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Exponential Stability of Discrete-Time Cellular Uncertain BAM Neural Networks with Variable Delays using Halanay-Type Inequality

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Abstract: This paper is concerned with the class of uncertain discrete time Bi-directional associative memory (BAM) cellular neural networks with variable delays. Here the result is enhanced to ensure the global stability in the sense of exponential for the addressed time delayed neural networks by employing a discrete analogue of Halanay-type inequality. This type of inequalities can be used as basic tool in the study of exponential stability of the equilibrium for certain generalized difference equations. An important feature presents in our paper is that, with the help of time-invariant perturbation matrix which is often called parameter uncertainties, the proposed stability conditions can be proved. [(i.e) It is allowed to be norm-bounded]. At last, three illustrative examples with simulations are provided for the addressed neural networks to demonstrate the usefulness and flexibility of the proposed design approach.

Keywords: Bidirectional associative memory; Globally exponential stable; Discrete-time cellular neural networks; Time variable delays; Halanay inequality; Uncertainty.

1 Introduction

In recent years, a lot of consideration from many researchers is on stability of nonlinear difference equations with time varying delays (see [5,16,26]). In order to obtain the stability conditions, the most commonly used theory in the field concerned is Lyapunov. On the other hand, by using discrete type inequalities, there are few stability conditions for nonlinear difference equalities. In [29], new stability conditions for nonlinear difference equations with time delays are obtained by using a generalized discrete Gronwall inequality. Generally, cellular neural networks (or) cellular non-linear networks are a parallel computing paradigm similar to neural networks that plays a key role in the study of computer science and machine learning. Chua and Yang [10] have found utilizations in vast areas such as image processing; signal processing, optimization technique and particularly solving partial differential equations. Hence, the problem with stability analysis has received much attention in cellular neural networks (see [14,39,6],[41]-[45],). Basically, the neurons or cells are often referred as non-linear processing units. Mathematically, each cell can be modeled as a dissipative, non-linear dynamical system where information is encoded through initial state, inputs and variables used to define its behavior. Furthermore, the dynamics of a network are usually continuous as in case of continuous time CNN processors but it can be discrete as in case of Discrete time CNN Processors.

Time delay is one of the major sources of causing instability to the neural system. Over the past few years, the study of time delay systems has received considerable attention, see for instance [17,19,20,18,34,47]. A great

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number of research results on time delays systems exist in the literature, which is often so called systems with after-effect or dead time, hereditary systems, equations with deviating argument or differential difference equations, one can refer([21]-[23], [30,47]). Kosko [15], first coined Bidirectional Associative Memory(BAM). It is just an extension of the unidirectional auto associator of hopfield type networks. Here, the neurons are arranged in two layers: one layer in first neuron always interconnected with other layer in next neuron. Mainly, artificial intelligence is one of the potential applications of BAM neural networks. Authors Maharajan et al and Huang et al proposed the concept of BAM neural networks in [13] and [28]. Halanay introduced an inequality in the year 1966 to establish the exponential stability of solutions of delay differential equations. Then, in ([2,4,7,8,9,11,46]), the scholars proved an asymptotic formula for the solutions of differential inequality involving the maximum functional and applied it in the stability theory of linear systems with delay. Such type of inequality was so called Halanay inequality. Intuitively, the authors in [3], [12], [24,27,32,40] considered a discrete Halanay type inequalities to study some discrete version of differential equations.

Moreover, parameter uncertainties and stochastic disturbances are contemplated to follow the most dynamical behaviors of the systems. Parameter uncertainties enter into all system matrices whereas stochastic disturbances are prescribed in the form of Brownian motion. Due to the modeling inaccuracies and/or changes in the environment of the model, parameter uncertainties can be often confronted in real systems as well as neural networks. In the past few years, to solve the obstacles brought by parameter uncertainty, robustness analysis for distinct uncertain systems has received substantial attention, see for example [33,36,37].

Inspired by the aforementioned works, in this paper, we aim at addressing the exponential stability analysis problem for discrete-time cellular BAM neural network with variable delays and the presence of Halanay type inequality. These types of inequalities can act as a basic tool for solving generalized difference equations. The conditions derived in this paper are easy to check out and are useful for designing BAM Neural networks. Additionally, we have provided two numerical examples with simulations that shows the less conservatism of the structured BAM neural network.

The main contribution of our proposed work lies in five aspects:

–This paper investigates the problem of Globally Exponential Stability of Discrete time Cellular Uncertain BAM Neural Networks with Variable delays using Halanay-type Inequality. Here the neural networks are assumed to be BAM type. Because feed forward neural networks are easy to check stability. But BAM has more complications to check the stability of the neural network system.

–Regarding Research, the Halanay-type Inequality with variable delays in discrete time is fewer in the literature. However, up to our knowledge, the Cellular neural network problem for discrete-time is considered only with feed forward neural network system and BAM is not taken into account.

–Using MATLAB LMI toolbox, the trajectory shows that the system of this paper leads to stable which is delivered in Example 5.1. The obtained result shows a new contribution in the field of discrete-time neural networks.

–Furthermore, when the stochastic disturbances or noise affects the BAM neural network system, how the system affects and leads to stable can be checked with the help of MATLAB LMI toolbox which is obtained in Remark 4.2.

–Time-invariant parameter uncertainty is used first time in the Halanay type inequality to check the considered BAM neural networks is exponentially stable.

The rest of the paper can be well organized as follows. In Section 2, we have provided the problem formulation and preliminaries. The essence of the main results are devoted in Section 3. Section 4 address the problem of Uncertain BAM neural networks. Section 5 contains examples with simulation results which help us to illustrate the effectiveness of the proposed method. Finally, conclusions are drawn in Section 6.

Notations. The notations in this paper are quite standard. R^n denotes the n-dimensional Euclidean space. I^* be the identity matrix with appropriate dimensions and diag $(.)$ denotes the diagonal matrix. The superscript ”T” denotes the transpose of the matrix and $||.||$ stands for Euclidean norm in R^n. $\lambda_{\max}(x)$ (respectively $\lambda_{\min}(x)$) stands for maximum (resp minimum) eigenvalue of matrix X.

2 System Description And Assumption

In this section, we consider the discrete time BAM neural networks with discrete delay as

$$\begin{aligned} u_i(n+1) =& a_i u_i(n) + \sum_{j=1}^{m_2} b_{ji}\tilde{f}_j(v_j(n)) \\ &+ \sum_{j=1}^{m_2} c_{ji}\tilde{f}_j(v_j(n-k_1(n))) + I_i \\ v_j(n+1) =& l_j v_j(n) + \sum_{i=1}^{m_1} m_{ij}\tilde{g}_i(u_i(n)) \\ &+ \sum_{i=1}^{m_1} n_{ij}\tilde{g}_i(u_i(n-k(n))) + J_i, \end{aligned} \tag{1}$$

where $m_1, m_2 \in \mathbb{N}$ and the vector form is
$u(n+1) = Au(n) + B\tilde{f}(v(n)) + C\tilde{f}(v(n-k_1(n))) + I$

$$v(n+1) = Lv(n) + M\tilde{g}(u(n)) + N\tilde{g}(u(n-k(n))) + J, \quad (2)$$

for $n = \{0,1,...\}$, where $u(n) = [u_1(n), u_2(n), ..., u_{m_1}(n)]^T$ and $v(n) = [v_1(n), v_2(n), ..., v_{m_2}(n)]^T$ are the neural state vector; $A = diag\{a_1, a_2, ..., a_{m_1}\}$ and $L = diag\{l_1, l_2, ..., l_{m_2}\}$ are the state feedback co-efficient matrices; $B = [b_{ij}]_{m_1\times m_2}, C = [c_{ij}]_{m_1\times m_2}, M = [m_{ij}]_{m_1\times m_2}, N = [n_{ij}]_{m_1\times m_2}$ are the discretely delayed connection weight matrices; $\tilde{f}(v(n)) = [\tilde{f}_1(v(n)), \tilde{f}_2(v(n)), ..., \tilde{f}_{m_1}(v(n))]^T$ and $\tilde{g}(u(n)) = [\tilde{g}_1(u(n)), \tilde{g}_2(u(n)), ..., \tilde{g}_{m_2}(u(n))]^T$ be the neuron activation functions. The constant vectors $I = [I_1, I_2, ..., I_{m_1}]^T$ and $J = [J_1, J_2, ...J_{m_2}]^T$ are the external inputs from outside the system.

