]_\alpha,[Lw]_\alpha\big).$$

Then there exists some $v \in \mathbb{X}$ such that $v \in [Kv]_\alpha \cap [Lv]_\alpha$.

Proof. Let w_0 be an arbitrary point in $\mathbb{X}$ and $w_1 \in [Kw_0]_\alpha$, then from (iv) with $w = w_0$ and $x = w_1$ we have

$$\varphi_1(w_0)d(w_0,w_1) + \varphi_2(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_3(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_4(w_0)\frac{d\big(w_1,[Kw_0]_\alpha\big)d\big(w_0,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_5(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)}{1+d(w_0,w_1)} \in s\big([Kw_0]_\alpha,[Lw_1]_\alpha\big).$$

By Lemma 2.5 (iii), we have

$$\varphi_1(w_0)d(w_0,w_1) + \varphi_2(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_3(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_4(w_0)\frac{d\big(w_1,[Kw_0]_\alpha\big)d\big(w_0,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_5(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)}{1+d(w_0,w_1)} \in s\big(w_1,[Lw_1]_\alpha\big).$$

By definition there exists some $w_2 \in [Lw_1]_\alpha$ such that

$$\varphi_1(w_0)d(w_0,w_1) + \varphi_2(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_3(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_4(w_0)\frac{d\big(w_1,[Kw_0]_\alpha\big)d\big(w_0,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_5(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)}{1+d(w_0,w_1)} \in s\big(d(w_1,w_2)\big),$$

that is,

$$d(w_1,w_2) \preceq \varphi_1(w_0)d(w_0,w_1) + \varphi_2(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_3(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_4(w_0)\frac{d\big(w_1,[Kw_0]_\alpha\big)d\big(w_0,[Lw_1]_\alpha\big)}{1+d(w_0,w_1)} + \varphi_5(w_0)\frac{d\big(w_0,[Kw_0]_\alpha\big)}{1+d(w_0,w_1)}.$$

By Definition 2.10, we have

$$d(w_1,w_2) \preceq \varphi_1(w_0)d(w_0,w_1) + \varphi_2(w_0)\frac{d(w_0,w_1)d(w_1,w_2)}{1+d(w_0,w_1)} + \varphi_3(w_0)\frac{d(w_0,w_1)d(w_1,w_2)}{1+d(w_0,w_1)} + \varphi_4(w_0)\frac{d(w_1,w_1)d(w_0,w_2)}{1+d(w_0,w_1)} + \varphi_5(w_0)\frac{d(w_0,w_1)}{1+d(w_0,w_1)}.$$

Taking absolute on both sides

$$\begin{aligned}\big|d(w_1,w_2)\big| \le\ & \varphi_1(w_0)\big|d(w_0,w_1)\big| \\ & + \varphi_2(w_0)\Big|\frac{d(w_0,w_1)d(w_1,w_2)}{1+d(w_0,w_1)}\Big| \\ & + \Big|\varphi_3(w_0)\frac{d(w_0,w_1)d(w_1,w_2)}{1+d(w_0,w_1)}\Big| \\ & + \varphi_4(w_0)\Big|\frac{d(w_1,w_1)d(w_0,w_2)}{1+d\big(w_0,w_1\big)}\Big| \\ & + \varphi_5(w_0)\Big|\frac{d(w_0,w_1)}{1+d(w_0,w_1)}\Big|,\end{aligned}$$

which implies that

$$\begin{aligned}\big|d(w_1,w_2)\big| \le\ & \varphi_1(w_0)\big|d(w_0,w_1)\big| + \varphi_2(w_0)\big|d(w_1,w_2)\big| \\ & + \varphi_3(w_0)\big|d\big(w_1,w_2\big)\big| + \varphi_5(w_0)\big|d(w_0,w_1)\big|.\end{aligned}$$

Thus we have

$$|d(w_1,w_2)| \le \frac{\varphi(w_0)+\varphi_5(w_0)}{1-(\varphi_2(w_0)+\varphi_3(w_0))}|d(w_0,w_1)|.$$

Since $\Sigma_{i=1}^{5}\varphi_i(w) < 1$, so $\rho = \frac{\varphi_1(w_0)+\varphi_5(w_0)}{1-(\varphi_2(w_0)+\varphi_3(w_0))} < 1$. So

$$|d(w_1,w_2)| \le \rho|d(w_0,w_1)|. \tag{2}$$

Now for $w_2 \in [Lw_1]_\alpha$, consider

$$\begin{aligned}&\varphi_1(w_2)d(w_2,w_1)+\varphi_2(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_3(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)}+ \\ &\varphi_4(w_2)\frac{d\big(w_1,[Kw_2]_\alpha\big)d\big(w_2,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_5(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)}{1+d(w_2,w_1)} \in s\big([Kw_2]_\alpha,[Lw_1]_\alpha\big).\end{aligned}$$

Again by using Lemma 2.5 (iii), we have

$$\begin{aligned}&\varphi_1(w_2)d(w_2,w_1)+\varphi_2(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_3(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\frac{\varphi_4(w_2)d\big(w_1,[Kw_2]_\alpha\big)d\big(w_2,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_5(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)}{1+d(w_2,w_1)} \in s\big([Kw_2]_\alpha,w_2\big).\end{aligned}$$

By definition there exists some $w_3 \in [Kw_2]_\alpha$ such that

$$\begin{aligned}&\varphi_1(w_2)d(w_2,w_1)+\varphi_2(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_3(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_4(w_2)\frac{d\big(w_1,[Kw_2]_\alpha\big)d\big(w_2,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ &+\varphi_5(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)}{1+d(w_2,w_1)} \in s\big(w_3,w_2\big),\end{aligned}$$

that is,

$$\begin{aligned}d(w_3,w_2) \preceq\ & \varphi_1(w_2)d(w_2,w_1)+ \\ & \varphi_2(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ & +\varphi_3(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)d\big(w_1,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)}+ \\ & \varphi_4(w_2)\frac{d\big(w_1,[Kw_2]_\alpha\big)d\big(w_2,[Lw_1]_\alpha\big)}{1+d(w_2,w_1)} \\ & +\varphi_5(w_2)\frac{d\big(w_2,[Kw_2]_\alpha\big)}{1+d(w_2,w_1)}.\end{aligned}$$

Using Definition 2.10, we get

$$\begin{aligned}d(w_3,w_2) \preceq\ & \varphi_1(w_2)d(w_2,w_1)+\varphi_2(w_2)\frac{d(w_2,w_3)d(w_1,w_2)}{1+d(w_2,w_1)} \\ & +\frac{\varphi_3(w_2)d(w_2,w_3)d(w_1,w_2)+\varphi_4(w_2)d(w_1,w_3)d(w_2,w_2)}{1+d(w_2,w_1)} \\ & +\varphi_5(w_2)\frac{d(w_2,w_3)}{1+d(w_2,w_1)},\end{aligned}$$

taking again absolute on both sides. We obtain

$$\begin{aligned}|d(w_3,w_2)| \le\ & \varphi_1(w_2)|d(w_2,w_1)|+\varphi_2(w_2)\Big|\frac{d(w_2,w_3)d(w_1,w_2)}{1+d(w_2,w_1)}\Big| \\ & +\Big|\frac{\varphi_3(w_2)d(w_2,w_3)d(w_1,w_2)+\varphi_4(w_2)d(w_1,w_3)d(w_2,w_2)}{1+d(w_2,w_1)}\Big| \\ & +\varphi_5(w_2)\Big|\frac{d(w_2,w_3)}{1+d(w_2,w_1)}\Big|,\end{aligned}$$

which implies that

$$\begin{aligned}\big|d(w_3,w_2)\big| \le\ & \varphi_1(w_2)\big|d(w_2,w_1)\big| + \varphi_2(w_2)\big|d(w_2,w_3)\big| \\ & + \varphi_3(w_2)\big|d(w_2,w_3)\big| + \varphi_5(w_2)\big|d(w_2,w_3)\big|.\end{aligned}$$

Using Lemma 3.1, we get

$$\begin{aligned}\big|d(w_3,w_2)\big| \le\ & \varphi_1(w_0)\big|d(w_2,w_1)\big| + \varphi_2(w_0)\big|d(w_2,w_3)\big| \\ & + \varphi_3(w_0)\big|d(w_2,w_3)\big| + \varphi_5(w_0)\big|d(w_2,w_3)\big|.\end{aligned}$$

This yields

$$\big|d(w_3,w_2)\big| \le \frac{\varphi_1(w_0)}{1-(\varphi_2(w_0)+\varphi_3(w_0)+\varphi_5(w_0))}\big|d(w_2,w_1)\big|.$$

Since $\Sigma_{i=1}^{5}\varphi(w) < 1$, so $\frac{\varphi_1(w)}{1-(\varphi_2(w)+\varphi_3(w)+\varphi_5(w))} < 1$.

