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Comparison of Two Kinds of Modified Prediction-Correction Methods for Pseudomonotone Variational Inequalities

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Abstract: In this paper, we propose the modified prediction-correction methods for solving pseudomonotone variational inequalities problems. The proposed methods can be viewed as an extension of the method of He et al. [11, B.S. He and X.M. Yuan, Comparison of two kinds of prediction–correction methods for monotone variational inequalities, Computational Optimization and Applications, Vol. 27, pp. 247–267 (2004).] by additional projection step at each iteration under the relaxed condition where the mapping is pseudomonotone. The convergence of the proposed method is proved. The numerical results are given to verify the efficiency of the modified methods.

Keywords: extra-gradient method, forward-backward splitting method, pseudomonotone, variational inequality

1 Introduction

In the past decades, researchers have developed a variety of efficient algorithms for solving variational inequality problem(VI): Find a vector $x^* \in \Omega$ such that

$$(x-x^*)^T F(x^*) \geq 0,\ \forall x \in \Omega, \tag{1.1}$$

where Ω is assumed to be a nonempty closed convex subset of $\mathbb{R}^n$ and F is assumed to be a mapping from $\mathbb{R}^n$ into itself. The notation VI(F,Ω) denotes the solution set of (1.1).

The projection method, proposed by Goldstein [6], and Levitin and Polyak [14], is one of the most basic solution algorithms for solving VI problems due to its simplicity. This method, beginning with any starting point, generates a new point via following formula:

$$x^{k+1} = P_\Omega[x^k - \beta_k F(x^k)],$$

where $P_\Omega(\cdot)$ is the projection from $\mathbb{R}^n$ onto Ω. The convergence of the projection method can be guaranteed under a strong condition that the mapping F is strongly monotone and Lipschitz continuous. In fact, it may be very expensive to estimate the strongly monotone modulus m. These strict conditions make it difficult for the applications of VI problems in some cases. To overcome this limitation, Korpelevich [13] proposed the extra-gradient method, which dispenses with the strong monotonicity. It updates the iterations by the following recursions:

$$\bar{x} = P_\Omega[x^k - \beta_k F(x^k)],$$

$$x^{k+1} = P_\Omega[x^k - \beta_k F(\bar{x})].$$

Under the assumptions that the mapping F is monotone and Lipschitz continuous, and $0 < \beta_k < 1/L$, the sequence $\{x^k\}$ converges to a solution of VI problems.

Recently, some researchers have proposed numerous modified projection and extra-gradient-type methods [1, 7, 9, 10, 11, 15, 16] for solving VI problems. Most of these methods were designed to improve the efficiency by changing the step size parameters based on some appropriate principles and removing the Lipschitz constant with line search technique. The prediction-correction method developed by He et al. [11] adopted the following iterative scheme:

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For given $x^k \in \mathbb{R}^n$ and $\beta_k > 0$, the new point is obtained by

$$x^{k+1} = P_\Omega[x^k - \gamma\alpha_k g(x^k, \beta_k)], 0 < \gamma < 2$$

where

$$e(x^k, \beta_k) = x^k - P_\Omega[x^k - \beta_k F(x^k)],$$
$$g(x^k, \beta_k) = e(x^k, \beta_k) - \beta_k[F(x^k) - F(x^k - e(x^k, \beta_k))],$$
$$\alpha_k = \frac{e(x^k, \beta_k)^T g(x^k, \beta_k)}{\|g(x^k, \beta_k)\|^2}.$$

The convergence of this method can be guaranteed under the conditions that F is monotone and β_k suitably chosen.

On the other hand, Bnouhachem et al. [2] (see also [4, 5]) introduced a new kind of extra-gradient method by additional projection step at each iteration and the convergence of this method can be guaranteed by the monotonicity of F.

Inspired by the above mentioned references, we will introduce a modified general forward-backward splitting method and a modified general extra-gradient splitting method. We also prove the convergent theorem for solving pseudomonotone variational inequalities.

Finally, some numerical results are presented to show the efficiency of the new method.

2 Preliminaries

In this section, some definitions and lemmas from the literature are presented which are used throughout the paper. For convenience, we consider the projection under the Euclidean norm.

