Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120313

An Inverse Source Problem in Time-Space Fractional Differential Equations

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Received: 30 Feb. 2018, Revised: 19 Apr. 2018, Accepted: 21 Apr. 2018
Published online: 1 May 2018

Abstract: In this paper we study a one dimensional inverse source problem. Distinguishability according to the source function in time-space fractional equation $D_t^{\beta}u(x,t) = D_x^{2\alpha}u(x,t) + f(x)$, $\frac{1}{2} < \alpha \leq 1$, $0 < x < 1$, $0 < \beta \leq 1$ with Dirichlet boundary conditions is investigated. In addition to this, the measured output data $h(t)$ determined analytically by a series representation.

Keywords: Inverse source problem, Time-space fractional equation

1 Introduction

Recent decades have witnessed a fast growing applications of fractional partial differential equations to diverse scientific and engineering fields regarding anomalous diffusion, constitutive modeling in viscoelasticity, signal processing and control, fluid mechanics, image processing, aerodynamics, electro-dynamics of complex medium and polymer rheology.

Compared to integer-order calculus, fractional partial differential equations have the capacity of providing a more simple and accurate description of complex mechanical and physical processes featuring history dependency and space non locality and has thus induced the occurrences of a series of fractional differential equations [1,2,3,4,5,6,7].

In this paper, we consider time-space fractional inverse problem with Dirichlet boundary conditions:

$$\begin{aligned} &D_t^{\beta}u(x,t) = D_x^{2\alpha}u(x,t) + f(x) \\ &\qquad \tfrac{1}{2} < \alpha \leq 1, 0 < x < 1, 0 < \beta \leq 1 \\ &u(x,0) = g(x) \quad 0 \leq x \leq 1 \\ &u(0,t) = \Psi_0(t) \quad u(1,t) = \Psi_1(t), 0 \leq t \leq T \end{aligned} \tag{1}$$

The fractional orders $\beta \in (0,1)$ and $\alpha \in \left(\frac{1}{2},1\right)$ are related to the parameters specifying the large-time behavior of the waiting-time distribution or long-range behaviour of the particle jump distribution. In hydrological studies, the parameter α is used to characterize the heterogeneity of porous medium.

In theory, these parameters can be determined from the underlying stochastic model, but often in practice, they are determined from experimental data.

The notation D_t^{β} is the Djrbashian-Caputo derivative operator of order $\beta \in (0,1)$ in the time variable t, and $D_t^{2\alpha}$ denotes the Djrbashian-Caputo derivative of order $\alpha \in \left(\frac{1}{2},1\right)$ in the space variable x.

$D_t^{\beta}u(x,t) = (I^{1-\beta}u')(t)$, $0 < \beta \leq 1$, I^{β} being the Riemann-Liouville fractional integral,

$$(I^{\beta}f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\beta-1}\cdot f(\tau)d\tau & 0 < \beta \leq 1 \\ f(t) & \beta = 0 \end{cases} \tag{2}$$

$\Psi_0(t)$ and $\Psi_1(t)$ are the left and right boundary value functions, respectively. They belong to $C^1[0,T]$. $g(x) \in C^1[0,1]$ is the function that satisfies the following conditions:
$g(0) = \Psi_0(0)$, $g(1) = \Psi_1(0)$.

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Under the conditions (C1) and (C2), the initial-boundary value problem (1) has the unique solution $u(x,t)$. Caputo's fractional derivatives are more widely used in initial value problems of differential equations and have stronger physical interpretations. The Caputo fractional derivative can better reconcile the well-established and polished mathematical theory with practical needs.

For a real number $n-1<\gamma<n$, $n\in\mathbb{N}$ and $f\in H^n(0,1)$ the left-sided Djrbashian-Caputo derivative $D_x^\gamma f$ of order γ is defined by

$$D_x^\gamma f=\frac{1}{\Gamma(n-\gamma)}\int_0^x (x-s)^{n-1-\gamma}f^{(n)}(s)ds, \qquad (3)$$

where $\Gamma(z)$ denotes Euler's Gamma function defined by

$$\Gamma(z)=\int_0^\infty s^{z-1}e^{-s}ds \quad \Re(z)>0 \qquad (4)$$

The Djrbashian-Caputo derivative was first introduced by mathematician Mkhitar M. Djrbashian for studies on space of analytical functions and integral transforms in 1960's (see [8]). Geophysicist Michele Caputo independently proposed the use of the derivative for modeling the dynamics of viscoelastic materials in 1967. We note that there are different definitions of fractional derivatives, notably the Riemann-Liouville fractional derivative, which formally is obtained from (3) by inter charging the order of integration and differentiation, that is the left-sided Riemann-Liouville fractional derivative $D_x^\gamma f$ of order $\gamma\in(n-1,n)$, $n\in\mathbb{N}$ is defined by

$$D_x^\gamma f=\frac{d^n}{dx^n}\frac{1}{\Gamma(n-\gamma)}\int_0^x (x-s)^{n-1-\gamma}f(s)ds \qquad (5)$$

In this work, we will focus on the Djrbashian-Caputo derivative since it allows a convenient treatment of the boundary and initial conditions.