Set $\sigma = \frac{\varphi_1(w_0)}{1-(\varphi_2(w_0)+\varphi_3(w_0)+\varphi_5(w_0))}$, so

$\big|d(w_2,w_3)\big| \le \sigma\big|d(w_1,w_2)\big|$.

Inductively we can construct a sequence $\{w_q\}$ in $\mathbb{X}$ such that

$w_{2q+1} \in [Kw_{2q}]_\alpha, w_{2q+2} \in [Lw_{2q+1}]_\alpha$ for $q = 0,1,2,\ldots,$

$|d(w_{2q+1},w_{2q+2})| \le \rho|d(w_{2q},w_{2q+1})|,$

$|d(w_{2q+2},w_{2q+3})| \le \sigma|d(w_{2q+1},w_{2q+2})|.$

Consequently

$$\begin{aligned}|d(w_{2q+1},w_{2q+2})| &\leq \rho|d(w_{2q},w_{2q+1})| \\ &\leq \rho\sigma|d(w_{2q-1},w_{2q})| \\ &\leq \rho\sigma\rho|d(w_{2q-2},w_{2q-1})| \\ &\leq \cdots \leq \rho(\rho\sigma)^{q}|d(w_0,w_1)|,\end{aligned}$$

and

$$\begin{aligned}|d(w_{2q+2},w_{2q+3})| &\leq \sigma|d(w_{2q+1},w_{2q+2})| \\ &\leq \cdots \leq (\rho\sigma)^{q+1}|d(w_0,w_1)|.\end{aligned}$$

So if $l<\kappa$, then we obtain

$$\begin{aligned}d(w_{2l+1},w_{2\kappa+1}) \preceq\ & d(w_{2l+1},w_{2l+2})+d(w_{2l+2},w_{2l+3}) \\ &+ d(w_{2l+3},w_{2l+4})+\cdots+d(w_{2\kappa},w_{2\kappa+1}),\end{aligned}$$

which yields

$$\begin{aligned}\left|d(w_{2l+1},w_{2\kappa+1})\right| \leq\ & \left|d(w_{2l+1},w_{2l+2})\right|+\left|d(w_{2l+2},w_{2l+3})\right| \\ &+ \left|d(w_{2l+3},w_{2l+4})\right|+\cdots+\left|d(w_{2\kappa},w_{2\kappa+1})\right| \\ \leq\ & \left[\rho\sum_{p=l}^{\kappa-1}(\rho\sigma)^p+\sum_{p=l+1}^{\kappa}(\rho\sigma)^p\right]\left|d(w_0,w_1)\right|.\end{aligned}$$

Similarly, we have

$$d(w_{2l},w_{2\kappa+1}) \leq \left[\sum_{p=l}^{\kappa}(\rho\sigma)^p+\rho\sum_{p=l}^{\kappa-1}(\rho\sigma)^p\right]\left|d(w_0,w_1)\right|,$$

$$d(w_{2l},w_{2\kappa}) \leq \left[\sum_{p=l}^{\kappa-1}(\rho\sigma)^p+\rho\sum_{p=l}^{\kappa-1}(\rho\sigma)^p\right]\left|d(w_0,w_1)\right|,$$

$$d(w_{2l+1},w_{2\kappa}) \leq \left[\rho\sum_{p=l}^{\kappa-1}(\rho\sigma)^p+\sum_{p=l+1}^{\kappa-1}(\rho\sigma)^p\right]\left|d(w_0,w_1)\right|.$$

Since $(\sigma\rho)<1$ which implies that $\{w_q\}$ is a Cauchy sequence in $\mathbb{X}$. From the completeness of $\mathbb{X}$, there exists $\nu\in\mathbb{X}$ such that $w_q\to\nu$ as $q\to\infty$. Now we have to show that $\nu\in[K\nu]_\alpha$ and $\nu\in[L\nu]_\alpha$, we get

$$\begin{aligned}&\varphi_1(w_{2q})d(w_{2q},\nu)+\varphi_2(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d\big(\nu,[Kw_{2q}]_\alpha\big)d\big(w_{2q},[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)}{1+d(w_{2q},\nu)} \in s\big([Kw_{2q}]_\alpha,[L\nu]_\alpha\big).\end{aligned}$$

Since $w_{2q+1}\in[Kw_{2q}]_\alpha$, by Lemma 2.5 (iii), we have

$$\begin{aligned}&\varphi_1(w_{2q})d(w_{2q},\nu)+\varphi_2(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d\big(\nu,[Kw_{2q}]_\alpha\big)d\big(w_{2q},[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)}{1+d(w_{2q},\nu)} \in s\big(w_{2q+1},[L\nu]_\alpha\big).\end{aligned}$$

By definition there exists some $y_q\in[L\nu]_\alpha$ such that

$$\begin{aligned}&\varphi_1(w_{2q})d(w_{2q},\nu)+\varphi_2(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d\big(\nu,[Kw_{2q}]_\alpha\big)d\big(w_{2q},[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)}{1+d(w_{2q},\nu)} \in s\big(d(w_{2q+1},y_q)\big),\end{aligned}$$

that is,

$$\begin{aligned}d(w_{2q+1},y_q) \preceq\ & \varphi_1(w_{2q})d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3\frac{(w_{2q})d\big(w_{2q},[Kw_{2q}]_\alpha\big)d\big(\nu,[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d\big(\nu,[Kw_{2q}]_\alpha\big)d\big(w_{2q},[L\nu]_\alpha\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d\big(w_{2q},[Kw_{2q}]_\alpha\big)}{1+d(w_{2q},\nu)}.\end{aligned}$$

Using Definition 2.10, we have

$$\begin{aligned}d(w_{2q+1},y_q) \preceq\ & \varphi_1(w_{2q})d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q})\frac{d\big(w_{2q},w_{2q+1}\big)d\big(\nu,y_q\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3(w_{2q})\frac{d(w_{2q},w_{2q+1})d(\nu,y_q)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d(\nu,w_{2q+1})d(w_{2q},y_q)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d(w_{2q},w_{2q+1})}{1+d(w_{2q},\nu)}.\end{aligned}$$

By triangular inequality, we get

$$\begin{aligned}d(\nu,y_q) \preceq\ & d(\nu,w_{2q+1})+d(w_{2q+1},y_q) \\ \preceq\ & d(\nu,w_{2q+1})+\varphi_1(w_{2q})d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q})\frac{d\big(w_{2q},w_{2q+1}\big)d\big(\nu,y_q\big)}{1+d(w_{2q},\nu)} \\ &+ \varphi_3(w_{2q})\frac{d(w_{2q},w_{2q+1})d(\nu,y_q)}{1+d(w_{2q},\nu)} \\ &+ \varphi_4(w_{2q})\frac{d(\nu,w_{2q+1})d(w_{2q},y_q)}{1+d(w_{2q},\nu)} \\ &+ \varphi_5(w_{2q})\frac{d(w_{2q},w_{2q+1})}{1+d(w_{2q},\nu)}.\end{aligned}$$

Using Lemma 3.1, we get

$$|d(v,y_q)| \leq |d(v,w_{2q+1})| + \varphi_1(w_0)|d(w_{2q},v)| + \varphi_2(w_0)\frac{|d(w_{2q},w_{2q+1})||d(v,y_q)|}{1+|d(w_{2q},v)|} + \varphi_3(w_0)\frac{|d(w_{2q},w_{2q+1})||d(v,y_q)|}{1+d(w_{2q},v)}| + \varphi_4(w_0)\frac{|d(v,w_{2q+1})||d(w_{2q},y_q)|}{1+|d(w_{2q},v)|} + \varphi_5(w_0)\frac{|d(w_{2q},w_{2q+1})|}{1+|d(w_{2q},v)|}.$$

Let $q \to \infty$, the above inequality implies that $|d(v,y_q)| \to 0$. We have $y_q \to v$ as $q \to \infty$. Since $[Lv]_\alpha$ is closed, so $v \in [Lv]_\alpha$. By similar process we can obtain that $v \in [Kv]_\alpha$. Thus $v \in [Kv]_\alpha \cap [Lv]_\alpha$.

Corollary 3.6. Let $(\mathbb{X},d)$ be complete complex-valued metric space, and $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings with glb property, such that for each $w \in \mathbb{X}$ related to some $\alpha \in (0,1]$ there exists $[Kw]_\alpha, [Lw]_\alpha$ non-empty closed bounded subsets of $\mathbb{X}$ such that

$$\mu d(w,x) + \gamma\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} + \lambda\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} + \varsigma\frac{d(x,[Kw]_\alpha)d(w,[Lx]_\alpha)}{1+d(w,x)} + \delta\frac{d(w,[Kw]_\alpha)}{1+d(w,x)} \in s([Kw]_\alpha,[Lx]_\alpha),$$

where $\mu,\gamma,\lambda,\delta,\varsigma$ are non-negative reals with $\mu+\gamma+\lambda+\delta+\varsigma < 1$. Then there exists some $v \in \mathbb{X}$ such that $v \in [Kv]_\alpha \cap [Lv]_\alpha$..