Assumption I. The activation function $\tilde{f}_j$, and $\tilde{g}_i$, $j = 1,2,3,...,m_2$, $i = 1,2,3,...,m_1$ are bounded and gratified conditions as follows

$$\begin{aligned} |\tilde{f}_j(\zeta_1) - \tilde{f}_j(\zeta_2)| &\le P_j|\zeta_1 - \zeta_2|, \quad \forall \zeta_1, \zeta_2 \in \mathbb{R} \\ |\tilde{g}_i(\gamma_1) - \tilde{g}_i(\gamma_2)| &\le Q_i|\gamma_1 - \gamma_2|, \quad \forall \gamma_1, \gamma_2 \in \mathbb{R} \end{aligned} \quad (3)$$

Since each functions $\tilde{f}_j$, $\tilde{g}_i$ satisfies the hypothesis (3) and satisfies

$$\begin{aligned} |\tilde{f}_j(\zeta_1)| &\le P_j|\zeta_1|, \\ |\tilde{g}_i(\gamma_1)| &\le Q_i|\gamma_1|. \end{aligned} \quad (4)$$

Remark 2.1 This type of activation function is clearly more general than both the usual sigmoid activation functions and the piecewise linear function:

$$\begin{aligned} \tilde{f}_j(y) &= \frac{1}{2}(y+1-y-1) \, and \\ \tilde{g}_i(x) &= \frac{1}{2}(x+1-x-1) \end{aligned} \quad (5)$$

Which is used in [10].

Definition 2.2 A vector $u^* = (u_1^*, u_2^*, u_3^*, ..., u_m^*)^T$ and $v^* = (v_1^*, v_2^*, v_3^*, ..., v_m^*)^T$ is called trivial solution of a NNs (1) if the below conditions satisfied.

$$u^* = a_i u_i^* + \sum_{i=1}^{n} b_{ji}\tilde{f}_j(v_j^*) + \sum_{i=1}^{n} c_{ji}\tilde{f}_j(v_j^*) + I_i$$
$$v^* = l_j v_j^* + \sum_{i=1}^{n} m_{ij}\tilde{g}_j(u_i^*) + \sum_{i=1}^{n} n_{ij}\tilde{g}_j(u_i^*) + J_j$$

To shorten our proof, we transfer u^* and v^* to the origin.

Let $x(n) = u(n) - u^*$ and $y(n) = v(n) - v^*$, then system (2) can be transformed to

$$\begin{aligned} x(n+1) &= Ax(n) + Bf(y(n)) + Cf(y(n-k_1(n))) \\ y(n+1) &= Ly(n) + Mg(x(n)) + Ng(x(n-k(n))). \end{aligned} \quad (6)$$

Definition 2.3 With the pioneering conditions $u_i(s) = \varphi_i(s)$ and $v_j(\tilde{s}) = \omega_j(\tilde{s})$, the equilibrium point of (7) is forenamed to be exponentially stable if for every solution $(u_i(n,\varphi_i), v_j(n,\omega_j))$for $s, \tilde{s} \in [-k, 0]$, there exists a scalar $\varepsilon \in (0,1)$ and $H \ge 1$ such that

$$\sum_{i=1}^{m_1} x_i^2(n) + \sum_{j=1}^{m_2} y_j^2(n) \le H[\|\phi\|^2 + \|\psi\|^2]\varepsilon^n, \; n \ge 0, \quad (7)$$

where $\|\phi\| = \max_{s\in[-k,0]}\left\{\sum_{i=1}^{m_1}\varphi_i^2(s)\right\}^{\frac{1}{2}}$ and $\|\psi\| = \max_{\tilde{s}\in[-k_1,0]}\left\{\sum_{j=1}^{m_2}\omega_j^2(\tilde{s})\right\}^{\frac{1}{2}}$.

In order to acquire our main result, we need the following lemma.

Lemma 2.4 Ref [25]

Let $r > 0$ be a natural number, let $\{z(n)\}_{n\ge -r}$ be a sequence of real numbers satisfying the inequality

$$\Delta z(n) \le -pz(n) + q\max\left\{z(n), z(n-1), ..., z(n-r)\right\} \quad (8)$$

where $n \ge 0$, If $0 < q < p \le 1$, so that

$$z(n) \le \max\left\{0, z(0), z(-1), ..., z(-r)\right\}\lambda_0^n$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0,1)$ of equation

$$\lambda^{r+1} + (p-1)\lambda^r - q = 0. \quad (9)$$

Remark 2.5 Discrete or continuous Halanay-type inequalities have been extensively applied to attain the globally exponential or asymptotic stability of the equilibria of dynamical systems with considerable delays, notably dynamical systems of neural networks.

3 Main Results

In this section, a new Halanay type inequality is used to check the stability of the discrete-time BAM neural network (6) with time varying delays.

Theorem 3.1 The zero solution of equation (6) is globally exponentially stable if

$$3\lambda_{\max}(G^TG)O_{\max}^2 < \min\left\{1 - 3k_{\tilde{l}}^2 - 3\lambda_{\max}(F^TF)r_{\tilde{l}}^2\right\},$$

where $\lambda_{\max}(G), \lambda_{\max}(F)$ denotes the largest eigen value of matrix G, F and $P_{\max} = \max_{1\le j\le m_2}\{p_j\}$,

$Q_{\max} = \max_{1\le i\le m_1}\{q_i\}$ respectively.

Proof. Consider the following function

$$z(n) = x^T(n)x(n) + y^T(n)y(n) \qquad (10)$$

Then, we obtain

$$\begin{aligned}
&z(n+1)\\
&= x^T(n+1)x(n+1) + y^T(n+1)y(n+1)\\
&= \Big[Ax(n) + Bf(y(n)) + Cf(y(n-k(n)))\Big]^T\\
&\quad \times\Big[Ax(n) + Bf(y(n)) + Cf(y(n-k(n)))\Big]\\
&\quad + \Big[Ly(n) + Mg(x(n)) + Ng(x(n-k_1(n)))\Big]^T\\
&\quad \Big[Ly(n) + Mg(x(n)) + Ng(x(n-k_1(n)))\Big]\\
&= \Big[Ax^T(n) + Bf^T(y(n)) + Cf^T(y(n-k(n)))\Big]^T\\
&\quad \Big[Ax(n) + Bf(y(n)) + Cf(y(n-k(n)))\Big]\\
&\quad + \Big[Ly^T(n) + Mg^T(x(n)) + Ng^T(x(n-k_1(n)))\Big]^T\\
&\quad \Big[Ly(n) + Mg(x(n)) + Ng(x(n-k_1(n)))\Big]\\
&= x^T(n)A^2x(n) + 2x^T(n)ABf(y(n)) + 2x^T(n)A\\
&\quad \times Cf(y(n-k(n))) + f^T(y(n))B^TBf(y(n))\\
&\quad + 2f^T(y(n))B^TCf(y(n-k(n))) + f^T(y(n-k(n)))\\
&\quad \times C^TCf(y(n-k(n))) + y^T(n)L^2y(n) + 2y^T(n)L\\
&\quad \times Mg(x(n)) + 2y^T(n)LNg(x(n-k_1(n))) + g^T(x(n))\\
&\quad \times M^TMg(x(n)) + 2g^T(x(n))M^TNg(x(n-k_1(n)))\\
&\quad + g^T(x(n-k_1(n)))N^TNg(x(n-k_1(n)))\\
&\le x^T(n)A^2x(n) + 2x^T(n)ABf(y(n)) + 2x^T(n)A\\
&\quad \times Cf(y(n-k(n))) + \lambda_{\max}(B^TB)\\
&\quad \times y^T(n)Q^2y(n) + 2f^T(y(n))B^TCf(y(n-k(n)))\\
&\quad + \lambda_{\max}(C^TC)Q^2_{\max}y^T(n-k(n))\\
&\quad \times y(n-k(n)) + y^T(n)L^2y(n) + 2y^T(n)LMg(x(n))\\
&\quad + 2y^T(n)LNg(x(n-k_1(n))) + \lambda_{\max}(M^TM)x^T(n)\\
&\quad \times P^2x(n) + 2g^T(x(n))M^TNg(x(n-k_1(n)))\\
&\quad + \lambda_{\max}(N^TN)P^2_{\max}x^T(n-k_1(n))x(n-k_1(n)) \qquad (11)
\end{aligned}$$