Definition 21*Let $F : \Omega \to \mathbb{R}^n$ and $\alpha > 0$. Then F is called*

(i) α-strongly monotone if

$$(x-y)^T(F(x) - F(y)) \geq \alpha\|x-y\|^2, \ \forall x, y \in \Omega.$$

(i) monotone if

$$(x-y)^T(F(x) - F(y)) \geq 0, \ \forall x, y \in \Omega.$$

(i) pseudomonotone if

$$(x-y)^T F(y) \geq 0 \ \Rightarrow \ (x-y)^T F(x) \geq 0, \ \forall x, y \in \Omega.$$

Remark 22*It is easy to check that a α-strongly monotone mapping is monotone and a monotone mapping is pseudomonotone.*

Example 23*Let K be a nonempty closed convex subset of $\mathbb{R}$ and $F : K \to \mathbb{R}$ be a mapping.*

(1) If we take $F(x) = 1 - x$ and $K = [0, 1]$, then, it is easy to check that the mapping F is a pseudomonotone mapping, neither a monotone mapping nor a strongly monotone mapping.

(2) If a mapping F is defined by $F(x) = c$, where c is a constant and $K = \mathbb{R}$. We observe that the mapping F is monotone, but not strongly monotone mapping.

Let C be a subset of $\mathbb{R}^n$. If C is nonempty, the distance from a point $x \in \mathbb{R}^n$ to C is $d_C(x) = \inf_{y \in C} \|x - y\|$; if C is also closed and convex, then for every $x \in \mathbb{R}^n$, there exists a unique point $P_C(x) \in C$ such that $\|x - P_C(x)\| = d_C(x)$. The point $P_C(x)$ is the metric projection of x onto C.

Lemma 24*[19] Let Ω be a closed convex set in $\mathbb{R}^n$, then the following statements hold:*

(1) $(y - P_\Omega(y))^T(x - P_\Omega(y)) \leq 0, \forall y \in \mathbb{R}^n$ and $\forall x \in \Omega$,

(2) $\|P_\Omega(y) - x\|^2 \leq \|y - x\|^2 - \|y - P_\Omega(y)\|^2, \forall y \in \mathbb{R}^n$ and $\forall x \in \Omega$,

(3) $\|P_\Omega(y) - P_\Omega(x)\|^2 \leq (y-x)^T(P_\Omega(y) - P_\Omega(x))$, $\forall y, x \in \mathbb{R}^n$.

Lemma 25*[17] Let Ω be a closed convex set in $\mathbb{R}^n$. Then x^* is a solution of $\mathrm{VI}(F, \Omega)$ if and only if*

$$x^* = P_\Omega[x^* - \beta F(x^*)], \ \forall \beta > 0. \tag{2.1}$$

From Lemma 25, it is clear that solving $\mathrm{VI}(F, \Omega)$ is equivalent to finding a zero point of the residue function

$$e(x, \beta) := x - P_\Omega[x - \beta F(x)], \ \forall \beta > 0. \tag{2.2}$$

Generally, the term $\|e(x, 1)\|$ is referred to as the error bound of $\mathrm{VI}(F, \Omega)$, since it measures the distance of x from the solution set.

Lemma 26*[3] For any $x \in \mathbb{R}^n$ and $\tilde{\beta} \geq \beta > 0$, we have*

$$\|e(x, \beta)\| \leq \|e(x, \tilde{\beta})\|, \tag{2.3}$$

and

$$\frac{\|e(x, \beta)\|}{\beta} \geq \frac{\|e(x, \tilde{\beta})\|}{\tilde{\beta}}. \tag{2.4}$$

3 Main Results

In this section, we describe the proposed methods. The proposed methods generate two predictors and evaluate F three times per iteration. We incorporate the algorithm with an Armijo-like line search similar to implicit method and improved prediction-correction method of [8] and [9], respectively in which β_k should satisfy two criteria. We also choose β_k with the same way in He et al. [11] to make it a good starting step size for the next iteration. And then investigate the strategy of how to choose the step size α_k.

