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Common Fixed-Point Theorems of Caristi-Type Mappings by Using its Absolute Derivative

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Abstract: In this article, we introduce common fixed-point theorems of Caristi-type mappings by using the absolute derivative of the mapping as a generator of its Caristi-type maps. The common fixed-point theorem that we obtain covers the single-valued and set-valued mappings. Some of the examples are given to support usability of our result.

Keywords: Common fixed point, Caristi-type mapping, set-valued, absolute derivative.

1 Introduction

Development of the Caristi's fixed-point theorems [1] has been carried out by researchers through a variety of different ways such as combining the Banach's fixed point theorems to that Caristi's fixed-point theorems [2]. In 1996, Kada-Suzuki and Takahashi used the *w*-distance functions to characterize the Caristi-type mappings [3]. Further, there exist several results involving set-valued mappings into Caristi-type conditions (see [4] [5], [6]).

In 1981, Bhakta and Basu [7] introduced a common fixed-point theorems of Caristi-type mappings on complete metric spaces. In 2010, Obama and Kuroiwa [8] proved the same thing by using ω-distance function which was introduced by Kada et al [3] as a generalization of common fixed-point theorems of Bhakta and Basu. In 2015, Sitthikul and Saejung discussed the result by Obama with weaker assumption [9]. Moreover, L. Samih et al. introduced common fixed-point theorems of Caristi-type mappings in cone metric spaces [10].

Motivated by the above results, in particular, by Bhakta and Basu [7], in this article, we introduce a common fixed-point theorem of Caristi-type mapping by using the absolute derivative as a generator of its Caristi type. In previous articles, we characterized Caristi-type mapping by its absolute derivative but only for one mapping [11]. In this article, we obtain a common fixed-point theorem of Caristi-type mapping for two mappings. We also give some examples to illustrate the main results in this article.

2 Common fixed-point of Caristi-type mappings

For the convenience, in the next we recall the Caristi's fixed-point theorems as follows.

Let (X,d) be a complete metric space and $K \subset X$. Caristi's fixed-point theorem states that each mapping $f : K \longrightarrow K$ satisfies the condition: there exists a lower semi-continuous function $\varphi : K \longrightarrow [0,+\infty)$ such that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \tag{1}$$

for each $x \in X$ has a fixed point.

Some authors have mentioned that a mapping $f : K \longrightarrow K$ is called ***Caristi-type mappings*** if the inequalities (1) is satisfied.

Suppose (X,d) and (Y,ρ) are two metric spaces. Then we use the notation $\mathscr{P}_0(X)$ (resp. $\mathscr{CL}(X)$) as the family of all non-empty (resp. closed) subsets of X.

The mapping $F : X \longrightarrow \mathscr{P}_0(Y)$ is called ***set-valued functions*** where the maps $F(x) \in \mathscr{P}_0(Y)$ for each $x \in X$.

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We say that a point $z \in X$ is a fixed point of F if $z \in F(z)$. The function $f : X \longrightarrow Y$ is said to be ***selection*** of F if $f(x) \in F(x)$ for all $x \in X$.

By using Caristi's fixed-point theorems, in 1989, Mizoguchi and Takahashi [5] resulted in fixed-point theorem for set-valued mappings.

Theorem 1. Let $(X.d)$ be a complete metric space and $F : X \longrightarrow \mathscr{P}_0(X)$ be a set-valued mapping. If there exists $\varphi : X \longrightarrow [0,+\infty]$ is a lower semi continuous function such that for each $x \in X$, there exists $y \in F(x)$ such that

$$d(x,y) \leq \varphi(x) - \varphi(y), \tag{2}$$

then the set-valued map F has a fixed point.

In 1971 Ciric [12] introduced the notion of orbital continuity as follows

Definition 1. Let (X,d) be a metric space and $f : X \to X$ be a mapping. The set

$$\mathscr{O}\{x_0\} = \{x_n = f^n x_0 : n = 1,2,3\cdots\} \tag{3}$$

is called orbit of f at fixed point $x_0 \in X$, where $f^n = \underbrace{f \circ f \circ f \cdots \circ f}_{n-times}$. Then the mapping f is called orbitally continuous if $\lim_{k\to\infty} f^{m_k} x_0 = t$, then $\lim_{k\to\infty} f f^{m_k} x_0 = f(t)$.

