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A New Proof of Extended Watson Summation Theorem

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Abstract: In the theory of generalized hypergeometric series, classical Watson summation theorem play an important role. In 2010, Kim et al. have given two extensions of the classical Watson summation theorem. In this note, we aim to provide a new proof for one of the extended Watson summation formula for the series ${}_4F_3$.

Keywords: Generalized hypergeometric function, Watson theorem, Extended Watson theorem, Gauss theorem

1 Introduction

The hypergeometric functions are particular types among the special functions and can be represented by the hypergeometric series. These functions have many applications in mathematical as well as in physical sciences. In particular, the classical summation theorems for the generalized hypergeometric series such as Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; similarly, Watson, Dixon and Whipple for the series ${}_3F_2$ play key roles in theory and applications. Bailey in [1] pointed out the several interesting applications by using the above summation theorems. Further, in 2010, these summation theorems were extended by Kim *et al.* [4].

In the present study we recall some of results that we need for the future development of our proof.

1.1 Extension of Gauss second summation theorem, [4]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \delta+1\\ \tfrac{1}{2}(\alpha+\beta+3), & \delta\end{matrix};\frac{1}{2}\right] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(\alpha+\beta+3))\Gamma(\frac{1}{2}(\alpha-\beta-1))}{\Gamma(\frac{1}{2}(\alpha-\beta-3))} \\
&\times\left\{\frac{\frac{1}{2}(\alpha+\beta-1)-\frac{\alpha\beta}{\delta}}{\Gamma(\frac{1}{2}(\alpha+1))\Gamma(\frac{1}{2}(\beta+1))}+\frac{\frac{\alpha+\beta+1}{\delta}-2}{\Gamma(\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)}\right\}.
\end{aligned}
\tag{1}
$$

For $\delta=\frac{1}{2}(\alpha+\beta+1)$, (1) reduces to the well known Gauss second summation theorem [2,8] viz.

$$
{}_2F_1\left[\begin{matrix}\alpha, & \beta\\ \tfrac{1}{2}(\alpha+\beta+1)\end{matrix};\frac{1}{2}\right]=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(\alpha+\beta+1))}{\Gamma(\frac{1}{2}(\alpha+1))\Gamma(\frac{1}{2}(\beta+1))}. \tag{2}
$$

1.2 Extension of Watson summation theorem, [4]

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\alpha, & \beta, & \gamma, & \delta+1\\ \tfrac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta\end{matrix};1\right] \\
&= \frac{2^{\alpha+\beta-2}\Gamma(\gamma+\frac{1}{2})\Gamma(\frac{1}{2}(\alpha+\beta+3))\Gamma(\gamma-\frac{1}{2}(\alpha+\beta+1))}{(\alpha-\beta-1)(\alpha-\beta+1)\Gamma(\frac{1}{2})\Gamma(\alpha)\Gamma(\beta)} \\
&\times\left\{\gamma_1\frac{\Gamma(\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)}{\Gamma(\gamma-\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\beta+\frac{1}{2})}+\gamma_2\frac{\Gamma(\frac{1}{2}(\alpha+1))\Gamma(\frac{1}{2}(\beta+1))}{\Gamma(\gamma-\frac{1}{2}\alpha)\Gamma(\gamma-\frac{1}{2}\beta)}\right\},
\end{aligned}
\tag{3}
$$

provided $\mathrm{Re}(2\gamma-\alpha-\beta)>1$.

Also, the constants γ_1 and γ_2 are given by

$$
\gamma_1=\alpha(2\gamma-\alpha)+\beta(2\gamma-\beta)-2\gamma+1-\frac{\alpha\beta}{\delta}(4\gamma-\alpha-\beta-1) \tag{4}
$$

and

$$
\gamma_2=\frac{4}{\delta}(\alpha+\beta+1)-8. \tag{5}
$$

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For $\delta = \frac{1}{2}(\alpha+\beta+1)$, (3) reduces to the classical Watson summation theorem [2] viz.

$$
{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma \\ \frac{1}{2}(\alpha+\beta+1), & 2\gamma\end{matrix};1\right] \tag{6}
$$
$$
= \frac{\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})\Gamma(\frac{1}{2}(\alpha+\beta+1))\Gamma(\gamma-\frac{1}{2}(\alpha+\beta-1))}{\Gamma(\frac{1}{2}(\alpha+1))\Gamma(\frac{1}{2}(\beta+1))\Gamma(\gamma-\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\beta+\frac{1}{2})},
$$

provided $\mathrm{Re}(2\gamma-\alpha-\beta) > -1$.

1.3 Gauss summation theorem, [2,8]

$$
{}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & \gamma\end{matrix};1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{7}
$$

provided $\mathrm{Re}(\gamma-\alpha-\beta) > 0$.

A special case of (7) [5, p49]

$$
{}_2F_1\left[\begin{matrix}-\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2} \\ & \gamma+\frac{1}{2}\end{matrix};1\right] = \frac{2^n(\gamma)_n}{(2\gamma)_n}. \tag{8}
$$

It is interesting to mention here that Kim *et al.* [4] established the result (3) by using contiguous functions relation together with Watson summation theorem (6) and its contiguous result which was obtained by Lavoie *et al.* [7] and Cho et al. [3] established by using [9] and [8].

In this note, we aim to provide a new proof of the extended Watson summation theorem (3).

2 Derivation of (3)

In order to derive (3), we proceed as follows. Consider the integral

$$
I = \int_0^\infty e^{-t}t^{\beta-1}\,{}_3F_3\left[\begin{matrix}\alpha, & \gamma, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta\end{matrix};t\right]dt
$$

for $\mathrm{Re}(\beta) > 0$.

