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Numerical Solution of a Linear Third Order Multi-Point Boundary Value Problems Using Fixed Point Iterative Method

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Abstract: In this paper, we propose a fixed point iteration method similar to Mann iteration process for solving two and three point boundary value problems described by a linear third order differential equation. The convergence of the method is proved and examples are given to demonstrate the accuracy of the method.

Keywords: Fixed point iteration method, Mann iteration method, linear third order differential equations, multipoint boundary value problems

1 Introduction

Many problems in mathematics and engineering sciences are formulated in boundary value problems for third order differential equations as in physical oceanography and in the frame work of variational theory, the deflection of curved beam having a constant or varying cross-section, three-layer beam, the motion of rocket, chemical engineering, underground water flow, plasma physics, electromagnetic waves, the study of stellar interiors. For more detail see [1]. A number of physical of technological problems lead to the question of formulating mathematical models for describing a given process or a given structure. From mathematical point of view, these problems often lead to differential equations. The problems of regulation and control of some actions by a control lever or by a signal reduce to solving third order equation [2]. Several studies have been conducted on the solutions of third order boundary value problems. For example, the study Akram et al.[3] developed an approximate method for the solution of third order differential equation with two and three point boundary conditions using iterative reproducing kernel while the study Hossain et al.[4] presented Galerkin weighted residual method for constructing the numerical solution of third order linear and non-linear boundary value problems with two point boundary conditions. Abd El-Salam et al.[5] developed a second and fourth order convergent methods based on Quartic non-polynomial Spline function for the numerical solution of a third order two point boundary value problem.In the study [6], a multiple finite difference method from a continuous k-step linear multistep method is derived and applied to solve third order boundary value problem. The convergence of the method is established through consistency and zero-stability by expressing them as a block method. In Sahi et al.[7] a fourth order derivative method with continuous coefficients is derived and used to obtain main method and additional method. The additional method is used to solve third order boundary value problem. Taha and Khledha [8] proposed the numerical scheme for the numerical solution of the third order two point boundary value problems using non-polynomial Spline method with finite difference method (FDM). In same vein, the study [9], investigates the existence and uniqueness of the solution of the general boundary value problem for the third order nonlinear ordinary differential equation. While our study [10], proposed a fixed point iterative method for the solution of two point boundary value problem for a second order differential equations. Recently Abushammala et al.[1] have developed a new alternative uniformly convergent iterative scheme for the solution of an extended class of linear and nonlinear third order two point boundary value problem. The method is based on

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embedding Green?s functions into well fixed point iterations, including Picard?s and Kranoselskii-Mann?scheme. In this paper, we propose a fixed point iterative method without constructing Green?s functions which approximates the solution of two and three point boundary value problems for linear third order differential equations. The rest of the paper is organized as follows: Section 2 discusses the fixed point iteration method while Section 3 addresses the implementation of the method through numerical examples. Section 4 concludes the paper.

2 Fixed Point Iteration Method

Given the following three-point boundary value problem:

$$y''' = f(x,y,y',y''), (x,y,y',y'') \in (a_1,a_3)X\Re^3 \quad (1)$$

$$\begin{aligned} \alpha_1 y(a_1) + \alpha_2 y'(a_1) + \alpha_3 y''(a_1) &= A_1 \\ \beta_1 y(a_2) + \beta_2 y'(a_2) + \beta_3 y''(a_2) &= A_2 \\ \gamma_1 y(a_3) + \gamma_2 y'(a_3) + \gamma_3 y''(a_3) &= A_3 \end{aligned} \quad (2)$$

where a_i, α_i, β_i $\gamma_i, A_i \in \Re, i = 1,2,3, a_1 \prec a_2 \prec a_{3,} \sum_{i=1}^{3} |\alpha_i| \succ 0$ $\sum_{i=1}^{3} |\beta_i| \succ 0, \sum_{i=1}^{3} |\gamma_i| \succ 0$
We proposed the fixed point iteration method to approximate the solution of (1) and (2)

From [2], it is shown that any solution$y(x)$ of the boundary value problem (1) and (2) solves the integro-differential equation:

$$y(x) = \Psi(x) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(x,s) f(x,y(s),y'(s),y''(s))ds \quad (3)$$

and conversely, where $G_K(x,s)$is the particular Green's function for the boundary value problem $y''' = 0$ with the homogeneous boundary conditions obtained from (2) by putting $A_1 = A_2 = A_3 = 0$. $\Psi(x)$is a solution of the boundary value problem $y''' = 0$ with the boundary conditions (2).