Every continuous mapping $f : X \to X$ is orbitally continuous but not conversely [12].

In 1981, Bhakta and Basu [7] proved a common fixed-point theorem of the Caristi-type mapping for two mappings on complete metric spaces. The following theorem in question.

Theorem 2. Let (X,d) be a complete metric space and $f,g : X \longrightarrow X$ be two orbitally continuous mappings on X. If there are two mappings $\varphi, \psi : X \longrightarrow [0,\infty)$ satisfying the condition:

$$d(fx,gy) \leq \varphi(x) - \varphi(fx) + \psi(y) - \psi(gy) \tag{4}$$

for all $x,y \in X$, then f and g have a unique common fixed point.

Theorem 2 has been generalized by Obama [8] with using ω-distance function and then followed by Sitthikul with weaker requirement [9].

3 Absolute derivatives

In 1971, E. Braude introduced the derivative of the metric-valued function with abstract metric domains which is known as "metrically differentiable" (see [13]).

Definition 2. Let (X,d) and (Y,ρ) be two metric spaces and let $p \in X$ be a limit point. The mapping $f : X \longrightarrow Y$ is said ***metrically differentiable*** at p if a real number $f'(p)$ exists with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x,y \in X, x \neq y$ and $0 < d(x,p) < \delta, 0 < d(y,p) < \delta$, then

$$\left| \frac{\rho(f(x),f(y))}{d(x,y)} - f'(p) \right| < \varepsilon. \tag{5}$$

On the other hand, in 1975, K. Skaland defined the weaker form of Braude's definition.

Definition 3. Let (X,d) and (Y,ρ) be a metric spaces and let $p \in X$ be a limit point. The mapping $f : X \longrightarrow Y$ is said ***differentiable*** at p if real number $f'(p)$ exists with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in N_\delta(p)$ then

$$\left| \frac{\rho(f(x),f(p))}{d(x,p)} - f'(p) \right| < \varepsilon. \tag{6}$$

A non-negative real number $f'(p)$ is called *metrically derivative* [13] or *quasiderivative* [14] of the mapping f at the point $p \in X$.

Example 1. Let $X = [-1,1]$. The function $f : [-1,1] \longrightarrow \mathbb{R}$ with $f(x) = |x|$ for each $x \in [-1,1]$ is metrically differentiable on X. For $p = 0 \in [-1,1]$, we obtain

$$f'(0) = \lim_{x\to 0^-} \frac{||x| - 0|}{|x|} = \lim_{x\to 0^-} \frac{|-x|}{|x|} = 1,$$

and

$$f'(0) = \lim_{x\to 0^+} \frac{||x| - 0|}{|x|} = \lim_{x\to 0^+} \frac{|x|}{|x|} = 1.$$

For each $0 < x < 1$ and $-1 < x < 0$, we have $f'(x) = 1$. We know that f is not differentiable in the classical sense at $x = 0$.

Since the value of the derivative is always a non-negative real number, its derivative is called absolute derivative.

Throughout this paper, we use the notation f'_{abs} as an absolute derivative of the function f and a function differentiable in the sense of the metric is called metrically differentiable.

4 Existence of common fixed point

Our first main result modifies the common fixed-point theorem (Theorem 2). The modification is done by replacing two non-negative real functions φ and ψ on Theorem 2 by two absolute derivatives of the functions f and g provided that the function f and g are metrically differentiable.

Theorem 4. Let (X,d) be a complete metric space and $f,g : X \longrightarrow X$ be two orbitally continuous mappings on X. If f and g are metrically differentiablae on X such that the absolute derivative $f'_{abs}, g'_{abs} : X \longrightarrow [0,\infty)$ satisfying the condition:

$$d(fx,gy) \leq f'_{abs}(x) - f'_{abs}(fx) + g'_{abs}(y) - g'_{abs}(gy) \tag{7}$$

for all $x, y \in X$, then f and g have a unique common fixed point.