For any $a, b \in \mathbb{R}^m$, remember that $2a^Tb \le a^Ta + b^Tb$ satisfies.

$$\begin{aligned}
&2x^T(n)ABf(y(n))\\
&= 2(Ax(n))^TB(f(y(n)))\\
&\le x^T(n)A^2x(n) + f^T(y(n))B^TBf(y(n))\\
&\le x^T(n)A^2x(n) + \lambda_{\max}(B^TB)f^T(y(n))f(y(n))\\
&\le x^T(n)A^2x(n) + \lambda_{\max}(B^TB)y^T(n)Q^2y(n)
\end{aligned}$$

$$\begin{aligned}
&2x^T(n)ACf(y(n-k(n)))\\
&= 2(A(x(n)))^TC(f(y(n-k(n))))\\
&\le x^T(n)A^2x(n) + f^T(y(n-k(n)))C^TCf(y(n-k(n)))\\
&\le x^T(n)A^2x(n) + \lambda_{\max}(C^TC)f^T(y(n-k(n)))f(y(n-k(n)))\\
&\le x^T(n)A^2x(n) + \lambda_{\max}(C^TC)y^T(n-k(n))Q^2_{\max}y(n-k(n))
\end{aligned}$$

$$\begin{aligned}
&2f^T(y(n))BCf(y(n-k(n)))\\
&= 2(B(f(y(n))))^TC(f(y(n-k(n))))\\
&\le f^T(y(n))B^TBf(y(n)) + f^T(y(n-k(n)))C^TCf(y(n-k(n)))\\
&\le \lambda_{\max}(B^TB)f^T(y(n))f(y(n)) + \lambda_{\max}(C^TC)f^T(y(n-k(n)))\\
&\quad \times f(y(n-k(n)))\\
&\le \lambda_{\max}(B^TB)y^T(n)Q^2y(n) + \lambda_{\max}(C^TC)y^T(n-k(n))\\
&\quad \times Q^2_{\max}y(n-k(n))\\
&2y^T(n)LMg(x(n))\\
&= 2(L(y(n)))^TMg(x(n))\\
&\le y^T(n)L^2y(n) + g^T(x(n))M^TMg(x(n))\\
&\le y^T(n)L^2y(n) + \lambda_{\max}(M^TM)g^T(x(n))g(x(n))\\
&\le y^T(n)L^2y(n) + \lambda_{\max}(M^TM)x^T(n)P^2x(n)\\
&2y^T(n)LNg(y(n-k_1(n)))\\
&= 2(L(y(n)))^TNg(x(n-k_1(n)))\\
&\le y^T(n)L^2y(n) + g^T(x(n-k_1(n)))N^TNg(x(n-k_1(n)))\\
&\le y^T(n)L^2y(n) + \lambda_{\max}(N^TN)g^T(x(n-k_1(n)))g(x(n-k_1(n)))\\
&\le y^T(n)L^2y(n) + \lambda_{\max}(N^TN)x^T(n-k_1(n))P^2_{\max}x(n-k_1(n))\\
&2g^T(x(n))MNg(x(n-k_1(n)))\\
&= 2(M(g(x(n))))^TN(g(x(n-k_1(n))))\\
&\le g^T(x(n))M^TMg(x(n))^TM^TMg(x(n)) + g^T(x(n-k_1(n)))\\
&\quad \times N^TNg(x(n-k_1(n)))\\
&\le \lambda_{\max}(M^TM)g^T(x(n))g(x(n)) + \lambda_{\max}(N^TN)\\
&\quad \times g^T(x(n-k_1(n)))g(x(n-k_1(n)))\\
&\le \lambda_{\max}(M^TM)x^T(n)P^2x(n) + \lambda_{\max}(N^TN)x^T(n-k_1(n))\\
&\quad \times P^2_{\max}x(n-k_1(n)) \qquad (12)
\end{aligned}$$

where $P = diag\Big(p_1, p_2, ..., p_{m_1}\Big)$, $Q = diag\Big(q_1, q_2, ..., q_{m_2}\Big), m_1, m_2 \in \mathbb{R}$.

By substituting (12) into (11), we obtain

$$\begin{aligned}
z(n+1) &\le x^T(n)3A^2x(n) + y^T(n)3L^2y(n)\\
&\quad + 3\lambda_{\max}(B^TB)y^T(n)Q^2y(n) + 3\lambda_{\max}(M^TM)\\
&\quad \times x^T(n)P^2x(n) + 3\lambda_{\max}(C^TC)y^T(n-k(n))\\
&\quad \times Q^2_{\max}y(n-k(n)) + 3\lambda_{\max}(N^TN)\\
&\quad \times x^T(n-k_1(n))P^2_{\max}x(n-k_1(n))\\
&\le x^T(n)\Big[3A^2 + 3\lambda_{\max}(M^TM)P^2\Big]x(n) + y^T(n)\\
&\quad \Big[3L^2 + 3\lambda_{\max}(B^TB)\Big]y(n)\\
&\quad + 3\lambda_{\max}(C^TC)y^T(n-k(n))Q^2_{\max}y(n-k(n))\\
&\quad + 3\lambda_{\max}(N^TN)x^T(n-k_1(n))\\
&\quad \times P^2_{\max}x(n-k_1(n)) \qquad (13)
\end{aligned}$$

With the help of (6)

$$\Delta z(n) = z(n+1) - z(n)$$

$$
\begin{aligned}
&\le -x^T(n)\Big[I^* - 3A^2 - 3\lambda_{\max}(M^TM)P^2\Big]x(n)\\
&\quad -y^T(n)\Big[I^* - 3L^2 - 3\lambda_{\max}(B^TB)Q^2\Big]y(n)\\
&\quad +3\lambda_{\max}(C^TC)Q^2_{\max}y^T(n-k(n))y(n-k(n))\\
&\quad +3\lambda_{\max}(N^TN)P^2_{\max}x^T(n-k_1(n))x(n-k_1(n))
\end{aligned} \tag{14}
$$

Put, $k_{\tilde{l}} = \max\{a_i, l_j\}$, $r_{\tilde{l}} = \max\{p_j, q_i\}$ and

$$
\begin{aligned}
&\min\Big[1 - 3k^2_{\tilde{l}} - 3\lambda_{\max}(F^TF)r^2_{\tilde{l}}\Big]\\
&= \min\Bigg\{\min\Big\{1 - 3a_i^2 - 3\lambda_{\max}(M^TM)p_i^2\Big\},\\
&\min\Big\{1 - 3l_j^2 - 3\lambda_{\max}(B^TB)q_j^2\Big\}\Bigg\}, \ \tilde{l} = 1,2,\dots,N,\ N \in \mathbb{N}
\end{aligned}
$$

and put $O_{\max} = \max\{r\tilde{l}\}$ and

$$
\begin{aligned}
&3\lambda_{\max}(G^TG)O^2_{\max}\\
&= \max\Big\{(3\lambda_{\max}(C^TC)P^2_{\max}), (3\lambda_{\max}(N^TN)Q^2_{\max})\Big\}
\end{aligned}
$$