To construct the proposed fixed point iteration method, we first transform (1) and (2) into (4) or (5) as follows:

$$\begin{aligned} y'''_{n+1} &= (1-\alpha_n)y'''_n + \alpha_n y'''_n = (1-\alpha_n)y'''_n + \alpha_n f(x,y_n,y'_n,y''_n) \\ &\alpha_1 y_{n+1}(a_1) + \alpha_2 y'_{n+1}(a_1) + \alpha_3 y''_{n+1}(a_1) = A_1 \\ &\beta_1 y_{n+1}(a_2) + \beta_2 y'_{n+1}(a_2) + \beta_3 y''_{n+1}(a_2) = A_2 \\ &\gamma_1 y_{n+1}(a_3) + \gamma_2 y'_{n+1}(a_3) + \gamma_3 y''_{n+1}(a_3) = A_3 \end{aligned} \quad (4)$$

or

$$\begin{aligned} y'''_{n+1} &= \alpha_n y'''_n + (1-\alpha_n)y'''_n = \alpha_n y'''_n + (1-\alpha_n) f(x,y_n,y'_n,y''_n) \\ &\alpha_1 y_{n+1}(a_1) + \alpha_2 y'_{n+1}(a_1) + \alpha_3 y''_{n+1}(a_1) = A_1 \\ &\beta_1 y_{n+1}(a_2) + \beta_2 y'_{n+1}(a_2) + \beta_3 y''_{n+1}(a_2) = A_2 \\ &\gamma_1 y_{n+1}(a_3) + \gamma_2 y'_{n+1}(a_3) + \gamma_3 y''_{n+1}(a_3) = A_3 \end{aligned} \quad (5)$$

provided $0 \le \alpha_n \le 1$and $\sum_{n=0}^{\infty} \alpha_n = \infty$ (Chidume [11]) Then, we let $T : C^{(??)}[a_1,a_3] \to C^{(2)}[a_1,a_3]$ be defined by

$$T[y(x)] \quad = \Psi(x) + \textstyle\sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k \quad (6)$$

$$(x,s) f(x,y(s),y'(s),y''(s))ds \quad (7)$$

where T is an operator such that any *y(x)* solution of (1) and (2) is a fixed point.

2.1 Convergence

This section describes the convergence of the proposed method but first we state the existing method as follows:

Theorem 1((Mann Iteration Method)). *(Chidume [11]) Let D be a non-empty convex subset of a real Banach Space X and $T : D \to D$ be a mapping. The sequence $\{y_n\} \subset D$ is called Mann iteration method and is defined by*

$$y_0 \in D, y_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, n \ge 0 \quad (8)$$

where $\{\alpha_n\}$is a real sequence in $[0,1]$ that satisfies some additive condition ($\sum_{n=o}^{\infty} \alpha_n = \infty$).

Then, the proposed method (4) is compared with existing method (8) to find their equivalence. The equivalence is obtained as follows:

Firstly, we differentiate (6) twice to obtain

$$(T[y_n(x)])'' = \Psi''(x) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} \frac{\partial^2}{\partial x^2} G_K(x,s) f(s,y_n(s),y'_n(s),y''_n(s))ds \quad (9)$$

Secondly, (8) is differentiated three times to obtain

$$y'''_{n+1} = (1-\alpha_n)y'''_n + \alpha_n (Ty_n)''' \quad (10)$$

Thirdly, (9) is differentiated only once to obtain

$$(T[y_n(x)])''' = \Psi'''(x) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} \frac{\partial^3}{\partial x^3} G_K(x,s) f(s,y_n(s),y'_n(s),y''_n(s))ds \quad (11)$$