Proof. We take two points $x_0 \in X$ and $y_0 \in X$ fixed. Thus, we can form the sequences as follows

$$x_1 = fx_0, x_2 = fx_1 = f^2x_0, \cdots x_k = f^kx_0, \cdots$$

and

$$y_1 = gy_0, y_2 = gy_1 = g^2y_0, \cdots y_k = g^ky_0, \cdots$$

for $k \in \mathbb{N}$.

By inequalities (7), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} d(x_i, y_i) = \sum_{i=1}^{n} d(fx_{i-1}, gy_{i-1}) \\
&\leq \sum_{i=1}^{n} \{f'_{abs}(x_{i-1}) - f'_{abs}(fx_{i-1}) + g'_{abs}(y_{i-1}) - g'_{abs}(gy_{i-1})\} \\
&= \sum_{i=1}^{n} \{f'_{abs}(x_{i-1}) - f'_{abs}(x_i) + g'_{abs}(y_{i-1}) - g'_{abs}(y_i)\} \\
&= f'_{abs}(x_0) - f'_{abs}(x_n) + g'_{abs}(y_0) - g'_{abs}(y_n) \\
&\leq f'_{abs}(x_0) + g'_{abs}(y_0). \qquad (8)
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&\sum_{i=1}^{n} d(y_i, x_{i+1}) = \sum_{i=1}^{n} d(gy_{i-1}, fx_i) \\
&\leq \sum_{i=1}^{n} \{f'_{abs}(x_i) - f'_{abs}(fx_i) + g'_{abs}(y_{i-1}) - g'_{abs}(gy_{i-1})\} \\
&= \sum_{i=1}^{n} \{f'_{abs}(x_i) - f'_{abs}(x_{n+1}) + g'_{abs}(y_0) - g'_{abs}(y_n)\} \\
&= f'_{abs}(x_1) - f'_{abs}(x_{n+1}) + g'_{abs}(y_0) - g'_{abs}(y_n) \\
&\leq f'_{abs}(x_1) + g'_{abs}(y_0). \qquad (9)
\end{aligned}$$

From inequalities (8) and (9), we have the inequality as follows

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &\leq \sum_{i=1}^{n} \{d(x_i, y_i) + d(y_i, x_{i+1})\} \\
&\leq f'_{abs}(x_0) + f'_{abs}(x_1) + 2g'_{abs}(y_0).
\end{aligned}$$

Since partial sums $\sum_{i=1}^{n} d(x_i, x_{i+1})$ is a bounded, the series $\sum_{i=1}^{\infty} d(x_i, x_{i+1})$ is convergent. Consequently the sequence non-negative real number $\{d(x_i, x_{i+1})\}$ converges to zero (as $i \longrightarrow \infty$). For each $m, n \in \mathbb{N}$ with $m > n$, we obtain

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \longrightarrow 0$$

as $n \to \infty$. So, the sequence $\{x_n\}$ is a Cauchy sequence on X.

Similarly, in the same way, the sequence $\{y_n\}$ is also Cauchy sequence on X. Since X is complete, each of them is convergent, namely $x_n \to t \in X$ and $y_n \to s \in X$ as $n \to \infty$.

If $\lim_{n\to\infty} f(x_n) = t$ implies $\lim_{n\to\infty} f(fx_n) = ft$ and if $\lim_{n\to\infty} g(x_n) = s$ implies $\lim_{n\to\infty} g(gx_n) = gs$ by f and g are orbitally continuous. It allows the sequence $x_{n+1} \to f(t)$ and $y_{n+1} \to g(s)$ as $n \to \infty$. This gives that $ft = t$ and $gs = s$. So the point t is a fixed point of f and the point s is a fixed point of g. By inequalities (7), we obtain

$$\begin{aligned}
d(t,s) = d(ft, gs) &\leq f'_{abs}(t) - f'_{abs}(ft) + g'_{abs}(s) - g'_{abs}(gs) \\
&= f'_{abs}(t) - f'_{abs}(t) + g'_{abs}(s) - g'_{abs}(s) = 0.
\end{aligned}$$

This means $t = s$. In the other words, the point t is a common fixed point of f and g ($t = ft = gt$).