Consider the inverse problem of determining the distinguishability of the unknown source function $f(x)$ from the Neumann type of measured output data at the inner point $x=\frac{1}{2}$. Measured output data is taken at $x=\frac{1}{2}$ since it is the middle point but it can be taken at an inner point at which output data is measured exactly. It depends on the physical conditions of the systems.

$$\Phi[f]=D_x^{2\alpha}u(x,t;f)\mid_{x=\frac{1}{2}} \qquad (6)$$

We can formulate the measured output data $h(t)$ as follows:

$$\Phi[f]=h \quad h\in C^1(0,T] \qquad (7)$$

The inverse problem of determining an unknown source function $f(x)$ is reduced to the problem of invertibility of $\Phi[\cdot]$. This directs us to investigate the distinguishability of the source function via input-output mappings. The mapping $\Phi[\cdot]:K\longrightarrow C^1[0,T]$ has distinguishability property whenever $\Phi[f_1]\neq\Phi[f_2]$ implies $f_1(x)\neq f_2(x)$. Hence, that means injectivity of the inverse mappings Φ^{-1} and Ψ^{-1}. Here, measured output data of Neumann type at the inner point $x=\frac{1}{2}$ is used in the determination of the distinguishability of the unknown source function $f(x)$. As a result of these, in the distinguishability of the unknown source function $f(x)$, analytical results are obtained. Regardless of the validity of auxiliary function what are the sufficient conditions under which the distinguishability of the unknown source function is determined. It is known that approximate initial and boundary conditions there is a solution but by using auxiliary solution we check the closeness of the solution to the right solution.

2 Analysis of the Time-Space Fractional Inverse Problem with Measured Data

Determine the measured measured output data $g(t)$ at the inner point $x=\frac{1}{2}$. We used the Fourier method in the formulation of problem (1). We need to define an auxiliary function $v(x,t)$ as follows:

$$v(x,t)=u(x,t)-x\Psi_1(t)+(x-1)\Psi_0(t) \qquad (8)$$

$v(x,t)$ helps to transform problem (1) into a problem with homogeneous boundary conditions. Therefore (1) can be rewritten in terms of $v(x,t)$ in the following form:

$$\begin{aligned} D_t^\beta v(x,t)-D_t^{2\alpha}v(x,t)&=f(x)-xD_t^\beta\Psi_1(t)+(x-1)D_t^\beta\Psi_0(t)\\ &\quad+\Psi_1(t)D_x^{2\alpha}x-\Psi_0(t)D_x^{2\alpha}(x-1)\\ v(x,0)&=g(x)-x\Psi_1(0)+(x-1)\Psi_0(t)\\ v(0,t)&=v(1,t)=0 \end{aligned} \qquad (9)$$

We can get the unique solution of the initial boundary value problem by using Fourier method:

$$\begin{aligned} v(x,t)&=\sum_{n=1}^{\infty}<\zeta(\theta),\Phi_{n,\alpha}(\theta)>E_{\beta,1}(-\lambda_{n,\alpha}t^\beta)\Phi_{n,\alpha}(x)\\ &\quad+\sum_{n=1}^{\infty}\left\{\int_0^t s^{\beta-1}E_{\beta,\beta}(-\lambda_{n,\alpha}s^\beta)(<\xi(\theta,t-s),\Phi_{n,\alpha}(\theta)>\right.\\ &\quad\left.+<F(\theta),\Phi_{n,\alpha}(\theta)>)d_s\right\}\Phi_{n,\alpha}(x)\\ \zeta(x)&=g(x)-x\Psi_1(0)+(x-1)\Psi_0(t)\\ \xi(x,t)&=f(x)-xD_t^\beta\Psi_1(t)+(x-1)D_t^\beta\Psi_0(t)+\Psi_1(t)D_x^{2\alpha}x\\ &\quad-\Psi_0(t)D_x^{2\alpha}(x-1) \end{aligned} \qquad (10)$$

Moreover,

$$<\zeta(\theta),\Phi_{n,\alpha}(\theta)>=\int_0^1\Phi_{n,\alpha}(\theta)\zeta(\theta)d\theta\ , \qquad (11)$$

$E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function (with $\beta > 0$ and $\alpha \in \mathbb{R}$) defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+\alpha)} , \; z \in \mathbb{C}. \tag{12}$$

This function with $\beta = 1$ was first introduced by Mittag-Leffler in 1903 [16].