and

$$K^*(n) = \min\Big\{k(n), k_1(n)\Big\}$$

Then

$$
\begin{aligned}
\Delta z(n) &\le -\min\Big\{1 - 3k^2_{\tilde{l}} - 3\lambda_{\max}(F^TF)r_{\tilde{l}}\Big\}\\
&\quad \times\Big[x^T(n)x(n) + y^T(n)y(n)\Big] + 3\lambda_{\max}(C^TC)\\
&\quad \times Q^2_{\max}y^T(n-k(n))y(n-k(n)) + 3\lambda_{\max}(N^TN)\\
&\quad \times P^2_{\max}x^T(n-k_1(n))x(n-k_1(n))\\
&\le -\min\Bigg\{\min\Big\{1 - 3k^2_{\tilde{l}} - 3\lambda_{\max}(F^TF)r^2_{\tilde{l}}\Big\}\\
&\quad \times\Big[x^T(n)x(n) + y^T(n)y(n)\Big] + \Big\{3\lambda_{\max}(G^TG)\\
&\quad \times O^2_{\max}\Big\}\Big[x^T(n-k^*(n))x(n-k^*(n))\\
&\quad + y^T(n-k^*(n))y(n-k^*(n))\Big]\Bigg\}\\
&\le -\min\Big\{1 - 3k^2_{\tilde{l}} - 3\lambda_{\max}(F^TF)r^2_{\tilde{l}}\Big\}z(n)\\
&\quad + \Big\{3\lambda_{\max}(G^TG)O^2_{\max}\Big\}\Big\{z(n), z(n-1),\dots,\\
&\quad \times z(n-k^*)\Big\}\\
&= -p^*z(n) + q^*\max\Big\{z(n), z(n-1),\dots,z(n-k^*)\Big\}
\end{aligned} \tag{15}
$$

where $k^* = \max\{k^*(n)\}$, $p^* = \Big\{1 - 3k^2_{\tilde{l}} - 3\lambda_{\max}(F^TF)r^2_{\tilde{l}}\Big\}$ and $q^* = 3\lambda(G^TG)O^2_{\max}$, I^* be the identity matrix

By Lemma 2.4, if $q^* = 3\lambda_{\max}(G^TG)O^2_{\max} < p^*$ then $\exists$ a scalar λ_0 $(0 < \lambda_0 < 1)$ such that

$$
\begin{aligned}
z(n) &= x^T(n)x(n) + y^T(n)y(n)\\
&= \sum_{i=1}^{m_1} x_i^2 + \sum_{j=1}^{m_2} y_j^2(n)\\
&\le \max\Big\{0, z(0), z(-1),\dots,z(-k^*)\Big\}\lambda_0^n\\
&= \max\Big\{0, \sum_{i=1}^{m_1} x_i^2(0) + \sum_{j=1}^{m_2} y_j^2(0),\dots,\sum_{i=1}^{m_1} x_i^2(-k^*)\\
&\quad + \sum_{j=1}^{m_2} y_j^2(-k^*)\Big\}\lambda_0^n\\
&\le \Big[||\phi||^2 + ||\psi||^2\Big]\lambda_0^n
\end{aligned} \tag{16}
$$

Uncertain Discrete-time BAM Neural Networks using Halanay Type Inequality Uncertainties are undeniable in neural networks because of the existence of modeling errors and external disturbances in practical performances. To reflect such a phenomenon, while modeling the network, unavoidable parameter uncertainties should be taken into account.

In our manuscript, the stability of uncertain discrete-time BAM neural networks with parameter uncertainties are mentioned as,

$$
\begin{aligned}
x(n+1) &= \Big[A + \Delta A\Big]x(n) + \Big[B + \Delta B\Big]f(y(n))\\
&\quad + \Big[C + \Delta C\Big]f(y(n-k_1(n)))\\
y(n+1) &= \Big[L + \Delta L\Big]y(n) + \Big[M + \Delta M\Big]g(x(n))\\
&\quad + \Big[N + \Delta N\Big]g(x(n-k(n))).
\end{aligned} \tag{17}
$$

for $n = \{0,1,\dots\}$, where $x(n) = [x_1(n), x_2(n),\dots,x_{m_1}(n)]^T$ and $y(n) = [y_1(n), y_2(n),\dots,y_{m_2}(n)]^T$ are the neural state vector; $A = diag\{a_1, a_2,\dots,a_{m_1}\}$ and $L = diag\{l_1, l_2,\dots,\ l_{m_2}\}$ are the state feedback co-efficient matrices; $B = [b_{ij}]_{m_1\times m_2}, C = [c_{ij}]_{m_1\times m_2}, M = [m_{ij}]_{m_1\times m_2}, N = [n_{ij}]_{m_1\times m_2}$ are the discretely delayed connection weight matrices; $f(y(n)) = [f_1(y(n)), f_2(y(n)),\dots,f_{m_1}(y(n))]^T$ and $g(x(n)) = [g_1(x(n)), g_2(x(n)),\dots,g_{m_2}(x(n))]^T$ be the neuron activation functions.

ΔA, ΔB, ΔC, ΔL, ΔM, ΔN denotes the time-invariant structured uncertainties, which are of the following form,

$$\Big[\Delta A,\ \Delta B,\ \Delta C,\ \Delta L,\ \Delta M,\ \Delta N\Big] = W\ O\ \Big[E_a\ E_b\ E_c\ E_l\ E_m\ E_n\Big] \tag{18}$$

where $W, E_a, E_b, E_c, E_l, E_m, E_n$ are known real constant matrices of appropriate dimensions; O is unknown time-invariant matrix function satisfying

$$O^TO \le I^*. \tag{19}$$

Remark 4.1 The parameter uncertainty structure as in (18), (19) has been extensively abused in the problems of robust control and filtering of uncertain systems (see, e.g., [38] and the references therein). Many practical systems acquire parameter uncertainties which can be either explicitly modeled or overbounded by (19). Observe that the unknown matrix O in (18) can even be allowed to be state-dependent, i.e., $O = O(t,x(t))$, as long as (19) is satisfied.

We first introduce one lemma which are essential for the proof of the main result in this proposed work.

Lemma 4.2 Ref [35] Let a positive scalar ε and positive definite matrix $Q > 0$ such that

$$NQN^T < \varepsilon I$$

Assume that $\Delta A = MFN$, where $FF^T \leq I^*$, M and N are constant matrices with appropriate dimensions. Then

$$(A_n+\Delta A)Q(A_n+\Delta A)^T \leq A_n(Q^{-1}-\varepsilon^{-1}N^TN)^{-1}A_n^T+\varepsilon MM^T$$

holds for all admissible perturbations.

In next theorem, we establish the required conditions for the stability of uncertain discrete-time BAM neural networks

Theorem 4.3 The zero solution of equation (17) is globally exponentially stable if

$$3\lambda_{\max}\{F^*\} < \min\left\{1-D^*\right\},$$

where D^* and F^* are defined in the proof of the theorem and $P_{\max} = \max_{1\leq j\leq m_2}\{P_j\}$,

$Q_{\max} = \max_{1\leq i\leq m_1}\{Q_i\}$ respectively.