Proof. We can easily prove by applying theorem 3.5 and by setting $\varphi_1(w) = \mu, \varphi_2(w) = \gamma, \varphi_3(w) = \lambda, \varphi_4(w) = \delta, \varphi_3(w) = \varsigma$.

Corollary 3.7. Let $(\mathbb{X},d)$ be complete complex-valued metric space, and $K : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mapping with glb property, such that for each $w \in \mathbb{X}$ related to some $\alpha \in (0,1]$ there exist $[Kw]_\alpha$ a non-empty closed bounded subset of $\mathbb{X}$. If there exists $\varphi_i : \mathbb{X} \to [0,1), \quad i = 1,\cdots,5$ such that

$i)$ $\varphi_i(u) \preceq \varphi_i(w), i = 1,\cdots,5.$ for all $u \in [Kw]_\alpha$ for all $w \in \mathbb{X}$;

$ii)$ $\Sigma_{i=1}^5 \varphi_i(w) < 1$;

$iii)$

$$\varphi_1(w)d(w,x) + \varphi_2(w)\frac{d(w,[Kw]_\alpha)d(x,[Kx]_\alpha)}{1+d(w,x)} + \varphi_3(w)\frac{d(w,[Kw]_\alpha)d(x,[Kx]_\alpha)}{1+d(w,x)} + \varphi_4(w)\frac{d(x,[Kw]_\alpha)d(w,[Kx]_\alpha)}{1+d(w,x)} + \varphi_5(w)\frac{d(w,[Kw]_\alpha)}{1+d(w,x)} \in s([Kw]_\alpha,[Kx]_\alpha).$$

Then there exists some $v \in \mathbb{X}$ such that $v \in [Kv]_\alpha$.

Proof. By setting $K = L$. it can be easily proved in Theorem 3.5.

Corollary 3. 8. Let $(\mathbb{X},d)$ be complete complex-valued metric space, and $K : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mapping with glb property, such that for each $w \in \mathbb{X}$ related to some $\alpha \in (0,1]$ there exists $[Kw]_\alpha$ a non-empty closed bounded subset of $\mathbb{X}$ such that for all $w,x \in \mathbb{X}$ the following conditions are satisfied;

$$\mu d(w,x) + \gamma\frac{d(w,[Kw]_\alpha)d(x,[Kx]_\alpha)}{1+d(w,x)} + \lambda\frac{d(w,[Kw]_\alpha)d(x,[Kx]_\alpha)}{1+d(w,x)} + \varsigma\frac{d(x,[Kw]_\alpha)d(w,[Kx]_\alpha)}{1+d(w,x)} + \delta\frac{d(w,[Kw]_\alpha)}{1+d(w,x)} \in s([Kw]_\alpha,[Kx]_\alpha),$$

where $\mu,\gamma,\lambda,\delta,\varsigma$ are non-negative reals with $\mu+\gamma+\lambda+\delta+\varsigma < 1$. Then there exists some $v \in \mathbb{X}$ such that $v \in [Kv]_\alpha$.

Proof We can easily prove by applying Corollary 3.7 and by setting $\varphi_1(w) = \mu, \varphi_2(w) = \gamma, \varphi_3(w) = \lambda, \varphi_4 = \varsigma, \varphi_5(w) = \delta.$

In the following we have obtained fixed point results for control functions defined from $\mathbb{X} \times \mathbb{X}$ to $[0,1)$.

Theorem 3.9. Let $(\mathbb{X},d)$ be a complete complex-valued metric space Let $K,L : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be a fuzzy mapping with glb property and for $w \in \mathbb{X}$, related to some $\alpha \in (0,1]$ there exists $[Kw]_\alpha, [Lw]_\alpha$ non-empty closed bounded subsets of $\mathbb{X}$. If there exists mappings $\varphi_i : \mathbb{X}\times\mathbb{X} \to [0,1), \quad i = 1,\cdots,8$ such that $\forall \ w, \ x \in \mathbb{X}$ the following conditions are satisfied: $i)$ $\varphi_i(u,x) \leq \varphi_i(u,x)$ for all $u \in [Kw]_\alpha$and $\varphi_i(w,v) \leq \varphi_i(w,x) \ \ \forall v \in [Lw]_\alpha$;
$ii)$

$$\varphi_1(w,x)d(w,x) + \varphi_2(w,x)d(w,[Lx]_\alpha) + \varphi_3(w,x)d(x,[Kw]_\alpha) + \varphi_4(w,x)d(w,[Kw]_\alpha) + \varphi_5(w,x)d(x,[Lx]_\alpha) + \varphi_6(w,x)\frac{d(x,[Lx]_\alpha)[1+d(w,[Kw]_\alpha)]}{1+d(w,x)} + \varphi_7(w,x)\frac{d(w,[Lx]_\alpha)[1+d(w,[Kw]_\alpha)]}{1+d(w,x)} + \varphi_8(w,x)\frac{d(w,[Lx]_\alpha)[1+d(x,[Kw]_\alpha)]}{1+d(w,x)} \in s([Kw]_\alpha,[Lx]_\alpha), \tag{3}$$

where $\varphi_1(w,x) + \sum_{i=3}^6 \varphi_i(w,x) + 2[\varphi_2(w,x) + \varphi_7(w,x) + \varphi_8(w,x)] < 1$. Then there exists some $v \in \mathbb{X}$ such that $v \in [Kv]_\alpha \cap [Lv]_\alpha$.

Proof. Let w_0 be arbitrary point in $\mathbb{X}$ and $w_1 \in [Kw_0]_\alpha$.

From (3) with $w = w_0$ and $x = w_1$, we get

$$\begin{aligned}&\varphi_1(w_0,w_1)d(w_0,w_1)+\varphi_2(w_0,w_1)d(\omega_0,[Lw_1]_\alpha)\\ &+\varphi_3(w_0,w_1)d(w_1,[Kw_0]_\alpha)\\ &+\varphi_4(w_0,w_1)d(w_0,[Kw_0]_\alpha)\\ &+\varphi_5(w_0,w_1)d(w_1,[Lw_1]_\alpha)\\ &+\varphi_6(w_0,w_1)\frac{d(w_1,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_7(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_8(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_1,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &\in s\big([Kw_0]_\alpha,[Lw_1]_\alpha\big).\end{aligned}$$

By Lemma 2.5 (iii) we get

$$\begin{aligned}&\varphi_1(w_0,w_1)d(w_0,w_1)+\varphi_2(w_0,w_1)d(\omega_0,[Lw_1]_\alpha)\\ &+\varphi_3(w_0,w_1)d(w_1,[Kw_0]_\alpha)\\ &+\varphi_4(w_0,w_1)d(w_0,[Kw_0]_\alpha)\\ &+\varphi_5(w_0,w_1)d(w_1,[Lw_1]_\alpha)\\ &+\varphi_6(w_0,w_1)\frac{d(w_1,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_7(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_8(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_1,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &\in s\big(w_1,[Lw_1]_\alpha\big).\end{aligned}$$

By definition there exists some $w_2 \in [Lw_1]_\alpha$ such that

$$\begin{aligned}&\varphi_1(w_0,w_1)d(w_0,w_1)+\varphi_2(w_0,w_1)d(\omega_0,[Lw_1]_\alpha)\\ &+\varphi_3(w_0,w_1)d(w_1,[Kw_0]_\alpha)\\ &+\varphi_4(w_0,w_1)d(w_0,[Kw_0]_\alpha)\\ &+\varphi_5(w_0,w_1)d(w_1,[Lw_1]_\alpha)\\ &+\varphi_6(w_0,w_1)\frac{d(w_1,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_7(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_8(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_1,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &\in s\big(w_1,w_2)\big),\end{aligned}$$

that is,

$$\begin{aligned}d(w_1,w_2)\preceq{}&\varphi_1(w_0,w_1)d(w_0,w_1)\\ &+\varphi_2(w_0,w_1)d(\omega_0,[Lw_1]_\alpha)\\ &+\varphi_3(w_0,w_1)d(w_1,[Kw_0]_\alpha)\\ &+\varphi_4(w_0,w_1)d(w_0,[Kw_0]_\alpha)\\ &+\varphi_5(w_0,w_1)d(w_1,[Lw_1]_\alpha)\\ &+\varphi_6(w_0,w_1)\frac{d(w_1,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_7(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_0,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}\\ &+\varphi_8(w_0,w_1)\frac{d(w_0,[Lw_1]_\alpha)\left[1+d(w_1,[Kw_0]_\alpha)\right]}{1+d(w_0,w_1)}.\end{aligned}$$