Fourthly, we substitute (11) into (10) to obtain

$$y'''_{n+1} = (1-\alpha_n)y'''_n + \alpha_n \left(\Psi'''(x) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} \frac{\partial^3}{\partial x^3} G_k(x,s) f(s,y_n(s),y'_n(s),y''_n(s))ds \right) \quad (12)$$

Notice that the quantity inside the brackets of (12) is obtained by differentiating (3) three times. If we denoted it by y_n''', then (12) can be written in the form,

$$y_{n+1}''' = (1-\alpha_n)y_n''' + \alpha_n y_n''' = (1-\alpha_n)y_n''' + \alpha_n f(x, y_n, y_n', y_n'')$$

which is the same thing as the proposed fixed point iteration method (4) for the solution of (1) and (2).

Finally, we prove the convergence of the proposed method and to do that, we first state the result as follows:

Theorem 2.*Let $\{\alpha_n\}_{n\geq 0}$ be the sequence of real numbers satisfying the following conditions: (i) $0 \leq \alpha_n \leq 1$ and (ii) $\sum_{n=o}^{\infty} \alpha_n = \infty$. Also let the operator T defined in (6) be contractive with constant of contraction $L \in (0,1)$. Then the sequence $\{y_n\}_{n\geq 0}$ in $C^{(2)}[a_1, a_3]$ generated by $y_0 \in C^{(2)}[a_1, a_3]$, where y_0 is obtained from $y''' = 0$ and the boundary condition (2) and $y_{n+1} = (1-\alpha_n)y_n + \alpha_n(Ty_n)$ converges to the unique solution y^* in $C^{(??)}[a_1, a_3]$ of (1) and (2).*

*Proof.*Given $y_{n+1} = (1-\alpha_n)y_n + \alpha_n(Ty_n)$, we have to prove $y_n \to y^*$ as $n \to \infty$ Let

$$\rho_n = \|y_n - y^*\| \; \rho_{n+1} = \|y_{n+1} - y^*\|$$
$$= \|(1-\alpha_n)(y_n - y^*) + \alpha_n(Ty_n - y^*)\|$$
$$\leq \|(1-\alpha_n)(y_n - y^*)\| + \|\alpha_n(Ty_n - y^*)\|$$

Recall that if y^* is a solution to (1) and (2), this implies that $Ty^* = y^*$. Therefore (13) becomes

$$\rho_{n+1} = \|y_{n+1} - y^*\| \leq \|(1-\alpha_n)(y_n - y^*)\| + \|\alpha_n(Ty_n - y^*)\|$$
$$\leq (\,|1-\alpha_n| + |\alpha_n L|\,)\,\|(y_n - y^*)\|$$
$$= (1-\alpha_n(1-L))\,\|y_n - y^*\|$$

Therefore, $\rho_{n+1} \leq e^{-\alpha_n(1-L)}\rho_n$

$$\rho_1 \leq e^{-\alpha_0(1-L)}\rho_0$$
$$\rho_2 \leq e^{-\alpha_1(1-L)}\rho_1 \leq e^{-\alpha_1(1-L)}e^{-(1-L)(\alpha_0+\alpha_1)}\rho_0 = e^{-(1-L)(\alpha_0+\alpha_1+\alpha_2)}\rho_0$$
$$\rho_{n+1} = e^{-(1-L)(\alpha_0+\alpha_1+\alpha_2+\ldots+\alpha_n)}\rho_0 = e^{-(1-L)\sum_{i=0}^{n}\alpha_i}\rho_0$$

From (ii) in Theorem 2 we have $y_n \to y^*$ as $n \to \infty$

3 Numerical Results

In this section, three examples are solved to test the accuracy of the proposed method. The results obtained are compared with the exact solutions and Akram et al.[3]. All computations are carried out with Maple 13.