Remark 31*[11] The sequence $\{\beta_k\}$ is monotonically nonincreasing. However, this may cause a slow convergence if*

$$r_k := \frac{\beta_k\|(F(\tilde{x}_1^k) - F(\tilde{x}_2^k))\|}{\|\tilde{x}_1^k - \tilde{x}_2^k\|}$$

is too small. In order to solve this problem, enlarging the step size β for the next iteration is necessary. Therefore, in $k+1^{th}$ iteration, we take

$$\beta_{k+1} = \begin{cases} 2\beta_k/m_2, & \text{if } 2r_k \leq m_2; \\ \beta_k, & \text{otherwise.} \end{cases}$$

where $m_2 \in (0, \sqrt{2})$ is a constant.

Algorithm3.1.
Step 1: Let $x^0 \in \Omega,\ \varepsilon > 0,\ \beta_0 = 1,\ m_1 \in (0,1),\ m_2 \in (0,\sqrt{2}),\ \gamma \in (0,2)$ and $k = 0$.
Step 2 : If $\|e(x^k,1)\| \le \varepsilon$, then stop. Otherwise, go to Step 3.
Step 3 : (1) For a given $x^k \in \Omega$, calculate

$$\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)],$$

$$\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)].$$

(2) If β_k satisfies both

$$r_k := \frac{\|\beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k))\|}{\|\bar{x}_1^k - \bar{x}_2^k\|} \le m_1, \tag{3.1}$$

and

$$\|(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) - (x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))\| \le \frac{m_2^2\|\bar{x}_1^k - \bar{x}_2^k\|^2}{\beta_k} \tag{3.2}$$

then go to Step 4; otherwise, continue.
(3) Perform an Armijo-like line search via reducing β_k

$$\beta_k := 0.75 * \beta_k * min\{1, \frac{m_1}{r_k}\},$$

and go to Step 3.
Step 4 : Take the new iteration x^{k+1} by setting
(Modified general forward-backward splitting method)

$$x_{FB}^{k+1} = P_\Omega[x^k - \gamma\alpha_k g(\bar{x}_1^k, \bar{x}_2^k)], \tag{3.3}$$

and (Modified general extra-gradient splitting method)

$$x_{EG}^{k+1} = P_\Omega[x^k - \gamma\alpha_k\beta_k F(\bar{x}_2^k)], \tag{3.4}$$

where

$$0 < \gamma < 2,\ \alpha_k = \frac{(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k)}{\|g(\bar{x}_1^k, \bar{x}_2^k)\|},$$

$$g(\bar{x}_1^k, \bar{x}_2^k) = (\bar{x}_1^k - \bar{x}_2^k) - \beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k)).$$

Step 5: Choosing a suitable β_{k+1} for the next iteration (same as [11]).

$$\beta_{k+1} = \begin{cases} 2\beta_k/m_2, & \text{if } 2r_k \le m_2; \\ \beta_k, & \text{otherwise.} \end{cases}$$

Return to Step 2, with k replaced by $k+1$.

Lemma 32*[2] In the k^{th} iteration, if $\|e(x^k,1)\| \ge \varepsilon$, then the Armijo-like line search procedure with criteria (3.1) and (3.2) is finite.*

Remark 33*It is a natural question that how to choose a suitable optimal α_k is an important issue. The Criterion (3.2) only could ensure $\alpha_k > 0$. In order to obtain a lower bound (away from zero) on α_k, we need criterion (3.1). We will discuss these issues in this section.*

Remark 34*Since* $(x - x^*)^T F(x^*) \ge 0, \forall x \in \Omega$ where $x^* \in \mathrm{VI}(F,\Omega)$, *and*

$$\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)] \in \Omega,$$

we obtain

$$(\bar{x}_2^k - x^*)^T F(x^*) \ge 0.$$

By pseudomonotonicity of F, we get that

$$\beta_k(\bar{x}_2^k - x^*)^T F(\bar{x}_2^k) \ge 0. \tag{3.5}$$

Since $\bar{x}_1^k - \beta_k F(\bar{x}_1^k) \in \mathbb{R}^n$ and $x^* \in \Omega$, *by Lemma 24(1), we deduce that*

$$(\bar{x}_2^k - x^*)^T(\bar{x}_1^k - \bar{x}_2^k - \beta_k F(\bar{x}_1^k)) \ge 0. \tag{3.6}$$

By combining (3.5) and (3.6), we derive

$$(\bar{x}_2^k - x^*)^T(\bar{x}_1^k - \bar{x}_2^k - \beta_k F(\bar{x}_1^k) + \beta_k F(\bar{x}_2^k)) \ge 0.$$

This implies that

$$(x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \ge (x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k). \tag{3.7}$$

The next two lemmas are useful tools for proving the convergence results of the two methods.