Suppose f has the other fixed point $u \in X$ ($fu = u$). By applying (7), we have

$$\begin{aligned}
d(u,t) &= d(fu, gt) \\
&\leq f'_{abs}(u) - f'_{abs}(fu) + g'_{abs}(t) - g'_{abs}(gt) \\
&= f'_{abs}(u) - f'_{abs}(u) + g'_{abs}(t) - g'_{abs}(t) \\
&= 0,
\end{aligned}$$

which implies $u = t$ (unique). Hence, the point t is the unique fixed point of f. Similarly, we can show that t is also the unique fixed point of g. This completes the proof. $\square$.

Example 2. Let $X = [0.68, 1]$ endowed by usual metrics. Let $f, g : [0.68, 1] \to \mathbb{R}$ be a real function with $f(x) = x^{\frac{7}{2}}$ and $g(x) = -x + 2$ for all $x \in [0.68, 1]$. It is clear that f and g are orbitally continuous and metrically differentiable on $(0.68, 1)$ with derivative as follows

$$f'_{abs}(x) = \left|\frac{7x^{\frac{5}{2}}}{2}\right| = \frac{7x^3}{2} \qquad (10)$$

and

$$g'_{abs}(x) = |-1| = 1, \qquad (11)$$

respectively. From the equation (10) and (11) we obtain

$$f'_{abs}(fx) = \left|\frac{7x^{\frac{35}{4}}}{2}\right| = \frac{x^{\frac{35}{4}}}{2} \qquad (12)$$

and

$$g'_{abs}(gx) = |-1| = 1. \qquad (13)$$

Now, we investigate as follows: For $x = y = 0.68$, we obtain

$$\begin{aligned}
&|f(0.68) - g(0.68)| = 1.0608 < 1.2147 \\
&= f'_{abs}(0.68) - f'_{abs}f(0.68) + g'_{abs}(0.68) - g'_{abs}g(0.68). \qquad (14)
\end{aligned}$$

For $x = y = 1$, we obtain

$$|f(1) - g(1)| = 0 = f'_{abs}(1) - f'_{abs}f(1) + g'_{abs}(1) - g'_{abs}g(1). \qquad (15)$$

For $x=0.68$ and $y=1$, we obtain

$$\begin{aligned}&|f(0.68)-g(1)| = 0.7408 < 1.2147\\&= f'_{abs}(0.68) - f'_{abs}f(0.68) + g'_{abs}(1) - g'_{abs}g(1).\end{aligned} \quad (16)$$

For all $0.68 < x,y < 0.791$, we have

$$f'_{abs}x = \frac{7x^{\frac{5}{2}}}{2} \geq -y+2 = gy$$

and

$$f'_{abs}f(x) = \frac{7x^{\frac{35}{4}}}{2} \leq x^{\frac{7}{2}} = fx$$

so that

$$\begin{aligned}&f'_{abs}x - f'_{abs}fx + g'_{abs}y - g'_{abs}gy = \frac{7x^{\frac{5}{2}}}{2} - \frac{7x^{\frac{35}{4}}}{2} + 1 - 1\\&> (-y+2) - x^{\frac{7}{2}} = |(-y+2) - x^{\frac{7}{2}}| = |gy - fx|\\&= |fx - gy|.\end{aligned} \quad (17)$$

For all $0.791 < x,y < 0.878$, we have

$$f'_{abs}f(x) = \frac{7x^{\frac{35}{4}}}{2} \leq -y+2 = gy$$

and

$$f'_{abs}x = \frac{7x^{\frac{5}{2}}}{2} \geq x^{\frac{7}{2}} = fx$$

so that

$$\begin{aligned}f'_{abs}fx - f'_{abs}x &= \frac{7x^{\frac{35}{4}}}{2} - \frac{7x^{\frac{5}{2}}}{2}\\&< (-y+2) - x^{\frac{7}{2}} = |(-y+2) - x^{\frac{7}{2}}|.\end{aligned} \quad (18)$$