Using the glb property of K an L, we get

$$\begin{aligned}d(w_1,w_2)\preceq{}&\varphi_1(w_0,w_1)d(w_0,w_1)+\varphi_2(w_0,w_1)d(w_0,w_2)\\ &+\varphi_3(w_0,w_1)d(w_1,w_1)+\varphi_4(w_0,w_1)d(w_0,w_1)\\ &+\varphi_5(w_0,w_1)d(w_1,w_2)\\ &+\varphi_6(w_0,w_1)\frac{d(w_1,w_2)[1+d(w_0,w_1]}{1+d(w_0,w_1)}\\ &+\varphi_7(w_0,w_1)\frac{d(w_0,w_2)[1+d(w_0,w_1]}{1+d(w_0,w_1)}\\ &+\varphi_8(w_0,w_1)\frac{d(w_0,w_2)[1+d(w_1,w_1]}{1+d(w_0,w_1)d(w_1,w_1)}\\ \preceq{}&\varphi_1(w_0,w_1)d(w_0,w_1)+\varphi_2(w_0,w_1)d(w_0,w_2)\\ &+\varphi_4(w_0,w_1)d(w_0,w_1)+\varphi_5(w_0,w_1)d(w_1,w_2)\\ &+\varphi_6(w_0,w_1)d(w_1,w_2)+\varphi_7(w_0,w_1)d(w_0,w_2)\\ &+\varphi_8(w_0,w_1)d(w_0,w_2)\\ \preceq{}&\varphi_1(w_0,w_1)d(w_0,w_1)\\ &+\varphi_2(w_0,w_1)[d(w_0,w_1)+d(w_1,w_2)]\\ &+\varphi_4(w_0,w_1)d(w_0,w_1)+\varphi_5(w_0,w_1)d(w_1,w_2)\\ &+\varphi_6(w_0,w_1)d(w_1,w_2)\\ &+\varphi_7(w_0,w_1)[d(w_0,w_1)+d(w_1,w_2)]\\ &+\varphi_8(w_0,w_1)d(w_0,w_1)+\varphi_8(w_0,w_1)d(w_1,w_2).\end{aligned}$$

The former inequality implies that

$$\begin{aligned}|d(w_1,w_2)|\le{}&\varphi_1(w_0,w_1)|d(w_0,w_1)|\\ &+\varphi_2(w_0,w_1)[|d(w_0,w_1)|+|d(w_1,w_2)|]\\ &+\varphi_4(w_0,w_1)|d(w_0,w_1)|+\varphi_5(w_0,w_1)|d(w_1,w_2)|\\ &+\varphi_6(w_0,w_1)|d(w_1,w_2)|\\ &+\varphi_7(w_0,w_1)[|d(w_0,w_1)|+|d(w_1,w_2)|]\\ &+\varphi_8(w_0,w_1)|d(w_0,w_1)|+\varphi_8(w_0,w_1)d(w_1,w_2)\\ \le{}&\Xi|d(w_0,w_1)|,\end{aligned}$$

where

$$\Xi=\frac{E}{1-U}<1.$$

and

$$E=\sum\varphi_t(w_0,w_1),\ \ t=1,2,4,7,8$$

,

$$U=\sum\varphi_t(w_0,w_1),\ \ t=2,5,6,7,8.$$

Now for $w_2 \in [Lw_1]_\alpha$, consider

$$\begin{aligned}&\varphi_1(w_2,w_1)d(w_2,w_1)+\varphi_2(w_2,w_1)d(w_2,[Lw_1]_\alpha)\\&+\varphi_3(w_2,w_1)d(w_1,[Kw_2]_\alpha)+\varphi_4(w_2,w_1)d(w_2,[Kw_2]_\alpha)\\&+\varphi_5(w_2,w_1)d(w_1,[Lw_1]_\alpha)\\&+\varphi_6(w_2,w_1)\frac{d(w_1,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_7(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_8(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_1,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&\in s\big([Kw_2]_\alpha,[Lw_1]_\alpha\big).\end{aligned}$$

Again by using Lemma 2.5 (iii), we have

$$\begin{aligned}&\varphi_1(w_2,w_1)d(w_2,w_1)+\varphi_2(w_2,w_1)d(w_2,[Lw_1]_\alpha)\\&+\varphi_3(w_2,w_1)d(w_1,[Kw_2]_\alpha)+\varphi_4(w_2,w_1)d(w_2,[Kw_2]_\alpha)\\&+\varphi_5(w_2,w_1)d(w_1,[Lw_1]_\alpha)\\&+\varphi_6(w_2,w_1)\frac{d(w_1,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_7(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_8(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_1,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&\in s\big([Kw_2]_\alpha,w_2\big).\end{aligned}$$

By definition there exists some $w_3 \in [Kw_2]_\alpha$, such that

$$\begin{aligned}&\varphi_1(w_2,w_1)d(w_2,w_1)+\varphi_2(w_2,w_1)d(w_2,[Lw_1]_\alpha)\\&+\varphi_3(w_2,w_1)d(w_1,[Kw_2]_\alpha)+\varphi_4(w_2,w_1)d(w_2,[Kw_2]_\alpha)\\&+\varphi_5(w_2,w_1)d(w_1,[Lw_1]_\alpha)\\&+\varphi_6(w_2,w_1)\frac{d(w_1,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_7(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_8(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_1,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&\in s\big(d(w_3,w_2)\big),\end{aligned}$$

that is,

$$\begin{aligned}d(w_3,w_2)\preceq\;&\varphi_1(w_2,w_1)d(w_2,w_1)\\&+\varphi_2(w_2,w_1)d(w_2,[Lw_1]_\alpha)+\varphi_3(w_2,w_1)d(w_1,[Kw_2]_\alpha)\\&+\varphi_4(w_2,w_1)d(w_2,[Kw_2]_\alpha)+\varphi_5(w_2,w_1)d(w_1,[Lw_1]_\alpha)\\&+\varphi_6(w_2,w_1)\frac{d(w_1,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_7(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_2,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}\\&+\varphi_8(w_2,w_1)\frac{d(w_2,[Lw_1]_\alpha)\big[1+d(w_1,[Kw_2]_\alpha)\big]}{1+d(w_2,w_1)}.\end{aligned}$$

Using the glb property of K and L we get

$$\begin{aligned}d(w_3,w_2)\preceq\;&\varphi_1(w_2,w_1)d(w_2,w_1)+\varphi_2(w_2,w_1)d(w_2,w_2)\\&+\varphi_3(w_2,w_1)d(w_1,w_3)+\varphi_4(w_2,w_1)d(w_2,w_3)\\&+\varphi_5(w_2,w_1)d(w_1,w_2)\\&+\varphi_6(w_2,w_1)\frac{d(w_1,w_2)[1+d(w_2,w_3)]}{1+d(w_2,w_1)}\\&+\varphi_7(w_2,w_1)\frac{d(w_2,w_2)[1+d(w_2,w_3)]}{1+d(w_2,w_1)}\\&+\varphi_8(w_2,w_1)\frac{d(w_2,w_2)[1+d(w_1,w_3]}{1+d(w_2,w_1)},\end{aligned}$$

further simplifying

$$\begin{aligned}\big|d(w_3,w_2)\big|\le\;&\varphi_1(w_2,w_1)\big|d(w_2,w_1)\big|\\&+\varphi_3(w_2,w_1)\big|[d(w_1,w_2)+d(w_2,w_3)]\big|\\&+\varphi_4(w_2,w_1)\big|d(w_2,w_3)\big|+\varphi_5(w_2,w_1)\big|d(w_1,w_2)\big|\\&+\varphi_6(w_2,w_1)\big|d(w_1,w_2)\big|+\varphi_6(w_2,w_1)\big|d(w_2,w_3)\big|.\end{aligned}$$

Applying Lemma 3.3

$$\begin{aligned}\big|d(w_3,w_2)\big|\le\;&\varphi_1(w_0,w_1)\big|d(w_2,w_1)\big|\\&+\varphi_3(w_0,w_1)\big|[d(w_1,w_2)+d(w_2,w_3)]\big|\\&+\varphi_4(w_0,w_1)\big|d(w_2,w_3)\big|+\varphi_5(w_0,w_1)\big|d(w_1,w_2)\big|\\&+\varphi_6(w_0,w_1)\big|d(w_1,w_2)\big|+\varphi_6(w_0,w_1)\big|d(w_2,w_3)\big|.\end{aligned}$$