Lemma 35*(Modified general forward-backward splitting method)*
Let $\theta_{FB} := \|x^k - x^*\|^2 - \|x_{FB}^{k+1} - x^*\|^2$. Then we have

$$\begin{aligned}\theta_{FB} \ge\ & 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ & + \|x^k - x_{FB}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\ & - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2.\end{aligned} \tag{3.8}$$

*Proof.*Since $x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k) \in \mathbb{R}^n$ and $x^* \in \Omega$, by Lemma 24(2), we obtain

$$\begin{aligned}\|x_{FB}^{k+1} - x^*\|^2 \le\ & \|x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k) - x^*\|^2 \\ & - \|x^k - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k) - x_{FB}^{k+1}\|^2 \\ \le\ & \|x^k - x^*\|^2 - \|x^k - x_{FB}^{k+1}\|^2 \\ & - 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ & + 2\alpha_k(x^k - x_{FB}^{k+1})^T g(\bar{x}_1^k, \bar{x}_2^k),\end{aligned}$$

the second inequality directly follows from (3.7). By a simple manipulation we deduce

$$\begin{aligned}\theta_{FB} \ge\ & \|x^k - x_{FB}^{k+1}\|^2 + 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ & - 2\alpha_k(x^k - x_{FB}^{k+1})^T g(\bar{x}_1^k, \bar{x}_2^k) \\ =\ & 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) + \|x^k - x_{FB}^{k+1} - g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\ & - \|\alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2,\end{aligned}$$

and the lemma is proved.

Lemma 36*(Modified general extra-gradient method)*
Let $\theta_{EG} := \|x^k - x^*\|^2 - \|x_{EG}^{k+1} - x^*\|^2$. Then the following inequality holds

$$\begin{aligned}\theta_{EG} \ge\ & 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) + \|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\ & - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2.\end{aligned} \tag{3.9}$$

Proof. Since $x_{EG}^{k+1} = P_\Omega[x^k - \alpha_k\beta_k F(\bar{x}_2^k)]$ and $x^* \in \Omega$, it follows from Lemma 24(2) that

$$\|x_{EG}^{k+1} - x^*\|^2 \leq \|x^k - \alpha_k\beta_k F(\bar{x}_2^k) - x^*\|^2 - \|x^k - \alpha_k\beta_k F(\bar{x}_2^k) - x_{EG}^{k+1}\|^2,$$

and consequently we get

$$\theta_{EG} \geq \|x^k - x_{EG}^{k+1}\|^2 + 2\alpha_k(x^k - x^*)^T\beta_k F(\bar{x}_2^k) - 2\alpha_k(x^k - x_{EG}^{k+1})^T\beta_k F(\bar{x}_2^k)$$

Using $\beta_k F(\bar{x}_2^k) = g(\bar{x}_1^k, \bar{x}_2^k) - [(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)]$, we obtain that

$$\begin{aligned}\theta_{EG} &\geq 2\alpha_k(x^k - x^*)^T\beta_k F(\bar{x}_2^k) + \|x^k - x_{EG}^{k+1}\|^2 \\ &\quad -2\alpha_k(x^k - x_{EG}^{k+1})^T g(\bar{x}_1^k, \bar{x}_2^k) \\ &\quad +2\alpha_k(x^k - x_{EG}^{k+1})^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\geq 2\alpha_k(x^k - x^*)^T\beta_k F(\bar{x}_2^k) \\ &\quad +2\alpha_k(x^k - x_{EG}^{k+1})^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\quad +\|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\theta_{EG} &\geq 2\alpha_k(x^k - x^*)^T\{g(\bar{x}_1^k, \bar{x}_2^k) - [(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)]\} \\ &\quad +2\alpha_k(x^k - x_{EG}^{k+1})^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\quad +\|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\ &= 2\alpha_k(x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ &\quad -2\alpha_k(x^k - x^*)^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\quad +2\alpha_k(x^k - x_{EG}^{k+1})^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\quad +\|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \\ &= 2\alpha_k(x^k - x^*)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ &\quad +2\alpha_k[(x^k - x_{EG}^{k+1}) - (x^k - x^*)]^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \\ &\quad +\|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2 \quad (3.10)\end{aligned}$$