Proof. Consider the following function

$$z(n) = x^T(n)x(n)+y^T(n)y(n) \tag{20}$$

Then, we Obtain

$$\begin{aligned}
z(n+1) &= x^T(n+1)x(n+1)+y^T(n+1)y(n+1)\\
&= \Big[(A+\Delta A)x(n)+(B+\Delta B)f(y(n))\\
&\quad +(C+\Delta C)f(y(n-k(n)))\Big]^T\Big[(A+\Delta A)x(n)\\
&\quad +(B+\Delta B)f(y(n))+(C+\Delta C)f(y(n-k(n)))\Big]\\
&\quad +\Big[(L+\Delta L)y(n)+(M+\Delta M)\\
&\quad \times g(x(n))+(N+\Delta N)g(x(n-k_1(n)))\Big]^T\\
&\quad \times\Big[(L+\Delta L)y(n)+(M+\Delta M)g(x(n))\\
&\quad +(N+\Delta N)g(x(n-k_1(n)))\Big]\\
&= \Big[(A+\Delta A)x^T(n)+(B+\Delta B)f^T(y(n))\\
&\quad +(C+\Delta)f^T(y(n-k(n)))\Big]^T\Big[(A+\Delta A)x(n)\\
&\quad +(B+\Delta B)f(y(n))+(C+\Delta C)f(y(n-k(n)))\Big]\\
&\quad +\Big[(L+\Delta L)y^T(n)+(M+\Delta M)\\
&\quad \times g^T(x(n))+(N+\Delta N)g^T(x(n-k_1(n)))\Big]^T\\
&\quad \times\Big[(L+\Delta L)y(n)+(M+\Delta M)g(x(n))\\
&\quad +(N+\Delta N)g(x(n-k_1(n)))\Big]\\
&= x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)\\
&\quad +2x^T(n)\Big[(A+\Delta A)^T(B+\Delta B)\Big]f(y(n))+2x^T(n)\\
&\quad \times\Big[(A+\Delta A)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&\quad +f^T(y(n))\Big[(B+\Delta B)^T(B+\Delta B)\Big]f(y(n))\\
&\quad +2f^T(y(n))\Big[(B+\Delta B)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&\quad +f^T(y(n-k(n)))\Big[(C+\Delta C)^T\\
&\quad \times(C+\Delta C)\Big]f(y(n-k(n)))+y^T(n)\Big[(L+\Delta L)^T\\
&\quad \times(L+\Delta L)\Big]y(n)+2y^T(n)\Big[(L+\Delta L)^T\\
&\quad \times(M+\Delta M)\Big]g(x(n))+2y^T(n)\Big[(L+\Delta L)^T\\
&\quad \times(N+\Delta N)\Big]g(x(n-k_1(n)))+g^T(x(n))\\
&\quad \times\Big[(M+\Delta M)^T(M+\Delta M)\Big]g(x(n))+2g^T(x(n))\\
&\quad \times\Big[(M+\Delta M)^T(N+\Delta N)\Big]\\
&\quad \times g(x(n-k_1(n)))+g^T(x(n-k_1(n)))\Big[(N+\Delta N)^T\\
&\quad \times(N+\Delta N)\Big]g(x(n-k_1(n)))\\
&\leq x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)\\
&\quad +2x^T(n)\Big[(A+\Delta A)^T(B+\Delta B)\Big]f(y(n))\\
&\quad +2x^T(n)\\
&\quad \times\Big[(A+\Delta A)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&\quad +\lambda_{\max}\Big[(B+\Delta B)^T(B+\Delta B)\Big]y^T(n)Q^2y(n)\\
&\quad +2f^T(y(n))\Big[(B+\Delta B)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&\quad +\lambda_{\max}\Big[(C+\Delta C)^T(C+\Delta C)\Big]Q_{\max}^2\\
&\quad \times y^T(n-k(n))y(n-k(n))+y^T(n)\Big[(L+\Delta L)^T\\
&\quad \times(L+\Delta L)\Big]y(n)+2y^T(n)\\
&\quad \times\Big[(L+\Delta L)^T(M+\Delta M)\Big]g(x(n))+2y^T(n)\Big[(L\\
&\quad +\Delta L)^T(N+\Delta N)\Big]g(x(n-k_1(n)))\\
&\quad +\lambda_{\max}\Big[(M+\Delta M)^T(M+\Delta M)\Big]x^T(n)P^2x(n)
\end{aligned}$$

$$+2g^T(x(n))\Big[(M+\Delta M)^T(N+\Delta N)\Big] \times g(x(n-k_1(n)))\lambda_{\max}\Big[(N+\Delta N)^T \times(N+\Delta N)\Big]P^2_{\max}x^T(n-k_1(n))x(n-k_1(n)) \quad (21)$$

Recall that the inequality $2a^Tb \le a^Ta + b^Tb$ holds for any $a, b \in \mathbb{R}^m$, we have

$$\begin{aligned}
&2x^T(n)\Big[(A+\Delta A)^T(B+\Delta B)\Big]f(y(n))\\
&=2\Big[(A+\Delta A)x(n)^T\Big]\Big[B+\Delta B\Big](f(y(n)))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)\\
&\quad+f^T(y(n))\Big[(B+\Delta B)^T(B+\Delta B)\Big]f(y(n))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)\\
&\quad+\lambda_{\max}\Big[(B+\Delta B)^T(B+\Delta B)\Big]f^T(y(n))f(y(n))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)]x(n)\\
&\quad+\lambda_{\max}[(B+\Delta B)^T(B+\Delta B)\Big]y^T(n)Q^2y(n)
\end{aligned}$$

$$\begin{aligned}
&2x^T(n)\Big[(A+\Delta A)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&=2\Big[(A+\Delta A)^T(x(n))\Big]^T\Big[(C+\Delta C)\Big](f(y(n-k(n))))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)\\
&\quad+f^T(y(n-k(n)))\Big[(C+\Delta C)^T(C+\Delta C)\Big]f(y(n-k(n)))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)+\lambda_{\max}\Big[(C\\
&\quad+\Delta C)^T(C+\Delta C)\Big]f^T(y(n-k(n)))f(y(n-k(n)))\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)+\lambda_{\max}\Big[(C\\
&\quad+\Delta C)^T(C+\Delta C)\Big]y^T(n-k(n))Q^2_{\max}y(n-k(n))
\end{aligned}$$

$$\begin{aligned}
&2f^T(y(n))\Big[(B+\Delta B)(C+\Delta C)\Big]f(y(n-k(n)))\\
&=2\Big[(B+\Delta B)(f(y(n)))\Big]^T\Big[(C+\Delta C)\Big](f(y(n-k(n))))\\
&\le f^T(y(n))\Big[(B+\Delta B)^T(B+\Delta B)\Big]f(y(n))\\
&\quad+f^T(y(n-k(n)))[(C+\Delta C)^T(C+\Delta C)]f(y(n-k(n)))\\
&\le \lambda_{\max}\Big[(B+\Delta B)^T(B+\Delta B)\Big]f^T(y(n))f(y(n))\\
&\quad+\lambda_{\max}\Big[(C+\Delta C)^T(C+\Delta C)\Big]f^T(y(n-k(n)))f(y(n-k(n)))\\
&\le \lambda_{\max}\Big[(B+\Delta B)^T(B+\Delta B)\Big]y^T(n)Q^2y(n)+\lambda_{\max}\Big[(C\\
&\quad+\Delta C)^T(C+\Delta C)\Big]y^T(n-k(n))\times Q^2_{\max}y(n-k(n))
\end{aligned}$$

$$\begin{aligned}
&2y^T(n)\Big[(L+\Delta L)(M+\Delta M)\Big]g(x(n))\\
&=2\Big[(L+\Delta L)(y(n))\Big]^T\Big[M+\Delta M\Big]g(x(n))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)\\
&\quad+g^T(x(n))\Big[(M+\Delta M)^T(M+\Delta M)\Big]g(x(n))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)\\
&\quad+\lambda_{\max}\Big[(M+\Delta M)^T(M+\Delta M)\Big]g^T(x(n))g(x(n))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)+\lambda_{\max}\Big[(M\\
&\quad+\Delta M)^T(M+\Delta M)\Big]x^T(n)P^2x(n)
\end{aligned}$$

$$\begin{aligned}
&2y^T(n)\Big[(L+\Delta L)(N+\Delta N)\Big]g(y(n-k_1(n)))\\
&=2\Big[(L+\Delta L)(y(n))\Big]^T\Big[N+\Delta N\Big]g(x(n-k_1(n)))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)\\
&\quad+g^T(x(n-k_1(n)))\Big[(N+\Delta N)^T(N+\Delta N)\Big]g(x(n-k_1(n)))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)\\
&\quad+\lambda_{\max}\Big[(N+\Delta N)^T(N+\Delta N)\Big]g^T(x(n-k_1(n)))g(x(n-k_1(n)))\\
&\le y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n)+\lambda_{\max}\Big[(N\\
&\quad+\Delta N)^T(N+\Delta N)\Big]x^T(n-k_1(n))P^2_{\max}x(n-k_1(n))
\end{aligned}$$

$$\begin{aligned}
&2g^T(x(n))\Big[(M+\Delta M)(N+\Delta N)\Big]g(x(n-k_1(n)))\\
&=2\Big[(M+\Delta M)(g(x(n)))\Big]^T\Big[N+\Delta N\Big](g(x(n-k_1(n))))\\
&\le g^T(x(n))\Big[(M+\Delta M)(M+\Delta M)\Big]g(x(n))^T\Big[(M\\
&\quad+\Delta M)^T(M+\Delta M)\Big]g(x(n))+g^T(x(n-k_1(n)))\\
&\quad\times\Big[(N+\Delta N)^T(N+\Delta N)\Big]g(x(n-k_1(n)))\\
&\le \lambda_{\max}\Big[(M+\Delta M)^T(M+\Delta M)\Big]g^T(x(n))g(x(n))+\lambda_{\max}\Big[(N\\
&\quad+\Delta N)^T(N+\Delta N)\Big]g^T(x(n-k_1(n)))g(x(n-k_1(n)))\\
&\le \lambda_{\max}\Big[(M+\Delta M)^T(M+\Delta M)\Big]x^T(n)P^2x(n)\\
&\quad+\lambda_{\max}\Big[(N+\Delta N)^T(N+\Delta N)\Big]x^T(n-k_1(n))\\
&\quad\times P^2_{\max}x(n-k_1(n)) \qquad (22)
\end{aligned}$$

where $P = diag\Big(p_1, p_2, ..., p_{m_1}\Big), Q = diag\Big(q_1, q_2, ..., q_{m_2}\Big), m_1, m_2 \in \mathbb{R}$.