Expressing the ${}_3F_3$ as a series and changing the order of integration since the series converges uniformly, we have

$$
I = \sum_{n=0}^\infty \frac{(\alpha)_n(\gamma)_n(\delta+1)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(2\gamma)_n(\delta)_n n!}\int_0^\infty e^{-t}t^{\beta+n-1}\,dt.
$$

Evaluating the well known gamma integral and making use of the relation of following Pochhammer symbol with gamma function

$$
(\beta)_n = \frac{\Gamma(\beta+n)}{\Gamma(\beta)}
$$

we obtain, after some arrangement

$$
I = \Gamma(\beta)\sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n(\gamma)_n(\delta+1)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(2\gamma)_n(\delta)_n n!}. \tag{9}
$$

Now summing up the terms of series, we have

$$
I = \Gamma(\beta)\,{}_4F_3\left[\begin{matrix}\alpha, & \beta, & \gamma, & \delta+1 \\ & \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2}, & 2\gamma, & \delta\end{matrix};1\right]. \tag{10}
$$

On the other hand, writing (9) in the form

$$
I = \Gamma(\beta)\sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n(\delta+1)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(\delta)_n 2^n n!}\left\{\frac{2^n(\gamma)_n}{(2\gamma)_n}\right\}.
$$

Now, using the result (8), we have

$$
I = \Gamma(\beta)\sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n(\delta+1)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(\delta)_n 2^n n!}
\times{}_2F_1\left[\begin{matrix}-\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2} \\ & \gamma+\frac{1}{2}\end{matrix};1\right].
$$

Now, writing ${}_2F_1$ as a series, we have after some calculation

$$
I = \Gamma(\beta)\sum_{n=0}^\infty\sum_{m=0}^{[\frac{n}{2}]} \frac{(\alpha)_n(\beta)_n(\delta+1)_n(-\frac{1}{2}n)_m(-\frac{1}{2}n+\frac{1}{2})_m}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(\delta)_n 2^n(\gamma+\frac{1}{2})_m m!\,n!}.
$$

Using the identity

$$
(-n)_{2m} = 2^{2m}\left(-\frac{1}{2}n\right)_m\left(-\frac{1}{2}n+\frac{1}{2}\right)_m = \frac{n!}{(n-2m)!},
$$

we have

$$
I = \Gamma(\beta)
\sum_{n=0}^\infty\sum_{m=0}^{[\frac{n}{2}]} \frac{(\alpha)_n(\beta)_n(\delta+1)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_n(\gamma+\frac{1}{2})_m(\delta)_n 2^{2m+n} m!\,(n-2m)!}.
$$

Now substituting n by $n+2m$ and making use of the known result in [8, Eq.8, p57]

$$
\sum_{n=0}^\infty\sum_{m=0}^{[\frac{n}{2}]} C(m,n) = \sum_{n=0}^\infty\sum_{m=0}^\infty C(m,n+2m),
$$

we have

$$
I = \Gamma(\beta)
\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(\alpha)_{n+2m}(\beta)_{n+2m}(\delta+1)_{n+2m}}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_{n+2m}(\gamma+\frac{1}{2})_m(\delta)_{n+2m} 2^{n+4m} m!\,n!}.
$$

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Using the identity

$$(\beta)_{n+2m} = (\beta)_{2m}(\beta+2m)_n$$

and after some calculation, we have

$$\begin{aligned} I =& \Gamma(\beta)\sum_{n=0}^{\infty}\frac{(\alpha)_{2m}\,(\beta)_{2m}\,(\delta+1)_{2m}}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_{2m}\,(\gamma+\frac{1}{2})_m\,(\delta)_{2m}\,2^{4m}\,m!} \\ &\times\sum_{n=0}^{\infty}\frac{(\alpha+2m)_n\,(\beta+2m)_n\,(\delta+1+2m)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2}+2m)_n\,(\delta+2m)_n\,2^n\,n!}. \end{aligned}$$

Summing up the inner series, thus we have

$$\begin{aligned} I =& \Gamma(\beta)\sum_{n=0}^{\infty}\frac{(\alpha)_{2m}\,(\beta)_{2m}\,(\delta+1)_{2m}}{(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})_{2m}\,(\gamma+\frac{1}{2})_m\,(\delta)_{2m}\,2^{4m}\,m!} \\ &\times {}_3F_2\left[\begin{matrix}\alpha+2m, & \beta+2m, & \delta+1+2m \\ & \frac{1}{2}(\alpha+\beta+3)+2m, & \delta+2m\end{matrix};\frac{1}{2}\right]. \end{aligned}$$

We now observe that the ${}_3F_2$ appearing can be computed with the help of (1) and after much simplification, seperating into two terms and finally applying (7), we get

$$\begin{aligned} I =& \Gamma(\beta)\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2})\,\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{3}{2})} \qquad (11) \\ &\times\left\{\frac{\frac{1}{2}(\alpha+\beta-1)-\frac{\alpha\delta}{\delta}}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})}+\frac{\frac{\alpha+\beta+1}{\delta}-2}{\Gamma(\frac{1}{2}\alpha)\,\Gamma(\frac{1}{2}\beta)}\right\}. \end{aligned}$$

Finally, equating the results (10) and (11), we obtain the required result (3). The new proof of (3) is completed.

*Remark.*By considering the particular and special cases one can obtain many relationship between the present study and another special functions, see ([9] - [10]).

3 Conclusion

In this note, an attempt has been made to provide a new proof of extended Watson summation formula for the series ${}_4F_3$ due to Kim et al. It should be remarked here that whenever generalized hypergeometric functions reduce to gamma function, the results are important from the application point of view. We conclude this note by remarking that as an application, evaluation of finite single and double integrals are under investigations and will form a part of our subsequent paper in this direction.

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