Setting $y := \bar{x}_1^k - \beta_k F(\bar{x}_1^k)$ and $x := x_{EG}^{k+1}$ in Lemma 24(1), we get

$$[\bar{x}_1^k - \beta_k F(\bar{x}_1^k) - P_\Omega(\bar{x}_1^k - \beta_k F(\bar{x}_1^k))]^T [P_\Omega(\bar{x}_1^k - \beta_k F(\bar{x}_1^k)) - x_{EG}^{k+1}] \geq 0$$

and therefore

$$[(x^k - x_{EG}^{k+1}) - (x^k - x^*)]^T[(\bar{x}_1^k - \bar{x}_2^k) - \beta_k F(\bar{x}_1^k)] \geq 0. \quad (3.11)$$

Substituting (3.11) into (3.10) and by (3.7), it follows that

$$\theta_{EG} \geq 2\alpha_k(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) + \|x^k - x_{EG}^{k+1} - \alpha_k g(\bar{x}_1^k, \bar{x}_2^k)\|^2 - \alpha_k^2\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2,$$

it completes the proof.

For convenience of later analysis, we use the following notations:

$$\begin{aligned}\rho_1 &= (\bar{x}_1^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ &= \|\bar{x}_1^k - \bar{x}_2^k\|^2 - \beta_k(\bar{x}_1^k - \bar{x}_2^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k)), \quad (3.12)\end{aligned}$$

and

$$\begin{aligned}\rho_2 &= (x^k - \bar{x}_1^k)^T g(\bar{x}_1^k, \bar{x}_2^k) \\ &= (x^k - \bar{x}_1^k)^T(\bar{x}_1^k - \bar{x}_2^k) \\ &\quad -\beta_k(x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k)), \quad (3.13)\end{aligned}$$

then $(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k) = \rho_1 + \rho_2$.

Now, in order to prove the fact that α_k is bounded away from zero, we need the next lemma.

Lemma 37 *Assume $x^k \in \Omega$, $\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)]$ and $\bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)]$. Then the following inequality is true*

$$\begin{aligned}\rho_2 &\geq \|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) \\ &\quad -(x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))]. \quad (3.14)\end{aligned}$$

Lemma 38 *Let*
$\bar{x}_1^k = P_\Omega[x^k - \beta_k F(x^k)], \bar{x}_2^k = P_\Omega[\bar{x}_1^k - \beta_k F(\bar{x}_1^k)]$
and

$$\alpha_k = \frac{(x^k - \bar{x}_2^k)^T g(\bar{x}_1^k, \bar{x}_2^k)}{\|g(\bar{x}_1^k, \bar{x}_2^k)\|}.$$

Then α_k is bounded away from zero.

Proof. Applying Lemma 32 and criterion (3.2), we get

$$\begin{aligned}\rho_1 + \rho_2 &\geq \|\bar{x}_1^k - \bar{x}_2^k\|^2 - \beta_k(\bar{x}_1^k - \bar{x}_2^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k)) \\ &\quad +\|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) - F(\bar{x}_1^k)) \\ &\quad -(x^k - \bar{x}_1^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))] \\ &= 2\|\bar{x}_1^k - \bar{x}_2^k\|^2 + \beta_k[(\bar{x}_1^k - \bar{x}_2^k)^T(F(x^k) \\ &\quad -F(\bar{x}_1^k)) - (x^k - \bar{x}_2^k)^T(F(\bar{x}_1^k) - F(\bar{x}_2^k))] \\ &\geq 2\|\bar{x}_1^k - \bar{x}_2^k\|^2 - m_2^2\|\bar{x}_1^k - \bar{x}_2^k\|^2 \\ &= (2 - m_2^2)\|x^k - \bar{x}_2^k\|^2. \quad (3.15)\end{aligned}$$