Finally we get

$$|d(w_3,w_2)|\preceq\Omega|d(w_2,w_1)|,$$

where

$$\Omega=\frac{\varphi_1(w_0,w_1)+\varphi_3(w_0,w_1)+\varphi_5(w_0,w_1)+\varphi_6(w_0,w_1)}{1-(\varphi_3(w_0,w_1)+\varphi_4(w_0,w_1)+\varphi_6(w_0,w_1))}<1$$

Inductively we can construct a sequence $\{w_q\}$ in $\mathbb{X}$ such that
$w_{2q+1}\in[Kw_{2q}]_\alpha, w_{2q+2}\in[Lw_{2q+1}]_\alpha$ for $q=0,1,2\cdots$.

$$|d(w_{2q+1},w_{2q+2})|\le\Xi|d(w_{2q},w_{2q+1})|$$
$$|d(w_{2q+2},w_{2q+3})|\le\Omega|d(w_{2q+1},w_{2q+2})|.$$

The former inequality implies that

$$\begin{aligned}|d(w_{2q+1},w_{2q+2})|&\le\Xi|d(w_{2q},w_{2q+1})|\\&\le\Xi\Omega|d(w_{2q-1},w_{2q})|\\&\le\Xi\Omega\Xi|d(w_{2q-2},w_{2q-1})|\\&\le\cdots\le\Xi(\Xi\Omega)^q|d(w_0,w_1)|,\end{aligned}$$

and

$$\begin{aligned}|d(w_{2q+2},w_{2q+3})|&\le\Omega|d(w_{2q+1},w_{2q+2})|\\&\le\cdots\le(\Xi\Omega)^{q+1}|d(w_0,w_1)|.\end{aligned}$$

So if $l<\kappa$, then

$$\begin{aligned}d(w_{2l+1},w_{2\kappa+1})\preceq\;&d(w_{2l+1},w_{2l+2})+d(w_{2l+2},w_{2l+3})\\&+d(w_{2l+3},w_{2l+4})+\cdots+d(w_{2\kappa},w_{2\kappa+1}),\end{aligned}$$

which implies that

$$\begin{aligned}|d(w_{2l+1},w_{2\kappa+1})| \leq\ & |d(w_{2l+1},w_{2l+2})| \\ &+ |d(w_{2l+2},w_{2l+3})| + |d(w_{2l+3},w_{2l+4})| \\ &+ \cdots + |d(w_{2\kappa},w_{2\kappa+1})|\end{aligned}$$

$$\leq \left[\Xi \sum_{p=l}^{\kappa-1} (\Xi\Omega)^p + \sum_{p=l+1}^{\kappa} (\Xi\Omega)^p\right] |d(w_0,w_1)|.$$

Similarly, we obtain

$$d(w_{2l},w_{2\kappa+1}) \leq \left[\sum_{p=l}^{\kappa} (\Xi\Omega)^p + \Xi \sum_{p=l}^{\kappa-1} (\Xi\Omega)^p\right] |d(w_0,w_1)|,$$

$$d(w_{2l},w_{2\kappa}) \leq \left[\sum_{p=l}^{\kappa-1} (\Xi\Omega)^p + \Xi \sum_{p=l}^{\kappa-1} (\Xi\Omega)^p\right] |d(w_0,w_1)|,$$

$$d(w_{2l+1},w_{2\kappa}) \leq \left[\Xi \sum_{p=l}^{\kappa-1} (\Xi\Omega)^p + \sum_{p=l+1}^{\kappa-1} (\Xi\Omega)^p\right] |d(w_0,w_1)|.$$

Since $(\Omega\Xi) < 1$, therefore $\{w_q\}$ is a Cauchy sequence in $\mathbb{X}$. Since $\mathbb{X}$ is complete so there exists $\nu \in \mathbb{X}$ such that $w_q \to \nu$ as $q \to \infty$. Now we have to show that $\nu \in [K\nu]_\alpha$ and $\nu \in [L\nu]_\alpha$, we get

$$\begin{aligned}&\varphi_1(w_{2q},\nu)d(w_{2q},\nu) + \varphi_2(w_{2q},\nu)d(w_{2q},[L\nu]_\alpha) \\ &+ \varphi_3(w_{2q},\nu)d(\nu,[Kw_{2q}]_\alpha) + \varphi_4(w_{2q},\nu)d(w_{2q},[Kw_{2q}]_\alpha) \\ &+ \varphi_5(w_{2q},\nu)d(\nu,[L\nu]_\alpha) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(\nu,[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &\in s\big([Kw_{2q}]_\alpha,[L\nu]_\alpha\big).\end{aligned}$$

By Lemma 2.5 (iii), we have

$$\begin{aligned}&\varphi_1(w_{2q},\nu)d(w_{2q},\nu) + \varphi_2(w_{2q},\nu)d(w_{2q},[L\nu]_\alpha) \\ &+ \varphi_3(w_{2q},\nu)d(\nu,[Kw_{2q}]_\alpha) + \varphi_4(w_{2q},\nu)d(w_{2q},[Kw_{2q}]_\alpha) \\ &+ \varphi_5(w_{2q},\nu)d(\nu,[L\nu]_\alpha) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(\nu,[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &\in s\big(w_{2q+1},[L\nu]_\alpha\big).\end{aligned}$$

By definition thee exists some $y_q \in [L\nu]_\alpha$, such that

$$\begin{aligned}&\varphi_1(w_{2q},\nu)d(w_{2q},\nu) + \varphi_2(w_{2q},\nu)d(w_{2q},[L\nu]_\alpha) \\ &+ \varphi_3(w_{2q},\nu)d(\nu,[Kw_{2q}]_\alpha) + \varphi_4(w_{2q},\nu)d(w_{2q},[Kw_{2q}]_\alpha) \\ &+ \varphi_5(w_{2q},\nu)d(\nu,[L\nu]_\alpha) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(\nu,[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &\in s\big(d(w_{2q+1},y_q)\big),\end{aligned}$$

that is,

$$\begin{aligned}&d(w_{2q+1},y_q) \preceq \varphi_1(w_{2q},\nu)d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q},\nu)d(w_{2q},[L\nu]_\alpha) + \varphi_3(w_{2q},\nu)d(\nu,[Kw_{2q}]_\alpha) \\ &+ \varphi_4(w_{2q},\nu)d(w_{2q},[Kw_{2q}]_\alpha) + \varphi_5(w_{2q},\nu)d(\nu,[L\nu]_\alpha) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(w_{2q},[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},[L\nu]_\alpha)\left[1+d(\nu,[Kw_{2q}]_\alpha)\right]}{1+d(w_{2q},\nu)}.\end{aligned}$$

By using the glb property of K and L, we find

$$\begin{aligned}d(w_{2q+1},y_q) \preceq\ & \varphi_1(w_{2q},\nu)d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q},\nu)d(w_{2q},y_q) + \varphi_3(w_{2q},\nu)d(\nu,w_{2q+1}) \\ &+ \varphi_4(w_{2q},\nu)d(w_{2q},w_{2q+1}) + \varphi_5(w_{2q},\nu)d(\nu,y_q) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,y_q)[1+d(w_{2q},w_{2q+1})]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},y_q)[1+d(w_{2q},w_{2q+1})]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},y_q)[1+d(\nu,w_{2q+1})]}{1+d(w_{2q},\nu)}.\end{aligned}$$

Utilizing the triangular inequality, we obtain

$$\begin{aligned}d(\nu,y_q) \preceq\ & d(\nu,w_{2q+1}) + d(w_{2q+1},y_q) \\ \preceq\ & d(\nu,w_{2q+1}) + \varphi_1(w_{2q},\nu)d(w_{2q},\nu) \\ &+ \varphi_2(w_{2q},\nu)d(w_{2q},y_q) + \varphi_3(w_{2q},\nu)d(\nu,w_{2q+1}) \\ &+ \varphi_4(w_{2q},\nu)d(w_{2q},w_{2q+1}) + \varphi_5(w_{2q},\nu)d(\nu,y_q) \\ &+ \varphi_6(w_{2q},\nu)\frac{d(\nu,y_q)[1+d(w_{2q},w_{2q+1})]}{1+d(w_{2q},\nu)} \\ &+ \varphi_7(w_{2q},\nu)\frac{d(w_{2q},y_q)[1+d(w_{2q},w_{2q+1})]}{1+d(w_{2q},\nu)} \\ &+ \varphi_8(w_{2q},\nu)\frac{d(w_{2q},y_q)[1+d(\nu,w_{2q+1})]}{1+d(w_{2q},\nu)}.\end{aligned}$$