*Example 1.*Source [4]. Consider the following two point boundary value problem:

$$y''' + y'' + y' + y = x \tag{13}$$

$$y(0) = 1, y(1) = 0, y'(0) = 1 \tag{14}$$

The exact solution of the problem is:

$$y_E(x) = x + \frac{1}{e^x} - \frac{(\frac{1}{e} - 1)\sin x}{\sin 1 - 1} \tag{15}$$

Applying the proposed method (4), the problem is transformed and the following algorithm is obtained:

$$y_{n+1}''' = (1-\alpha_n)y_n''' + \alpha_n(x - y_n'' - y_n' - y_n) \tag{16}$$

$$y_{n+1}(0) = 1, y_{n+1}(1) = 0, y_{n+1}'(0) = 1 \tag{17}$$
$$y_0(x) = -2x^2 + x + 1 \tag{18}$$

Here $y_0(x)$is the solution of $y''' = 0$ with the boundary conditions$y(0) = 1, y(\mathbf{??}) = 0, y'(0) = 1$. Integrating (16) three times and imposing the boundary conditions (17), the required approximate solution $y_{n+1}$is obtained.

The results obtained for different values of the parameter α_n at 12^{th} iteration of the algorithm are presented in Table 3.1 and Table 3.2 along with the result of the exact solution. Table 3.1 shows the value of the approximate solutionsy_{12}, namely twelfth iteration at different points. From the tables, it observed that the results are comparable with exact solution and the choice of $\alpha_n = 1$gives the best accuracy.

*Example 2.*Source[3]. Consider the following third order three- point boundary value problem:

$$y''' + xy'' = -6x^2 + 3x - 6 \tag{19}$$

$$y(0) = 0, y'(0) = 0, y'(1) = y'(\frac{1}{2}) - \frac{3}{4} \tag{20}$$

The exact solution of this problem is:

$$y_E(x) = \frac{3}{2}x^2 - x^3$$

Applying the proposed method (4), the problem is transformed in the following algorithm:

$$y_{n+1}''' = (1-\alpha_n)y_n''' + \alpha_n(-6x^2 + 3x - 6 - xy_n'') \tag{21}$$

$$y_{n+1}(0) = \quad 1, y_{n+1}'(0) = 0, y_{n+1}'(1) = y_{n+1}'(\frac{1}{2}) - \frac{3}{4}$$
$$y_0(x) = -\frac{3}{4}x^2$$

Here $y_0(x)$is the solution of $y''' = 0$ with the boundary conditions (20). Integrating (21) three times and imposing the boundary conditions (22), the required approximate solution $y_{n+1}$is obtained.

Example 3.: Source [2]. Consider the Sandwich problem: The shear deformation $y(x)$ of Sandwich beams is governed by a linear third order differential equation with the boundary conditions at three different points is given as:

$$y''' - k^2y' + r = 0 \tag{22}$$

$$y'(0) = y'(1) = 0, y(\frac{1}{2}) = 0 \tag{23}$$

Table 1: Comparison of numerical results for (1) at 12^{th} iteration

x	Exact sol.	$\alpha_n = 1$	$\alpha_n = 0.99$	$\alpha_n = 0.979$
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.0743953317	1.074395331	1.074457194	1.074525117
0,2	1.0998344155	1.099834415	1.100051801	1.100290604
0.3	1,0798437219	1.079843721	1.080266301	1.080730761
0.4	1.0181389480	1.018138949	1.018774117	1.019472599
0.5	0.9186130796	0.918613079	0.9194279821	0.9203245901
0.6	0.7853202863	0.785320286	0.7862442096	0.7872613
0.7	0.6224560361	0.622456036	0.6233831638	0.6244043
0.8	0.434333836	0.4343338360	0.4351263802	0.43599980
0.9	0.225359018	0.225359018	0.2253590187	0.2263929
1.0	0.000000000	0.000000000	0.0000000000	0.0000000