Recalling the definition of

$$g(\bar{x}_1^k, \bar{x}_2^k) = (\bar{x}_1^k - \bar{x}_2^k) - \beta_k(F(\bar{x}_1^k) - F(\bar{x}_2^k))$$

and applying criterion (3.1), we conclude that

$$\begin{aligned}\|g(\bar{x}_1^k, \bar{x}_2^k)\| &\leq (\|\bar{x}_1^k - \bar{x}_2^k\| + \|\beta(F(\bar{x}_1^k) - F(\bar{x}_2^k))\|)^2 \\ &\leq (1 + m_1)^2\|\bar{x}_1^k - \bar{x}_2^k\|^2 \quad (3.16)\end{aligned}$$

Moreover, by using (3.15) together with (3.16), we get that

$$\alpha_k = \frac{\rho_1 + \rho_2}{\|g(\bar{x}_1^k, \bar{x}_2^k)\|^2} \geq \frac{2 - m_2^2}{(1 + m_1)^2} > 0, \quad (3.17)$$

where $m_2 \in (0, \sqrt{2})$. The proof is completed.

Next, we will show the convergence result of proposed method.

Theorem 39 *The sequences $\{x_{FB}^{k+1}\}$ and $\{x_{EG}^{k+1}\}$ generated by Algorithm 3.1 are bounded.*

Proof. From Lemma 35 and Lemma 36, we have

$$\theta_{FB} \text{ (and } \theta_{EG}) \geq 2\alpha_k (x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k) - \alpha_k^2 \|g(\tilde{x}_1^k, \tilde{x}_2^k)\|^2. \quad (3.18)$$

For simplicity, we denote θ_{FB} (and θ_{EG}) by $\theta(\alpha_k)$ and x_{FB}^{k+1} (and x_{EG}^{k+1}) by $x^{k+1}(\alpha_k)$. The right-hand side of the inequality given by (3.18) is a quadratic function of α_k which its maximum consists of

$$\alpha_k^* = \frac{(x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k)}{\|g(\tilde{x}_1^k, \tilde{x}_2^k)\|^2}. \quad (3.19)$$

Let $\gamma \in (0,2)$ be a relaxation factor and $\alpha_k = \gamma \alpha_k^*$, it follows that

$$\begin{aligned}\theta(\gamma\alpha_k^*) &\geq 2\gamma\alpha_k^* (x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k) - \gamma^2 \alpha_k^{*2} \|g(\tilde{x}_1^k, \tilde{x}_2^k)\|^2 \\ &= \gamma\alpha_k^* (2(x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k) - \gamma\alpha_k^* \|g(\tilde{x}_1^k, \tilde{x}_2^k)\|^2) \\ &= \gamma\alpha_k^* (2-\gamma)(x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k). \quad (3.20)\end{aligned}$$

By (3.15) and (3.17), we obtain

$$\alpha_k^* (x^k - \tilde{x}_2^k)^T g(\tilde{x}_1^k, \tilde{x}_2^k) \geq \frac{(2-m_2^2)^2}{(1+m_1)^2} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2. \quad (3.21)$$

By (3.20) and (3.21), we get

$$\theta(\gamma\alpha_k^*) \geq \gamma(2-\gamma) \frac{(2-m_2^2)^2}{(1+m_1)^2} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2.$$

This implies that

$$\begin{aligned}&\|x^k - x^*\|^2 - \| x^{k+1}(\gamma\alpha_k) - x^*\|^2 \\ &\qquad \geq \gamma(2-\gamma) \frac{(2-m_2^2)^2}{(1+m_1)^2} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2. \quad (3.22)\end{aligned}$$

Then

$$\begin{aligned}&\|x^{k+1}(\gamma\alpha_k) - x^*\|^2 \\ &\qquad \leq \|x^k - x^*\|^2 - \gamma(2-\gamma) \frac{(2-m_2^2)^2}{(1+m_1)^2} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2. \quad (3.23)\end{aligned}$$

According to (3.23), it follows that

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \leq \cdots \leq \|x^0 - x^*\|.$$

Hence the sequence $\{x^k\} \subset \mathbb{R}^n$ generated by Algorithm 3.1 is bounded.

Theorem 310 *Suppose that the solution set of* $\mathrm{VI}(F,\Omega)$ *is nonempty. Then the sequence* $\{x^k\} \subset \mathbb{R}^n$ *generated by Algorithm 3.1 converges to a solution of* $\mathrm{VI}(F,\Omega)$.