By substituting (22) into (21), we attain

$$\begin{aligned}
&z(n+1)\\
&\le x^T(n)\Big[(A+\Delta A)^T(A+\Delta A)\Big]x(n)+x^T(n)\Big[(A+\Delta A)^T
\end{aligned}$$

$$\times(A+\Delta A)]x(n)+y^T(n)\Big[\lambda_{\max}Q^2\Big[(B+\Delta B)^T \times(B+\Delta B)\Big]\Big]y(n)+x^T(n)\Big[(A+\Delta A)^T (A+\Delta A)\Big]x(n)+y^T(n-k(n)) \times\Big[\lambda_{\max}Q^2_{\max}\Big[(C+\Delta C)^T(C+\Delta C)\Big]\Big]y(n-k(n)) +\lambda_{\max}\Big[(B+\Delta B)^T(B+\Delta B)\Big]y^T(n) \times Q^2y(n)+y^T(n)\Big[\lambda_{\max}Q^2\Big[(B+\Delta B)^T(B +\Delta B)\Big]\Big]y(n)+y^T(n-k(n))\Big[\lambda_{\max}Q^2_{\max} \times\Big[(C+\Delta C)^T(C+\Delta C)\Big]\Big]y(n-k(n))+\lambda_{\max}\Big[(C +\Delta C)^T(C+\Delta C)\Big]Q^2_{\max}y^T(n-k(n)) \times y(n-k(n))+y^T(n)\Big[(L+\Delta L)^T(L +\Delta L)\Big]y(n)+y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)]y(n) x^T(n)\Big[\lambda_{\max}P^2\Big[(M+\Delta M)^T(M+\Delta M)\Big]\Big]x(n) +y^T(n)\Big[(L+\Delta L)^T(L+\Delta L)\Big]y(n) +x^T(n-k(n))\Big[\lambda_{\max}P^2_{\max}\Big[(N+\Delta N)^T(N+\Delta N)\Big]\Big] x(n-k(n))+\lambda_{\max}\Big[(M+\Delta M)^T(M+\Delta M)\Big] \times x^T(n)P^2x(n)+x^T(n)\Big[\lambda_{\max}P^2\Big[(M+\Delta M)^T(M+ \Delta M)\Big]\Big]x(n)+x^T(n-k(n))\Big[\lambda_{\max}P^2_{\max}\Big[(N+ \times\Delta N)^T(N+\Delta N)\Big]\Big]x(n-k(n))+\lambda_{\max}\Big[(N +\Delta N)^T(N+\Delta N)\Big]Q^2_{\max}x^T(n-k(n))x(n-k(n))$$

By using the Lemma 4.2, we get

$$\begin{aligned} & z(n+1) \\ \leq\ & x^T(n)\Big[3\Big(A(I^{-1}-\varepsilon^{-1}E_a^TE_a)^{-1}A^T +\varepsilon WW^T\Big)\Big]x(n)+y^T(n)\Big[3\Big(L(I^{-1}-\varepsilon^{-1}E_l^T \times E_l)^{-1}L^T+\varepsilon WW^T\Big)\Big]y(n) \\ & +y^T(n)\Big[3\lambda_{\max}\Big(B(I^{-1}-\varepsilon^{-1}E_b^TE_b)^{-1}B^T +\varepsilon WW^T\Big)\Big]Q^2y(n)+x^T(n) \\ & \times\Big[3\lambda_{\max}\Big(M(I^{-1}-\varepsilon^{-1}E_m^TE_m)^{-1}M^T +\varepsilon WW^T\Big)\Big]P^2x(n)+y^T(n-k(n)) \\ & \times\Big[3\lambda_{\max}\Big(C(I^{-1}-\varepsilon^{-1}E_c^TE_c)^{-1}C^T +\varepsilon WW^T\Big)\Big]Q^2_{\max}y(n-k(n))+x^T(n-k_1(n)) \\ & \times\Big[3\lambda_{\max}\Big(N(I^{-1}-\varepsilon^{-1}E_n^TE_n)^{-1}N^T +\varepsilon WW^T\Big)\Big]P^2_{\max}x(n-k_1(n)) \end{aligned}$$

$$\begin{aligned} & z(n+1 \\ \leq\ & x^T(n)\Big[A_1\Big]x(n)+y^T(n)\Big[B_1\Big]y(n)+\Big[B_2\Big]y^T(n)Q^2y(n) \\ & +\Big[A_2\Big]x^T(n)P^2x(n)+C_1y^T(n-k(n))Q^2_{\max} \\ & \times y(n-k(n))+D_1x^T(n-k_1(n))P^2_{\max}x(n-k_1(n)) \\ \leq\ & x^T(n)\Big[A_1+A_2\Big]x(n)+y^T(n)\Big[B_1+B_2\Big]y(n) \\ & +C_1Q^2_{\max}y(n-k(n))+D_1x^T(n-k_1(n)) \\ & \quad\times P^2_{\max}x(n-k_1(n)) \\ \leq\ & x^T(n)\Big[A^*\Big]x(n)+y^T(n)\Big[B^*\Big]y(n) \\ & +C_1Q^2_{\max}y(n-k(n))+D_1x^T(n-k_1(n)) \\ & \quad\times P^2_{\max}x(n-k_1(n)) \end{aligned} \tag{23}$$

where

$$A_1=\Big[3\Big(A(I^{-1}-\varepsilon^{-1}E_a^TE_a)^{-1}A^T+\varepsilon WW^T\Big)\Big]$$
$$A_2=\Big[3\lambda_{\max}\Big(M(I^{-1}-\varepsilon^{-1}E_m^TE_m)^{-1}M^T+\varepsilon WW^T\Big)\Big]$$
$$B_1=\Big[3\Big(L(I^{-1}-\varepsilon^{-1}E_l^TE_l)^{-1}L^T+\varepsilon WW^T\Big)\Big]$$

$$B_2=\Big[3\lambda_{\max}\Big(B(I^{-1}-\varepsilon^{-1}E_b^TE_b)^{-1}B^T+\varepsilon WW^T\Big)\Big]$$
$$C_1=\Big[3\lambda_{\max}\Big(C(I^{-1}-\varepsilon^{-1}E_c^TE_c)^{-1}C^T+\varepsilon WW^T\Big)\Big]$$
$$D_1=\Big[3\lambda_{\max}\Big(N(I^{-1}-\varepsilon^{-1}E_n^TE_n)^{-1}N^T+\varepsilon WW^T\Big)\Big]$$

For our convenience, we denote

$$A_1+A_2=A^*,\quad B_1+B_2=B^*$$

By the aid of (17)

$$\begin{aligned} \Delta z(n) &= z(n+1)-z(n) \\ &\leq -x^T(n)\Big[I^*-A^*\Big]x(n)-y^T(n)\Big[I^*-B^*\Big]y(n) \\ &\quad +C_1Q^2_{\max}y^T(n-k(n))y(n-k(n)) \\ &\quad +D_1P^2_{\max}x^T(n-k_1(n))x(n-k_1(n)) \end{aligned} \tag{24}$$