It can be verified that $E_{1,1}(z) = e^z$, $E_{2,1}(z) = \cosh\sqrt{z}$ and $E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}$. As a consequence, it represents a generalization of the exponential function in that $E_{1,1}(z) = e^z$. The functions $E_{\beta,1}(-\lambda_{n,\alpha}, t^\beta)$ and $s^{\beta-1}E_{\beta,\beta}(-\lambda_{n,\alpha}s^\beta)$ appear in the kernel of the time fractional diffusion problem with initial data and the right-hand side, respectively. Eigenfunctions also appear in the kernel of fractional Sturm-Liouville problem with a zero potential [9,10,11,12,13,14,15].

Assume that $\Phi_{n,\alpha}(x)$ is the solution of the following Sturm-Liouville problem:

$$\begin{aligned} D^{2\alpha}\Phi(x) &= \lambda\Phi(x) \quad 0 < x < 1 \\ \Phi(0) &= 0 \;,\; \Phi(1) = 0 \end{aligned} \tag{13}$$

where the eigenvalues are $\lambda_{n,\alpha} = (n\pi)^{2\alpha}$, $n = 1,2,\ldots$ and the associated eigenvalues are $\Phi_{n,\alpha} = \sin(n\Pi x)$.

The Neumann type of measured output data at $x = \frac{1}{2}$ can be written in terms of $v(x,t)$ in the following form

$$h(t) = D_x^{2\alpha} v\left(\tfrac{1}{2},t\right) + \Psi_1(t) - \Psi_0(t) \;, \quad t \in (0,T]. \tag{14}$$

In order to arrange (10), set the followings

$$\begin{aligned} z_n(t) &=< \zeta(\theta), \Phi_{n,\alpha}(\theta) > E_{\beta,1}(-\lambda_{n,\alpha}t^\beta) \\ w_n(t) &= \int_0^t s^{\beta-1}E_{\beta,\beta}(-\lambda_{n,\alpha}s^\beta) < \xi(\theta,t-s), \Phi_{n,\alpha}(\theta) > \\ y_n(t) &= \int_0^t s^{\beta-1}E_{\beta,\beta}(-\lambda_{n,\alpha}s^\beta) < F(\theta), \Phi_{n,\alpha}(\theta) > \end{aligned} \tag{15}$$

The solution in terms of $z_n(t), w_n(t), y_n(t)$ can be written in the following form:

$$\begin{aligned} v(x,t) = &\sum_{n=1}^{\infty} z_n(t)\Phi_{n,\alpha}(x) + \sum_{n=1}^{\infty} w_n(t)\Phi_{n,\alpha}(x) \\ &+ \sum_{n=1}^{\infty} y_n(t)\Phi_{n,\alpha}(x). \end{aligned} \tag{16}$$

Now, let us differentiate both sides of (16) with respect to x

$$\begin{aligned} D_x^{2\alpha} v(x,t) = &\sum_{n=1}^{\infty} z_n(t) D_x^{2\alpha}\Phi_{n,\alpha}(x) + \sum_{n=1}^{\infty} w_n(t) D_x^{2\alpha}\Phi_{n,\alpha}(x) \\ &+ \sum_{n=1}^{\infty} y_n(t) D_x^{2\alpha}\Phi_{n,\alpha}(x) \end{aligned} \tag{17}$$

By substituting $x = \frac{1}{2}$ we obtain:

$$\begin{aligned} D_x^{2\alpha} v\left(\tfrac{1}{2},t\right) = &\sum_{n=1}^{\infty} z_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} w_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \\ &+ \sum_{n=1}^{\infty} y_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) . \end{aligned} \tag{18}$$

By considering the over-measured data

$$D_x^{2\alpha} v\left(\tfrac{1}{2},t\right) + \Psi_1(t) - \Psi_0(t) = h(t) \tag{19}$$

we get:

$$\begin{aligned} h(t) = &\Psi_1(t) - \Psi_0(t) + \sum_{n=1}^{\infty} z_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \\ &+ \sum_{n=1}^{\infty} w_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} y_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \end{aligned} \tag{20}$$

. (20) implies that $h(t)$ can be determined analytically.