Table 2: Comparison of absolute errors in numerical results for Example 1 at 12^{th} iteration

x	$\alpha_n = 1$	$\alpha_n = 0.99$	$\alpha_n = 0.979$
0.1	7E-10	6.18623E-05	1.297853E-04
0,2	5E-10	2.173855E-04	4.561885E-04
0.3	9E-10	4.225791E-04	8.870391E-04
0.4	1E-9	6.35169E-04	1.333651E-03
0.5	1E-10	8.149025E-04	1.7115105E-03
0.6	1E-10	9.239233E-04	1.941026E-04
0.7	3E-10	9.271277E-04	1.9483077E-03
0.8	0.0	7.929966E-04	1.6659641E-03
0.9	5E-10	4.917053E-04	1.0338843E-03
1.0	0.0	0.0	0.0

Table 3: Comparison of numerical results for Example 2 at 12^{th} Iteration

x	Exact sol.	$\alpha_n = 1$	$\alpha_n = 0.99$	$\alpha_n = 0.979$
0.1	0.01400000000	0.01400000000	0.01376547980	0.01350799010
0.2	0.05200000000	0.05200000002	0.05110405386	0.05012038885
0.3	0.10800000000	0.1080000000	0.1060814149	0.1039750063
0.4	0.1760000000	0.1760000001	0.1727654973	0.1692143032
0.5	0.2500000000	0.2500000002	0.2452251015	0.2399825401
0.6	0.3240000000	0.32400000002	0.3175285328	0.3104229549
0.7	0.3920000000	0.3920000003	0.3837423086	0.3746750786
0.8	0.4480000000	0.4480000004	0.4379299771	0.4268722846
0.9	0.4860000000	0.4860000000	0.4741510908	0.4611396513
1.0	0.500000000	0.500000004	0.48664603622	0.4715921951

Table 4: Comparison of absolute errors in numerical results for Example 2 at 12^{th} iteration

X	$\alpha_n = 1$	$\alpha_n = 0.99$	$\alpha_n = 0.979$	Akram et al. [2] (n = 100)
0.1	0.0	2.345202E-05	4.920099E-05	5.17E-09
0.2	2E-11	8.9594614E-05	1.87961115E-04	4.13E-08
0.3	0.0	1.918585E-04	4.0249937E-04	1.39E-07
0.4	1E-10	3.2345027E-04	6.7856968E-04	3.32E-07
0.5	2E-10	4.7778985E-04	1.0017459E-02	6.53E-07
0.6	2E-10	6.4714672E-04	1.3577045E-02	1.13E-06
0.7	3E-10	8.2576914E-04	1.7324921E-02	1.81E-06
0.8	4E-10	1.0070022E-02	2.1127715E-02	2.73E-06
0.9	0.0	1.1848909E-02	2.24860348E-02	3.94E-06
1.0	4E-09	1.3539637E-02	2.8407804E-02	5.48E-06

Table 5: Comparison of numerical results for Example 3 when r = k= 1 at 12^{th} Iteration

x	Exact sol.	$\alpha_n = 1$	$\alpha_n = 0.98$	$\alpha_n = 0.77$
0.0	-0.0378828427	-0.03788284274	-0.03788284274	-0.03788284274
0.1	-0.0357370808	-0.03573708084	-0.03573708084	-0.03573708084
0.2	-0.02994565300	-0.02994565322	-0.02994565322	-0.02994565322
0.3	-0.0214514309	-0.02145143092	-0.02145143092	-0.02145143092
0.4	-0.0111702344	-0.01117023454	-0.01117023454	-0.01117023454
0.5	0.0000000000	2.1299277E-12	3.155678734E-19	2.162127113E-13
0.6	0.0111702345	0.01117023454	0.01117023453	0.01117023453
0.7	0.0214514312	0.02145143090	0.02145143090	0.02145143089
0.8	0.02994456531	0.02994565318	0.02994565318	0.02994565317
0.9	0.0357370804	0.03573708086	0.03573708084	0.03573708082
1.0	0.03788284310	0.03788284274	0.03788284274	0.03788284274