Put, $D^*=\max\Big\{A^*,B^*\Big\}$ and

$$\min\Big[1-D^*\Big]=\min\Big\{\min\Big\{1-A_i^*\Big\},\min\Big\{1-B_i^*\Big\}\Big\},$$

and

$$F^*=\max\Big\{C_1,D_1\Big\}$$
and
$$K^*(n)=\min\Big\{k(n),k_1(n)\Big\}$$

Then

$$\begin{aligned}\Delta z(n) \leq & -\min\left\{1-D^*\right\}\left[x^T(n)x(n)+y^T(n)y(n)\right] \\ & +3\lambda_{\max}F^*[x^T(n-k^*(n))x(n-k^*(n)) \\ & +y^T(n-k^*(n))y(n-k^*(n))]+3\lambda_{\max}(N^TN) \\ \leq & -\min\left\{1-D^*\right\}z(n)+\left\{3\lambda_{\max}F^*\right\} \\ & \times\left\{z(n),z(n-1),...,z(n-k^*)\right\} \\ = & -p^*z(n)+q^*\max\left\{z(n),z(n-1),...,z(n-k^*)\right\}\end{aligned} \tag{25}$$

where $k^* = \max\{k^*(n)\}$, $p^* = \left\{1-D^*\right\}$ and $q^* = 3\lambda_{\max}F^*$, I^* be the identity matrix

By Lemma 4.2, if $q^* = 3\lambda_{\max}(F^*) < p^*$ then $\exists$ a scalar $\lambda_0 \in (0,1)$ such that

$$\begin{aligned}z(n) &= x^T(n)x(n)+y^T(n)y(n) \\ &= \sum_{i=1}^{m_1}x_i^2+\sum_{j=1}^{m_2}y_j^2(n) \\ &\leq \max\left\{0,z(0),z(-1),...,z(-k^*)\right\}\lambda_0^n \\ &= \max\left\{0,\sum_{i=1}^{m_1}x_i^2(0)+\sum_{j=1}^{m_2}y_j^2(0),...,\sum_{i=1}^{m_1}x_i^2(-k^*)\right. \\ &\quad \left.+\sum_{j=1}^{m_2}y_j^2(-k^*)\right\}\lambda_0^n \\ &\leq \left[||\phi||^2+||\psi||^2\right]\lambda_0^n\end{aligned} \tag{26}$$

Remark 4.4 In [31], [1], the authors considered parameter uncertainties as a time-varying matrix valued function. Further, the admissible parameters as a time-invariant structure uncertainties was discussed in [33], [36], [37]. However, in the proposed work, we dealt with the exponential stability problem of uncertain BAM neural networks with the help of Halanay type inequality for the first time and uncertainty is examined in the form of time-invariant matrix valued function.

Remark 4.5 With the research motivation of function of the human brain, artificial neural networks are able to learn from experience. These powerful problem solvers are highly effective where traditional, formal analysis would be complexity or impossible. Their strength lies in their ability to make sense out of difficult, noisy, or nonlinear data. Neural networks can provide robust solutions to problems in a wide range of disciplines, particularly areas involving filtering, pattern recognition, classification, prediction, function approximation and optimization. Time delay is unavoidable in real-world neural networks. In practical implementations by electrical circuits, time-delays occur in the finite switching speed amplifiers and the signal transmissions among neurons. As is well known that the existence of time-delays in discrete term causes poor performance, oscillation and instability of the concerned neural networks. Thus, by the help of different Lyapunov-Krasovskii functional, the oscillations and instability are controlled while checking the stability in exponential sense.

4 Illustrative Examples

In this section, two illustrative examples are introduced to demonstrate the less conservativeness of our proposed method.

Example 5.1 The discrete-time delayed Bi-directional associative cellular NNs (6) with the following parameters as below

$$G = \begin{pmatrix} 0.05 & 0.04 \\ 0.07 & 0.02 \end{pmatrix}, \; F = \begin{pmatrix} 0.04 & 0.08 \\ 0.01 & 0.03 \end{pmatrix}, \; K = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.06 \end{pmatrix}$$

and the activation function are taken to be $f(y) = \sin(0.6y)$ and $g(x) = \sin(0.6x)$ which satisfies the hypothesis (4) with

$O_1 = O_2 = 1$.

Furthermore, one can easily check that

$3\lambda_{\max}(G^TG)O_{\max}^2 < \min\{1-3k_i^2-3\lambda_{\max}(F^TF)r_i^2\}$

$3\lambda_{\max}(G^TG)O_{\max}^2 = 0.156$

$\min\{1-3k_i^2-3\lambda_{\max}(F^TF)r_i^2\} = 0.3241$

Therefore, we obtain

$$0.156 < 0.3241 \tag{27}$$

Hence from the Theorem 3.1 which is stated in the main result, it pursue that the neural networks (6) satisfies the definition 2.3. In order to execute the trajectory of the cellular NNs (6) we ratify the following inputs:

$$A = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.6 \end{pmatrix}, \; L = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.7 \end{pmatrix}, \; B = \begin{pmatrix} 0.06 & 1.35 \\ -0.2 & 0.5 \end{pmatrix}$$

$$C = \begin{pmatrix} -1.25 & -0.8 \\ 0.04 & 0.05 \end{pmatrix}, \; M = \begin{pmatrix} 0.7 & -0.8 \\ 0.02 & 0.9 \end{pmatrix},$$

$$N = \begin{pmatrix} 0.09 & 0.02 \\ 0.03 & -1.22 \end{pmatrix}$$

Here the activation functions are defined as

$f_1(y) = \sin(0.6y), \quad g_1(x) = \sin(0.6x)$

$f_2(y) = \sin(0.6y), \quad g_2(x) = \sin(0.6x)$

According to Theorem 3.1, we obtained that system (6) with the above given parameters are exponentially stable with the help of Halanay type inequality and the trajectories of the state variables $x_1(n)$, $x_n(k)$, $y_n(k)$, $y_n(k)$ of the discrete-time BAM neural network (6) are depicted in figure 1.

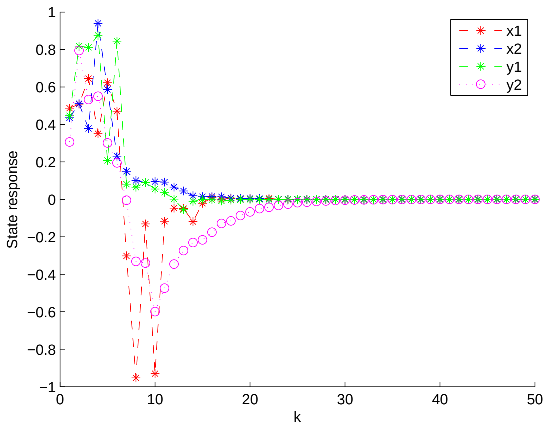

Fig. 1: The state response $x(n)$, $y(n)$ of (6) with BAM neural networks

Remark 5.2 Suppose, we consider a noise disturbance in Neural networks system (6), then it can be reformulated as follows:

$$
\begin{aligned}
x(n+1) &= Ax(n)+B\tilde{f}(y(n))+C\tilde{f}(y(n-k_1(n))) \\
&\quad +\delta\Big(x(n),\, y(n),\, y(n-k_1(n)),\, n\Big)\omega_1(k) \\
y(n+1) &= Ly(n)+M\tilde{g}(x(n))+N\tilde{g}(x(n-k(n))) \\
&\quad +\sigma\Big(y(n),\, x(n),\, x(n-k(n)),\, n\Big)\omega_2(k)
\end{aligned}
\tag{28}
$$

where δ: $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\longrightarrow\mathbb{R}^n$ and σ: $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\longrightarrow\mathbb{R}^n$ are non-linear vector function representing the disturbance intensities. $w_1(k)$ and $w_2(k)$ are the weiner process on the probability space $[\Omega,\,\mathscr{F},\,\beta]$

$$
\begin{aligned}
&\mathbb{E}[w_1(k)]=0,\ \mathbb{E}[w_1^2(k)]=1,\ \mathbb{E}[w_1(i)w_1(j)]=0\ (i\neq j), \\
&\mathbb{E}[w_2(k)]=0,\ \mathbb{E}[w_2^2(k)]=1,\ \mathbb{E}[w_2(i)w_2(j)]=0\ (i\neq j),
\end{aligned}
\tag{29}
$$

and $\mathbb{E}(.)$ being the mathematical expectation operator. The trajectory of the neural network system (28) with the following parameters:

$$
A=\begin{pmatrix}0.09 & 0\\ 0 & 0.6\end{pmatrix},B=\begin{pmatrix}0.7 & -0.8\\ 0.02 & 0.9\end{pmatrix},
$$

$$
C=\begin{pmatrix}-1.25 & -0.8\\ 0.04 & 0.05\end{pmatrix},L=\begin{pmatrix}0.08 & 0\\ 0 & 0.7\end{pmatrix},
$$

$$
M=\begin{pmatrix}0.06 & 1.35\\ -0.2 & 0.05\end{pmatrix},\quad N=\begin{pmatrix}0.09 & 0.02\\ 0.03 & -1.22\end{pmatrix},
$$

$$
\delta=\begin{pmatrix}0.04(\sinh(\frac{\pi}{2})) & 0\\ 0 & 0.06(\sinh(\frac{\pi}{2})-3)\end{pmatrix},
$$

$$
\sigma=\begin{pmatrix}0.08(\sinh(\frac{\pi}{2})) & 0\\ 0 & 0.05(\sinh(\frac{\pi}{2})-0.9)\end{pmatrix}
$$