The right-hand side of the above identity (20) defines the following input-output mapping:

$$\begin{aligned} \Phi[f](t) := &\Psi_1(t) - \Psi_0(t) + \sum_{n=1}^{\infty} z_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \\ &+ \sum_{n=1}^{\infty} w_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} y_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \end{aligned} \tag{21}$$

So we obtain the relation between the source functions and the corresponding outputs $f_j(t) := D_x^{2\alpha} u\left(\frac{1}{2},t;f_j\right)$ by providing the following lemma.

Lemma 1.*Let $v_1(x,t) = v(x,t;f_1)$ and $v_2(x,t) = v(x,t;f_2)$ be the solutions of the direct problem (9) corresponding to the admissible source functions $f_1(x), f_2(x) \in K$. If $h_j(t) = D_x^{2\alpha}(\frac{1}{2},t;f_j) + \Psi_1(t) - \Psi_0(t)$ $j = 1,2$ are the corresponding outputs, the outputs $h_j(t)$, $j = 1,2$, satisfy the following series identity*

$$\Delta h(t) = \sum_{n=1}^{\infty} \Delta w_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} \Delta y_n(t) D_x^{2\alpha}\Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \tag{22}$$

for each $t \in (0,T]$ where $\Delta h(t) = h_1(t) - h_2(t)$, $\Delta w_n(t) = w_n^1(t) - w_n^2(t)$, $\Delta f(t) = f_1(t) - f_2(t)$ and $\Delta y_n(t) = y_n^1 - y_n^2$.

*Proof.*By using identity (20) we can write the measured output data as follows, respectively [16,17,18,19,20].

$$h_j(t) = D_x^{2\alpha} v\left(\tfrac{1}{2},t\right) + \Psi_1(t) - \Psi_0(t) \quad j = 1,2 \tag{23}$$

$$h_1(t) = \Psi_1(t) - \Psi_0(t) + \sum_{n=1}^{\infty} z_n^1(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} w_n^1(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} y_n^1(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \quad (24)$$

$$h_2(t) = \Psi_1(t) - \Psi_0(t) + \sum_{n=1}^{\infty} z_n^2(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} w_n^2(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} y_n^2(t) D_x^{2\alpha} \Phi_{n,\alpha}\left(\tfrac{1}{2}\right) \quad (25)$$

Therefore, the desired result is the difference of these two formulas.

Corollary 1. *Let the conditions of Lemma hold. In addition*

$$< f_1(x) - f_2(x), \Phi_{n,\alpha}(x) >= 0 \quad (26)$$

$\forall\, t \in (0,T]$, $\forall n = 0,1,2,\ldots$ *holds, then* $h_1(t) = h_2(t)$ $\forall\, t \in (0,T]$

Since $\Phi_{n,\alpha}(x)$, $\forall\, n = 0,1,2,\ldots$ *form a basis for the space and* $D_x^{2\alpha}\Phi_{n,\alpha}(x) \neq 0$ $\forall\, n = 0,1,2,\ldots$ *then* $f_1(x) \neq f_2(x)$ *implies that* $< f_1(x) - f_2(x), \Phi_{n,\alpha}(x) >\neq 0$ *at least for some* $n \in N$. *Hence by Lemma 1 we conclude that* $h_1(t) \neq h_2(t)$ *which leads us to the following consequence:* $\Phi[f_1] \neq \Phi[f_2] \Longrightarrow f_1(x) \neq f_2(x)$

Theorem 1. *Assume that* $\Phi[\cdot] : K \longrightarrow C^1[0,T]$ *is the input-output mapping defined by (21) and corresponding to the measured output* $h(t) = D_x^{2\alpha} u\left(\frac{1}{2}, t\right)$. *Then we get that the mapping* $\Phi[f]$ *has the distinguishability in the class of admissible parameters* K*, that is*

$$\Phi[f_1] \neq \Phi[f_2]\ \forall\ f_1, f_2 \in K \Longrightarrow f_1(x) \neq f_2(x) \quad (27)$$

3 Conclusion

Mapping $\Phi[\cdot] : K \to C^1[0,1]$ which is determined by the measured output data at the inner point $x = \frac{1}{2}$ in the time-space fractional diffusion equation.We came to conclude that the distinguishability of the input-output mapping holds, which implies the injectivity of the inverse mapping Φ^{-1}. The measured output data $h(t)$ is obtained analytically by a series representation which leads to the explicit form of the input-output mapping $\Phi[\cdot]$.

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