Table 6: Comparison of absolute errors in numerical results for Example 3 at 12^{th} iteration

x	$\alpha_n = 1$	$\alpha_n = 0.98$	$\alpha_n = 0.77$
0.0	4E-11	4E-11	4E-11
0.1	4E-11	4E-11	4E-11
0.2	2.2E-10	2.2E-10	2.2E-10
0.3	2E-11	2E-11	2E-11
0.4	1.4E-10	1.4E-10	1.4E-10
0.5	2.12E-12	3.15E-19	2.16E-13
0.6	4E-11	3E-11	3E-11
0.7	3E-10	3E-10	3.1E-10
0.8	8E-11	8E-11	7E-11
0.9	4.6E-10	4.4E-10	4.2E-10
1.0	3.6E-10	3.6E-10	3.6E-10

Table 7: Comparison of the results for the Exact solution and the proposed method for Example 4 at 4^{th} iteration.

x	Exact sol.	$\alpha_n = 1$	Abso.err
0.1	-0.08985007515	-0.0898500749	1.7E-10
0.2	-0.1589354694	-0.1589354646	4.8E-09
0.3	-0.2068641772	-0.2068641447	3.25E-08
0.4	-0.2336510423	-0.2336510054	3.69E-08
0.5	-0.239712823	-0.2397127693	5.37E-08
0.6	-0.225857066	-0.2258569894	7.66E-08
0.7	-0.193265402	-0.1932653062	9.58E-08
0.8	-0.143471331	-0.1434712182	1.12E-07
0.9	-0.078332817	-0.0783326909	1.26E-07
1.0	0.0	1.27E-07	1.27E-07

where $k^2$and r are physical parameters depending on the elasticity of the layers. The exact solution of this problem is:

$$y(x) = \frac{r(k(2x-1) - 2\sinh(kx) + 2\cosh(kx)\tan(\frac{k}{2}))}{2k^3}$$

Applying the proposed method (4) to (22) and (23) with r=k=1, the problem is transformed into the following algorithm:

$$y'''_{n+1} = (1-\alpha_n)y'''_n + \alpha_n(y'_n - 1) \tag{24}$$

$$y'_{n+1}(0) = y'_{n+1}(1) = 0, y'_{n+1}(\frac{1}{2}) = 0 \tag{25}$$

$$y_0(x) = 0$$

Here $y_0(x)$is the solution of $y''' = 0$ with the boundary conditions (23).

Integrating (24) three times and imposing the boundary conditions (25), the approximate solution $y_{n+1}$is obtained. The results obtained for different values of the parameter α_n at 12^{th} iteration of the algorithm are reported in Table 3.5 and Table 3.6. From the tables, it is clear that the results are comparable with the exact solution and the values of $\alpha_n$give almost same accuracy.

*Example 4.*Source[5]. Consider the following boundary value problem:

$$y''' + y = (x+4)\sin x + (1-x)\cos x \tag{26}$$

$$y(0) = 0, y'(0) = -1, y'(1) = \sin 1 \tag{27}$$

The exact solution of this problem is

$$y_E(x) = (x-1)\sin x$$

Applying the proposed method (4), the problem is converted in the following algorithm:

$$y'''_{n+1} = (1-\alpha_n)y'''_n + \alpha_n((x-4)\sin x + (1-x)\cos x - y_n) \tag{28}$$

$$y_{n+1}(0) = 0, y'_{n+1}(1) = -1, y'_{n+1}(1) = \sin 1 \tag{29}$$

$$y_0(x) = -x + \frac{1+\sin 1}{2}x^2$$

Here $y_0(x)$ is the solution of $y''' = 0$ subject to boundary condition (27). Integrating Eq. (28) three times and imposing the boundary condition (29), the approximate y_{n+1} is obtained.

The results obtained for the value of $\alpha_n = 1$ at 4^{th} iteration of the algorithm are reported in Table 7 shows the comparison with the exact solution. It is observed from the table that with few iterations, the order of the error is encouraging, which indicates the assurance of the convergence.

4 Perspective

In this paper, a fixed point iteration method is proposed for solving linear third order multi-point boundary value problems. The results obtained are in excellent agreement with exact solution and some existing methods in the literature.

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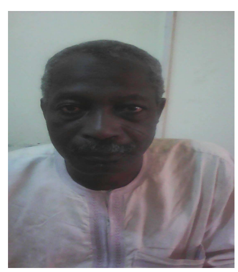

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