Applying Lemma 3.3, we get

$$\begin{aligned}|d(v,y_q)| \leq &|d(v,w_{2q+1})| + \varphi_1(w_0,v)|d(w_{2q},v)| \\ &+ \varphi_2(w_0,v)|d(w_{2q},y_q)| + \varphi_3(w_0,v)|d(v,w_{2q+1})| \\ &+ \varphi_4(w_0,v)|d(w_{2q},w_{2q+1})| + \varphi_5(w_0,v)|d(v,y_q)| \\ &+ \varphi_6(w_0,v)\frac{|d(v,y_q)|[1+|d(w_{2q},w_{2q+1})|]}{1+|d(w_{2q},v)|} \\ &+ \varphi_7(w_0,v)\frac{|d(w_{2q},y_q)|[1+|d(w_{2q},w_{2q+1})|]}{1+|d(w_{2q},v)|} \\ &+ \varphi_8(w_0,v)\frac{|d(w_{2q},y_q)|[1+|d(v,w_{2q+1})|]}{1+|d(w_{2q},v)|}.\end{aligned}$$

Let $q \to \infty$, the above inequality implies that

$$|d(v,y_q)| \leq \big[\varphi_5(w_0,v) + \varphi_6(w_0,v)\big]|d(v,y_q)|.$$

Since $\varphi_5(w_0,v) + \varphi_6(w_0,v) < 1$, so $|d(v,y_q)| \to 0$. We have $y_q \to v$ as $q \to \infty$. Since $[Lv]_\alpha$ is closed, so $v \in [Lv]_\alpha$. By similar process, We can obtain that $v \in [Lv]_\alpha$. Hence $v \in [Kv]_\alpha \cap [Lv]_\alpha$.

Corollary 3.10. Let $(\mathbb{X},d)$ be a complete complex-valued metric space Let $K : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be a fuzzy mapping with glb property and for $w,w \in \mathbb{X}$, related to some $\alpha \in (0,1]$ there exists $[Kw]_\alpha$ non-empty closed bounded subset of $\mathscr{X}$. If there exists mappings $\varphi_i : \mathbb{X}\times\mathbb{X} \to [0,1), i = 1,\cdots,8$ such that $\forall\ w,x \in \mathbb{X}$ the following conditions are satisfied:
$i)$ $\varphi_i(u,x) \leq \varphi_i(w,x)\ \ \forall\ u \in [Kw]_\alpha$;
$ii)$

$$\begin{aligned}&\varphi_1(w,x)d(w,x) + \varphi_2(w,x)d(w,[Kx]_\alpha) \\ &+ \varphi_3(w,x)d(x,[Kw]_\alpha) + \varphi_4(w,x)d(w,[Kw]_\alpha) \\ &+ \varphi_5(w,x)d(x,[Kx]_\alpha) + \varphi_6(w,x)\frac{d(x,[Kx]_\alpha)\left[1+d(w,[Kw]_\alpha)\right]}{1+d(w,x)} \\ &+ \varphi_7(w,x)\frac{d(w,[Kx]_\alpha)\left[1+d(w,[Kw]_\alpha)\right]}{1+d(w,x)} \\ &+ \varphi_8(w,x)\frac{d(w,[Kx]_\alpha)\left[1+d(x,[Kw]_\alpha)\right]}{1+d(w,x)} \\ &\in s\big([Kw]_\alpha,[Kx]_\alpha\big),\end{aligned}$$

where $\varphi_1(w,x) + \sum_{i=3}^{6}\varphi_i(w,x) + 2[\varphi_2(w,x) + \varphi_7(w,x) + \varphi_8(w,x)] < 1$. Then there exist some $v \in \mathbb{X}$ with $v \in [Kw]_\alpha$

Proof. By setting $K = L$ in Theorem 3.9, it can be easily proved.

4 Application

In this section we discussed applications of our derived results to multivalued mappings and demonstrate an appropriate example to show the validity of our main results.

Theorem 4.1. Let $(\mathbb{X},d)$ be a complete complex-valued metric space and $K,L : \mathbb{X} \to \mathfrak{CB}(\mathbb{X})$ be multivalued mappings with glb property such that for each $w,x \in \mathbb{X}$
$i)$ $\varphi_i(u) \preceq \varphi_i(w), i = 1\cdots 5\ \ \forall\ u \in Kw,\ \ \forall\ w \in \mathbb{X}$;
$ii)$ $\varphi_i(v) \preceq \varphi_i(w), i = 1\cdots 5\ \ \forall\ v \in Lw,\ \ \forall\ w \in \mathbb{X}$;
$iii)$ $\sum_{i=1}^{5}\varphi(w) < 1$;
$iv)$

$$\begin{aligned}&\varphi_1(w)d(w,x) + \varphi_2(w)\frac{d(w,Kw)d(x,Lx)}{1+d(w,x)} \\ &+ \varphi_3(w)\frac{d(w,Kw)d(x,Lx) + \varphi_4(w)d(x,Kw)d(w,Lx)}{1+d(w,x)} \\ &+ \varphi_5(w)\frac{d(w,Kw)}{1+d(w,x)} \in s(Kw,Lx).\end{aligned}$$

Then there exists some $z \in \mathbb{X}$ such that $z \in Kz \cap Lz$.

Proof. Let $S,T : \mathbb{X} \to \mathcal{F}(\mathbb{X})$ be fuzzy mappings defined by

$$Sw = \begin{cases}\alpha\ \ if\ \ w \in Kw \\ 0\ \ if\ \ w \notin Kw.\end{cases}$$

$$Tw = \begin{cases}\alpha\ \ if\ \ w \in Lw \\ 0\ \ if\ \ w \notin Lw.\end{cases}$$

So for arbitrary $\alpha \in (0,1], [Sw]_\alpha = Kw$ and $[Tw]_\alpha = Lw$. Since for every $w,x \in \mathbb{X}$, $s([Sw]_\alpha,[Tx]_\alpha) = s(Kw,Lx)$,
therefore, Theorem 3.5 can be applied to obtain a some $z \in \mathbb{X}$ such that $z \in K(z) \cap L(z)$.

Corollary 4.2. Let $(\mathbb{X},d)$ be a complete complex-valued metric space and $K,L : \mathbb{X} \to \mathfrak{CB}(\mathbb{X})$ be multivalued mapping with glb property such that for each $w,x \in \mathbb{X}$

$$\begin{aligned}&\mu d(w,x) + \gamma\frac{d(w,Kw)d(x,Lx)}{1+d(w,x)} \\ &+ \frac{\lambda d(w,Kw)d(x,Lx) + \varsigma d(x,Kw)d(\omega,Lx)}{1+d(w,x)} \\ &+ \delta\frac{d(w,Kw)}{1+d(w,x)} \in s(Kw,Lx),\end{aligned}$$

where $\mu,\gamma,\lambda,\delta,\varsigma$ are non-negative reals, with $\mu+\gamma+\lambda+\delta+\varsigma < 1$. Then there exists some $z \in \mathbb{X}$ such that $z \in Kz \cap Lz$.

Proof It can be easily proved by applying Theorem 4.1 by setting
$\varphi_1(w) = \mu, \varphi_1(w) = \gamma, \varphi_3(w) = \lambda, \varphi_4(w) = \varsigma, \varphi_5(w) = \delta$

Remark 4.3. 1) By setting $\lambda = \delta = 0$ in Corollary 3.6, we get theorem 12 of [12].
2) By setting $\gamma = \delta = 0$ in Corollary 4.2 above corollary we get theorem 12 of [9].
3) By setting $\lambda = \delta = \varsigma = 0$ in Corollary 4.2 we get Theorem10 of [1].

Theorem 4.4. Let $(\mathbb{X},d)$ be a complete complex-valued metric space and $K : \mathbb{X} \to \mathfrak{CB}(\mathbb{X})$ be multivalued mapping with glb property such that for each $w,x \in \mathbb{X}$
$i)$ $\varphi_i(u) \preceq \varphi_i(w), i = 1\cdots 5\ \ \forall\ u \in Kw,\ \ \forall\ w \in \mathbb{X}$;
$ii)$ $\varphi_1(w) + \varphi_2(w) + \varphi_3(w) + \varphi_4(w) + \varphi_5(w) < 1$;

iv)

$$\varphi_1(w)d(w,x)+\varphi_2(w)\frac{d(w,Kw)d(x,Kx)}{1+d(w,x)}$$
$$+\ \frac{\varphi_3(w)d(w,Kw)d(x,Kx)+\varphi_4(w)d(x,Kw)d(w,Kx)}{1+d(w,x)}$$
$$+\ \varphi_5(w)\frac{d\big(w,Kw\big)}{1+d(w,x)}\in s(Kw,Kx).$$