Proof. Let x^* be a solution of $\mathrm{VI}(F,\Omega)$. First, it follows from (3.23) that

$$\sum_{k=0}^{\infty} \frac{\gamma(2-\gamma)(2-m_2^2)^2}{(2+m_1)^2} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2 \leq \|x^0 - x^*\|^2 < +\infty,$$

which means that

$$\lim_{k\to\infty} \|\tilde{x}_1^k - \tilde{x}_2^k\|^2 = 0.$$

According to Theorem 39, the sequence $\{x^k\}$ is bounded. Thus, it has at least one cluster point. Assume that x^* is a cluster point of $\{x^k\}$. Then there exists a subsequence $\{x^{k_j}\}$ that converges to x^*. It follows from the continuity of e and (2.3) that

$$\begin{aligned}\|e(x^*,\beta)\| &= \lim_{k_j\to\infty} \|e(x^{k_j},\beta)\| \\ &\leq \lim_{k_j\to\infty} \|e(x^{k_j},\beta_{k_j})\| = \lim_{k_j\to\infty} \|x_1^{k_j} - x_2^{k_j}\| = 0.\end{aligned}$$

Therefore, x^* is a solution of $\mathrm{VI}(F,\Omega)$.

In the following, we prove that the sequence $\{x^k\}$ has exactly one cluster point. Assume that $\bar{x}$ is another cluster point, and denotes $\delta := \|\bar{x} - x^*\| > 0$. Since x^* and $\bar{x}$ are cluster point of the sequence $\{x^k\}$, there is a $k_1 \in \mathbb{N}$ such that

$$\|x^k - x^*\| \leq \frac{\delta}{2}, \ \forall k \geq k_1,$$

and there is a $k_2 \in \mathbb{N}$ such that

$$\|x^k - \bar{x}\| \leq \frac{\delta}{2}, \ \forall k \geq k_2,$$

and choose $k_0 = \max\{k_1, k_2\}$. On the other hand, since $x^* \in \mathrm{VI}(F,\Omega)$, thus

$$\|x^k - x^*\| \leq \|x^{k_0} - x^*\|, \forall k \geq k_0,$$

It follows that

$$\|x^k - \bar{x}\| \geq \|\bar{x} - x^*\| - \|x^{k_0} - x^*\| \geq \frac{\delta}{2}, \forall k \geq k_0,$$

This is a contradiction with $\bar{x}$ is a cluster point, thus the sequence $\{x^k\}$ converges to $x^* \in \mathrm{VI}(F,\Omega)$.

However, the following theorem shows that in each iterative step, we may expect the modified general extra-gradient method to get closer with the solution than the modified general forward-backward splitting method.

Theorem 311 *Let* x_{FB}^{k+1} *and* x_{EG}^{k+1} *be defined as in (3.3) and (3.4). Then*

$$\begin{aligned}|x_{FB}^{k+1} - x_{EG}^{k+1}\|^2 \leq \|x^k &- \alpha_k g(\tilde{x}_1^k, \tilde{x}_2^k) - x_{EG}^{k+1}\|^2 \\ &- \|x^k - \alpha_k g(\tilde{x}_1^k, \tilde{x}_2^k) - x_{FB}^{k+1}\|^2.\end{aligned}$$

Proof. Set $y := x^k - \alpha_k g(\tilde{x}_1^k, \tilde{x}_2^k)$ and $x := x_{EG}^{k+1}$, by applying Lemma 24(2), we obtain

$$\begin{aligned}\|x_{FB}^{k+1} - x_{EG}^{k+1}\|^2 \leq \|x^k &- \alpha_k g(\tilde{x}_1^k, \tilde{x}_2^k) - x_{EG}^{k+1}\|^2 \\ &- \|x^k - \alpha_k g(\tilde{x}_1^k, \tilde{x}_2^k) - x_{FB}^{k+1}\|^2.\end{aligned}$$

This leads us to the conclusion.

Table 1: Example 1: Numerical results for starting point $x^0 = (1,0,1,0,\ldots)^T$

TAB 1	method in [11]		Modified FB		Modified EG	
dimN	it	cpu	it	cpu	it	cpu
10	112	0.031	178	0.062	55	0.000
50	497	0.046	344	0.062	131	0.046
100	1035	0.187	489	0.171	183	0.093
200	2117	0.437	676	0.265	267	0.125
500	5367	1.796	1054	0.656	427	0.312

4 Numercal experiments

In this section, we use two examples given in [7] and [18] to show the efficiency of the proposed new algorithm. All codes are written in Matlab 7.12 and run on a desktop computer(CPU:Intel Pentium 4 3.00 GHz, Memory:1.00 GB).