$\omega1=0.07$; $\omega2=0.09$;. The activation functions are taken as

$$
f_1(y)=g_1(x)=\begin{pmatrix}\tan h(-0.6y)\\ \tan h(-0.6y)\\ \tan h(-0.6y)\end{pmatrix},
$$

$$
f_2(y)=g_2(x)=\begin{pmatrix}\tan h(-0.6x)\\ \tan h(-0.6x)\\ \tan h(-0.6x)\end{pmatrix}
$$

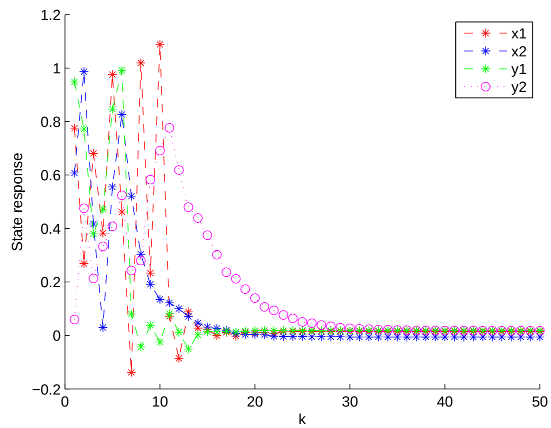

Fig. 2: For ω_1= 0.07, ω_2 = 0.09, the state response $x(n)$, $y(n)$ of (17) with Stochastic Disturbances

According to Remark 5.2, we obtained that system (28) with the above given parameters are stable and the trajectories of $x_1(n)$, $x_2(n)$, $y_1(n)$, $y_2(n)$ of the discrete-time Stochastic BAM neural network (28) are portrayed in figure 2.

Example 5.3 Consider the discrete time uncertain BAM cellular neural networks (17) with the following parameters

$$
A=\begin{pmatrix}0.5 & 0.4\\ 0.7 & 0.08\end{pmatrix},\ B=\begin{pmatrix}0.8 & 0.1\\ 0.07 & 0.05\end{pmatrix},\ C=\begin{pmatrix}0.04 & 0.08\\ 0.4 & 0.06\end{pmatrix}
$$

$$
L=\begin{pmatrix}0.09 & 0.04\\ 0.2 & 0.06\end{pmatrix},\ M=\begin{pmatrix}0.07 & 0.3\\ 0.8 & 0.06\end{pmatrix},\ N=\begin{pmatrix}0.4 & 0.9\\ 0.03 & 0.56\end{pmatrix}
$$

$$
E_a=E_b=E_c=E_l=E_m=E_n=\begin{pmatrix}0.2 & 0.2\\ 0.2 & 0.2\end{pmatrix},
$$

$$
I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ W=\begin{pmatrix}0.1 & 0.1\\ 0.1 & 0.1\end{pmatrix}
$$

$\varepsilon=0.05.$

Furthermore, one can easily check that

$3\lambda_{\max}(F^*) < \min\{1-D^*\}$

$3\lambda_{\max}(F^*) = 0.546$

$\min\{1-D^*\} = 0.728$

Therefore, we obtain

$$0.546 < 0.728 \qquad (30)$$

Hence from the Theorem 4.3, which is stated in the main result, it pursue that the neural networks (17) is satisfied. In order to execute the trajectory of cellular neural networks(17) we follow the following inputs:

$$A = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.0.08 \end{pmatrix}, \quad L = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.09 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.06 & 1.35 \\ -0.2 & 0.05 \end{pmatrix}, C = \begin{pmatrix} -1.25 & -0.8 \\ 0.04 & 0.05 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.7 & -0.8 \\ 0.02 & 0.9 \end{pmatrix}, \quad N = \begin{pmatrix} 0.09 & 0.02 \\ 0.03 & -1.22 \end{pmatrix}$$

Here the activation functions are defined as

$f_1(y) = \exp(-6y), \quad g_1(x) = \sin(0.6x)$

$f_2(y) = \sin(0.6y), \quad g_2(x) = \sin(0.6x)$

According to Theorem 4.3, we obtained that system (17) with the above given parameters are exponentially stable with the help of Halanay type inequality and the trajectories of the state variables $x_1(n), x_2(k), y_1(k), y_2(k)$ of the cellular BAM neural network (17) are depicted in figure .

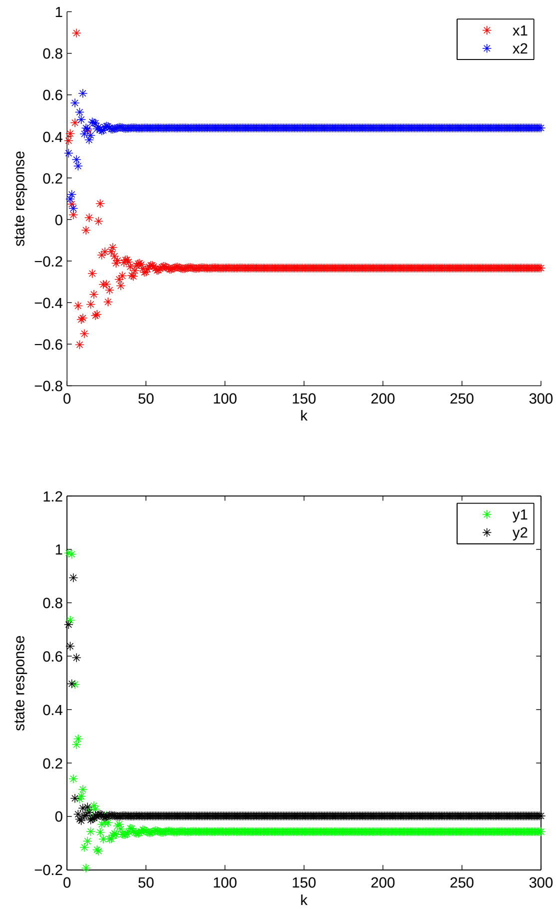

Fig. 3: The state response $x(k)$, $y(k)$ of(ref(18))with uncertain parameters

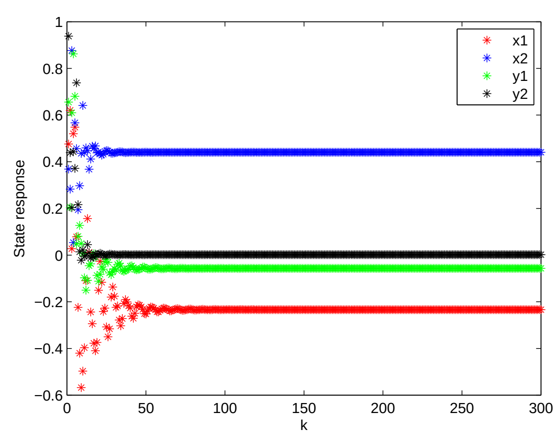

Fig. 4: The state response $x(n)$, $y(n)$ of (17) with BAM neural networks

5 Conclusions

We have investigated a class of discrete time BAM cellular neural networks with time varying delays in this manuscript. Here we introduced a new discrete type Halanay inequalities and helps to study the discretized system of functional difference equation to derive the global stability conditions for non-linear difference equations by using the inequalities obtained. Uncertain parameters with time invariant is introduced using halanay type inequality. Finally, we have given three numerical examples to show the advantages and fruitfulness of our obtained theoretical results.

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