Then there exists some $z\in\mathbb{X}$ such that $z\in Kz$.
Proof Let $T:\mathbb{X}\to\mathcal{F}(\mathbb{X})$ be fuzzy mapping defined by

$$Tw=\begin{cases}\alpha \ \ if\ w\in Kw\\ 0\ \ if\ w\notin Kw.\end{cases}$$

Then for arbitrary $\alpha\in(0,1]$ we have $[Tw]_\alpha=Kw$. Since for every $\omega,x\in\mathbb{X}$, $s([Tw]_\alpha,[Tx]_\alpha)=s(Kw,Kx)$, therefore by applying Theorem 3 we get some point $z\in\mathbb{X}$ such that $z\in K(z)$.
Corollary 4.5 Let $(\mathbb{X},d)$ be a complete complex-valued metric space and $K:\mathbb{X}\to\mathfrak{CB}(\mathbb{X})$ be multivalued mappings with glb property such that for each $w,x\in\mathbb{X}$

$$\mu d(w,x)+\gamma\frac{d(w,Kw)d(x,Kx)}{1+d(w,x)}$$
$$+\ \frac{\lambda d(w,Kw)d(x,Kx)+\varsigma d(x,Kw)d(w,Kx)}{1+d(w,x)}$$
$$+\ \delta\frac{d\big(w,Kw\big)}{1+d(w,x)}\in s(K,Kx),$$

where $\mu,\gamma,\lambda,\delta,\varsigma$ are non-negative reals, with $\mu+\gamma+\lambda+\delta+\varsigma<1$; Then there exists some $z\in\mathbb{X}$ such that $z\in Kz$.
Proof It can be easily proved by applying Theorem 4 by setting
$\varphi_1(w)=\mu,\varphi_2(w)=\gamma,\varphi_3(w)=\lambda,\varphi_4(w)=\varsigma,\varphi_5(w)=\delta$
Theorem 4.6 Let $(\mathbb{X},d)$ be a complete complex-valued metric space. Let $K,L:\mathbb{X}\to\mathfrak{CB}(\mathbb{X})$ be multivalued mappings with glb property and for $w,x\in\mathbb{X}$. If there exists mappings $\varphi_i:\mathbb{X}\times\mathbb{X}\to[0,1), i=1,\cdots,8$ such that for all $w,x\in\mathbb{X}$ the following conditions are satisfied:
i) $\varphi_i(u,x)\le\varphi_i(w,x)\quad\forall u\in Kw$ and $\varphi_i(w,v)\le\varphi_i(w,x)\ \ \forall v\in Lx$;
ii)

$$\varphi_1(w,x)d(w,x)+\varphi_2(w,x)d(w,Lx)+\varphi_3(w,x)d(x,Kw)$$
$$+\ \varphi_4(w,x)d(w,Kw)+\varphi_5(w,x)d(x,Lx)$$
$$+\ \varphi_6(w,x)\frac{d(x,Lx)\big[1+d(w,Kw)\big]}{1+d(w,x)}$$
$$+\ \varphi_7(w,x)\frac{d(w,Lx)\big[1+d(w,Kw)\big]}{1+d(w,x)}$$
$$+\ \varphi_8(w,x)\frac{d(w,Lx)\big[1+d(x,Kw)\big]}{1+d(w,x)}\in s(Kw,Lx)$$

where $\varphi_1(w,x)+\sum_{i=3}^{6}\varphi_i(w,x)+2[\varphi_2(w,x)+\varphi_7(w,x)+\varphi_8(w,x)]<1$. Then there exists unique $z\in\mathbb{X}$ such that $z\in Kz\cap Lz$.
Proof. Assume that the fuzzy mappings $S,T:\mathbb{X}\to\mathcal{F}(\mathbb{X})$ are defined as

$$Sw=\begin{cases}\alpha\ \ if\ w\in Kw\\ 0\ \ if\ w\notin Kw.\end{cases}$$

$$Tw=\begin{cases}\alpha\ \ if\ w\in Lw\\ 0\ \ if\ w\notin Lw.\end{cases}$$

Then for any $\alpha\in(0,1],[Sw]_\alpha=Kw$ and $[Tw]_\alpha=Lw$. Since for every $w,x\in\mathbb{X}$, $s([Sw]_\alpha,[Tx]_\alpha)=s(Kw,Lx)$ therefore, by applying Theorem 3 we obtain some $z\in\mathbb{X}$ such that $z\in K(z)\cap L(z)$.
Example 4.7. Let $\mathbb{X}=[0,1]$ and $d:\mathbb{X}\times\mathbb{X}\to\mathfrak{C}$ be complex-valued metric space defined by
$d(w,x)=|w-x|e^{i\frac{\pi}{6}}$, for all $w,x\in\mathbb{X}$.
Let $\alpha\in(0,1]$ and $K,L:\mathbb{X}\to\mathcal{F}(\mathbb{X})$ are fuzzy mappings defined as follows:

$$K(0)(a)=\begin{cases}1\ \ if\ a=0\\ \frac{1}{4}\ \ if\ 0<a\le\frac{w}{190}\\ 0\ \ if\ \frac{w}{190}<a\le1,\end{cases}$$

$$L(0)(a)=\begin{cases}1\ \ if\ a=0\\ \frac{1}{2}\ \ if\ 0<a\le\frac{w}{100}\\ 0\ \ if\ \frac{w}{100}<a\le1,\end{cases}$$

if $w\neq0$,

$$K(w)(a)=\begin{cases}\alpha\ \ if\ 0\le a\le\frac{w}{60}\\ \frac{\alpha}{3}\ \ if\ \frac{w}{60}<a\le\frac{w}{40}\\ \frac{\alpha}{7}\ \ if\ \frac{w}{40}<a\le\frac{w}{20}\\ \frac{\alpha}{4}\ \ if\ \frac{w}{20}<a\le1,\end{cases}$$

$$L(w)(a)=\begin{cases}\alpha\ \ if\ 0\le a\le\frac{w}{20}\\ \frac{\alpha}{2}\ \ if\ \frac{w}{20}<a\le\frac{w}{10}\\ \frac{\alpha}{3}\ \ if\ \frac{w}{10}<a\le\frac{w}{5}\\ \frac{\alpha}{5}\ \ if\ \frac{w}{5}<a\le1.\end{cases}$$

Let $\varphi_i:\mathbb{X}\to[0,1)\ i=1,\cdots,5$ be defined by
$\varphi_1(w)=\frac{w+1}{60},\varphi_2(w)=\frac{w}{10},\varphi_3(w)=\frac{w}{20},\varphi_5(w)=\frac{w}{30},\varphi_4(w)=\frac{w}{60}$.
Then for $w=0,[K0]_1=[L0]_1=\{0\}$ and $\forall w,x\neq0$ $[Kw]_\alpha=[0,\frac{w}{60}]$ and $[Lw]_\alpha=[0,\frac{w}{20}]$.
Clearly $\varphi_i(u)\le\varphi_i(w), i=1,\cdots,5,$ for all $u\in[Kw]_\alpha, x\in[0,1)$ and

$\varphi_i(v) \leq \varphi_i(w), i = 1, \cdots, 5$, for all $v \in [Lw]_\alpha, x \in [0,1)$.
It can be easily seen that for $u = w/60 \in [Kw]_\alpha$,

$$\begin{aligned}\varphi_1(\frac{(w/60)+1}{60}) &= \varphi_1(\frac{w+60}{3600}) \\ &= \varphi_1(w/3600 + 60/3600) \\ &\leq \varphi_1(w/60 + 1/60) \\ &= \varphi_1(\frac{w+1}{60}) = \varphi_1(w)\end{aligned}$$

$$d\big(x,[Kw]_\alpha\big) = \begin{cases} 0 \quad if\ x \leq \frac{w}{60} \\ (x - \frac{w}{60})e^{i\frac{\pi}{6}} \quad if\ x > \frac{w}{60}, \end{cases}$$

and

$$d\big(w,[Lx]_\alpha\big) = \begin{cases} 0 \quad if\ w \leq \frac{x}{20} \\ (w - \frac{x}{20})e^{i\frac{\pi}{6}} \quad if\ w > \frac{x}{20}, \end{cases}$$

also $d\big(w,[Kw]_\alpha\big) = (\frac{59w}{60})e^{i\frac{\pi}{6}}$,
and $d\big(x,[Lx]_\alpha\big) = (\frac{19x}{20})e^{i\frac{\pi}{6}}$.
Moreover if $\mathfrak{P}_{xw} \in \mathscr{C}$ such that

$$\mathfrak{P}_{wx} = \left|\frac{w}{60} - \frac{x}{20}\right|\sqrt{2}e^{i\frac{\pi}{6}},$$

then

$$s\big([Kw]_\alpha,[Lx]_\alpha\big) = \{\mathfrak{P} \in \mathscr{C} : \mathfrak{P}_{wx} \preceq \mathfrak{P}\}.$$