Two examples of the linear complementarity problem are adopted in this paper:

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0,$$

where $F(x) = Mx + q$.

Example1

$$M = \begin{bmatrix} 1 & 2 & \cdots & \cdots & 2 \\ 0 & 1 & 2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, q = (-1,-1,\cdots,-1)^T.$$

Example2

$$M = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \ddots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4(n-1)+1 \end{bmatrix},$$

$q = (-1,-1,\cdots,-1)^T$.

In our test we take $\Omega = \mathbb{R}^n_{++}$, $\beta = 1$, $m_1 = 0.9$, $m_2 = 0.3$, $\alpha = 1.8$, $\mu^2 = 2/3$ and the stop as soon as $\|e(x^k,1)\| \leq 10^{-7}$. We are going to compare the method generated by algorithm 3.1 with the method proposed by He et al. [11].The test results for Example 1 are reported in tables 1 and 2 and the test results for Example 2 are reported in tables 3 and 4. "it" denotes the number of iterations and "cpu" is computation time (second).

Table4

The numerical results show that the method generated by algorithm 3.1 is more effective than the method presented in [11], which can achieve the solution with fewer iteration and time. Moreover, it seems that the modified general extra-gradient method can converge faster than the modified general forward-backward method which is consistent with Theorem 311. We confirm that all examples coding by accord to the theory.

Table 2: Example 1: Numerical results for starting point $x^0 = (-2,-2,-2,\ldots)^T$

TAB 2	method in [11]		Modified FB		Modified EG	
dimN	it	cpu	it	cpu	it	cpu
10	156	0.046	191	0.031	58	0.015
50	704	0.062	375	0.046	131	0.031
100	1393	0.234	514	0.187	183	0.093
200	2769	0.546	711	0.265	262	0.109
500	6900	2.515	1081	0.750	399	0.265

Table 3: Example 2: Numerical results for starting point $x^0 = (-5,5,-5,\ldots)^T$

TAB 3	method in [11]		Modified FB		Modified EG	
dimN	it	cpu	it	cpu	it	cpu
10	256	0.031	210	0.046	66	0.031
50	32805	1.656	399	0.078	160	0.062
100	138985	20.328	550	0.171	228	0.109
200	-	-	766	0.296	332	0.140
500	-	-	1191	0.796	526	0.296

" − " represents that the CPU time is longer than 60 s.

Table 4: Example 2: Numerical results for starting point $x^0 = (3,3,3,\ldots)^T$

TAB 4	method in [11]		Modified FB		Modified EG	
dimN	it	cpu	it	cpu	it	cpu
10	2596	0.171	66	0.031	55	0.015
50	65295	3.718	154	0.046	126	0.031
100	261237	38.031	180	0.109	219	0.078
200	-	-	250	0.109	311	0.140
500	-	-	381	0.312	489	0.281

" − " represents that the CPU time is longer than 60 s.

Next, in Example 3 is an example of mapping F which is not monotone but is pseudomonotone (see [12]). Thus, our algorithm can apply to solve the problem but the algorithm in [11] can not.

Example3

$$F(x) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 - 4x_2x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_3 \end{bmatrix},$$

and $\Omega = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 5,\ i = 1,\ldots,4\}$.

The initial point is generated randomly. And because of the dimension of this problem is very small, the calculation is too fast, so it is not appropriate to compare performance of the methods by time. We compare the average of number of iterations for each method by

Table 5: Example 3: Numerical results for a random start point in $[-10,10]^4$

	method in [11]	Modified FB	Modified EG
Avg it	NAN	19	12

”NAN” represents that the iteration is endless calculation.

calculating 500 times, then estimate the average of ”it” for each method (Avg it). The test result for example 3 has been shown in the table below.

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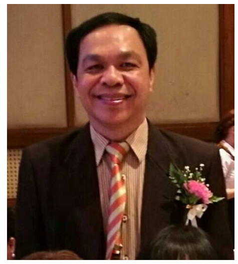

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