Consider

$$\begin{aligned}&\varphi_1(w)|d(w,x)| + \varphi_2(w)\frac{|d(w,[Kw]_\alpha)||d(x,[Lx]_\alpha)|}{1+|d(w,x)|} \\ &\quad + \varphi_3(w)\frac{|d(w,[Kw]_\alpha)||d(x,[Lx]_\alpha)|}{1+d(w,x)} + \\ &\varphi_4(w)\frac{|d(x,[Kw]_\alpha)||d(w,[Lx]_\alpha)|}{1+|d(w,x)|} + \varphi_5(w)\frac{|d\big(w,[Kw]\big)_\alpha|}{1+|d(w,x)|}.\end{aligned}$$

Clearly for $\varphi_1(w) = \frac{w+1}{60}, \varphi_2(w) = \frac{w}{10}, \varphi_3(w) = \frac{w}{20}, \varphi_5(w) = \frac{w}{30}, \varphi_4(w) = \frac{w}{60}$.

$$\begin{aligned}&|\frac{w}{60} - \frac{x}{20}| \leq \frac{w+1}{60}|w-x| + \frac{w}{10}\frac{|\frac{59w}{60}||\frac{19x}{20}|}{1+|w-x|} \\ &+ \frac{\frac{w}{20}|\frac{59w}{60}||\frac{19x}{20}| + \frac{w}{60}|x - \frac{w}{60}||w - \frac{x}{20}|}{1+|w-x|} + \frac{w}{30}\frac{|\frac{59w}{60}|}{1+|w-x|}. \end{aligned} \tag{4}$$

Since, one can easily calculate that,

$$\begin{aligned}\varphi_1(w)d(w,x) &= \frac{w+1}{60}|w-x| \\ &= |\frac{(w+1)w}{60} - \frac{(w+1)x}{60}| \\ &= |\frac{w^2}{60} + \frac{w}{60} - \frac{x}{60} - \frac{wx}{60}| \\ &= |\frac{w}{60} - \frac{x}{60} + \frac{w^2}{60} - \frac{wx}{60}| \\ &\succeq |\frac{w}{60} - \frac{x}{60} - \frac{wx}{60}| \\ &= |\frac{w}{60} - \frac{x(1+w)}{60}| \\ &= |\frac{w}{60} - \frac{x}{60}| \succeq |\frac{w}{60} - \frac{x}{20}| = \mathfrak{P}_{wx}.\end{aligned}$$

All the remaining terms on the right hand side of (4) are non-negative reals for all $x \in \mathbb{X}$. Consequently one can obtain

$$\begin{aligned}&\varphi_1(w)d(w,x) + \varphi_2(w)\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} \\ &+ \varphi_3(w)\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} + \varphi_4(w)\frac{d(x,[Kw]_\alpha)d(w,[Lx]_\alpha)}{1+d(w,x)} \\ &+ \varphi_5(w)\frac{d\big(w,[Kw]\big)_\alpha}{1+d(w,x)+d(x,[Lx]_\alpha)} \succeq \mathfrak{P}_{wx},\end{aligned}$$

therefore

$$\begin{aligned}&\varphi_1(w)d(w,x) + \varphi_2(w)\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} \\ &+ \varphi_3(w)\frac{d(w,[Kw]_\alpha)d(x,[Lx]_\alpha)}{1+d(w,x)} \\ &+ \varphi_4(w)\frac{d(x,[Kw]_\alpha)d(w,[Lx]_\alpha)}{1+d(w,x)} \\ &+ \varphi_5(w)\frac{d\big(w,[Kw]_\alpha\big)}{1+d(w,x)+d(x,[Lx]_\alpha} \in s([Kw]_\alpha,[Lx])_\alpha.\end{aligned}$$

Hence, all conditions of Theorem 3.5 are satisfied and $w = 0$ remains fixed under the mappings K and L.

5 Conclusion

With the help of classical result due to W. Shatanwi *et al.*, sequence and distance between closed bounded sets due to jamshed *et al.*, we have established adequate results for the existence of fixed point for fuzzy mappings in the setup of complex-valued metric space. Further the criteria has been extended for multivalued maps for the considered results. The established theoretical results have been demonstrated by apropriate example. Hence a conclusion, we state that aforementioned results play a basic role to the existence of fixed point for fuzzy mappings as well as multivalued mappings.

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References

[1] S. Nadarajah and S. Eljabri, The Kumaraswamy Generalized Pareto Distribution, Journal of Data Sci, Vol. 11, No. 2, pp. 739-766 (2013).

[2] R. Alfred, On measures of information and entropy, Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability, Vol. 1960, pp. 47-561 (1961) .

[3] T. Shams, The Kumaraswamy-Generalized Lomax Distribution, Middle-East J Sci Res, Vol. 17, No. 7, pp. 641-646 (2013).

[4] S. C. Arora and V. Sharma, Fixed point theorems for fuzzy mappings. *Fuzzy Sets and Systems*, **110**(1), (2000), 127-130.

[5] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces. *Numerical Functional Analysis and Optimization*, **32**(3), (2011), 243-253.

[6] A. Azam, M. Arshad and I. Beg, Banach contraction principle on cone rectangular metric spaces. *Applicable Analysis and Discrete Mathematics*, **3**(2), (2009), 236-241.

[7] A. Azam and I. Beg, Common fuzzy fixed points for fuzzy mappings. *Fixed Point Theory and Applications.*, **1** (2013), p 14.

[8] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intgrales. *Fundamenta Mathematicae*, **3**(1), (1922), 133-181.

[9] L. B. Ćirić, A generalization of Banachs contraction principle. *Proceedings of American Mathematical Society*, **45**(2), (1974), 267-273.

[10] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expression, *indian Journal of Pure and Applied Mathematics*, **6**, (1975), 1455-1458.

[11] V. D. Estruch and A. Vidal, A note on fixed fuzzy points for fuzzy mappings, *Rendiconti dell'Istituto di Mathematica dell' Universiyiá di Trieste*, **32**, (2001), 39-45.

[12] V. Gregori and J. Pastor, A fixed point theorem for fuzzy contraction mappings, *Rendiconti dell'Istituto di Mathematica dell' Universiyiá di Trieste*, **XXX**, (1999), 103-109.

[13] S. Heilpern, Fuzzy mappings and fixed point theorem. *Journal of Mathematical Analysis and Applications.*, **83**(2), (1981), 566-569.

[14] V. Joshi, N. Singh and D. Singh, ϕ-contractive multivalued mappings in complex valued metric spaces and remarks on some recent papers, *Cogent Mathematics*, **3**, 1162484, (2016).

[15] R. P. Agarwal, E. Karapinar, D. O'Regan and A.F. Roldán-López-de-Hierro, Further Fixed Point Results on G-Metric Spaces. In *Fixed Point Theory in Metric Type Spaces*, Springer Cham. (2015), 107-173.

[16] C. Klin-eam and C. Suanoom, Some common fixed point theorems for generalized contractive type mappings on complex valued metric spaces, *Abstract and Applied Analysis*, **6**, Article ID 604215, (2013).

[17] W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications. *Journal of Inequalities and Applications*, **84**(1), (2012).

[18] P. Kumam, M. Sarwar and M.B. Zada, Fixed Point Results Satisfying Rational Type Contractive Conditions in Complex Valued Metric Spaces. *Annales Mathematicae Silesianae* **30**(1), (2016), 89-110.

[19] M. A. Kutbi, J. Ahmad, A. Azam and N. Hussain, (2014) On fuzzy fixed points for fuzzy maps with generalized weak property, *Journal of Applied Mathematics*, **2014**, (2014).

[20] R. K. Bose, D. Sahani, Fuzzy mapping and fixed point theorems.*Fuzzy sets and systems*, **21**(1), (1987), 53-58.

[21] W. Shatanawi, V. C. Rajić, S. Radenović and Al-Rawashdeh, A. Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces. *Fixed Point Theory and Applications*, **106**(1), (2012).

[22] M. D. Weiss, Fixed points separation and induced topologies for fuzzy sets. *Journal of Mathematical Analysis and Applications*, **50**(1), (1975), 142-150.

[23] D. Turkoglu and B. E. Rhoades, A fixed fuzzy point for fuzzy mapping in complete metric spaces. *Mathematical Communications*, **10**(2), (2005), 115-121.

[24] Y. Tian and Z. Bai, Existence results for the three-point impulsive boundary value problem involving fractional differential equations. *Computers and Mathematics with Applications*, **59**(8), (2010), 2601-2609.

[25] L. Zhang and G. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations*, **7**, (2011), 1-11.

[26] Zadeh, L. A. Fuzzy sets. *Information and control*, **8**(3),)1965), 338-353.

[27] Humaira, M. Sarwar and G. N. V Kishore, Fuzzy fixed point results for Φ contractive mapping with applications *Complexity* **2018**, (2018).

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