If both sides are multiplied by the number -1, then we have

$$\begin{aligned}f'_{abs}x - f'_{abs}fx &> (-1)|(-y+2) - x^{\frac{7}{2}}|\\&= |x^{\frac{7}{2}} - (-y+2)| = |f(x) - g(y)|.\end{aligned} \quad (19)$$

For all $0.878 < x,y < 1$, we have $\frac{7x^{\frac{35}{4}}}{2} \geq -y+2 > 0$ and $\frac{7x^{\frac{5}{2}}}{2} \geq x^{\frac{7}{2}} > 0$ so that

$$\begin{aligned}&f'_{abs}x - f'_{abs}fx + g'_{abs}y - g'_{abs}gy = \frac{7x^{\frac{5}{2}}}{2} - \frac{7x^{\frac{35}{4}}}{2} + 1 - 1\\&> (-y+2) - x^{\frac{7}{2}} = |(-y+2) - x^{\frac{7}{2}}| = |f(x) - gy|.\end{aligned} \quad (20)$$

Since the inequality (7) is satisfied, the function f and g have a unique fixed point, namely $1 = f(1) = g(1)$.

Let $\mathscr{F} = \{f \mid f : X \to X\}$ be a collection of all metrically differentiable.

Corolary 1. Let (X,d) be a complete metric space. If two mappings $f,g \in \mathscr{F}$ such that the absolute derivative f'_{abs} and g'_{abs} satisfying the following condition:

$$d(fx,gy) \leq f'_{abs}(x) - f'_{abs}(fx) + g'_{abs}(y) - g'_{abs}(gy)$$

for all $x,y \in X$, then f and g have a unique common fixed point.

Proof By Theorem 4, it is clear f and g have a unique common fixed point $x_0 \in X$. If h is the other mapping in $\mathscr{F}$, then f and h have a unique common fixed point $u \in X$ by Theorem 4. Since $x_0 \in X$ is the unique fixed point of the mapping f, hence $x_0 = u$. So the point x_0 is a unique common fixed point of f, g and h. Of course, x_0 is a unique common fixed point of the mappings in $\mathscr{F}$ because h is an arbitrary mapping in the collection $\mathscr{F}$. □

Theorem 5. Let (X,d) be a complete metric space and $f,g : X \longrightarrow X$ be two mappings on X. If f and g are metrically differentiable on X such that the absolute derivative $f'_{abs}, g'_{abs} : X \longrightarrow [0,\infty)$ satisfying the condition :

$$\begin{aligned}&d(x,y) + d(x,fx) + d(y,gy)\\&\leq f'_{abs}(x) - f'_{abs}(fx) + g'_{abs}(y) - g'_{abs}(gy)\end{aligned} \quad (21)$$

for all $x,y \in X$, then f and g have a unique common fixed point.

Proof Now consider two points $x_0 \in X$ and $y_0 \in X$ as fixed. Then, we can form sequences as follows.

$$x_1 = fx_0, x_2 = fx_1 = f^2x_0, \cdots x_k = f^kx_0, \cdots$$

and

$$y_1 = gy_0, y_2 = gy_1 = f^2y_0, \cdots y_k = g^ky_0, \cdots$$

for $k \in \mathbb{N}$.

By inequalities (21), we obtain

$$\begin{aligned}&\sum_{i=1}^{n} d(x_{i-1},x_i)\\&\leq \sum_{i=1}^{n} \{d(x_{i-1},y_{i-1}) + d(x_{i-1},x_i) + d(y_{i-1},y_i)\}\\&= \sum_{i=1}^{n} \{d(x_{i-1},y_{i-1}) + d(x_{i-1},fx_{i-1}) + d(y_{i-1},gy_{i-1})\}\\&\leq \sum_{i=1}^{n} \{f'_{abs}(x_{i-1}) - f'_{abs}(x_i) + g'_{abs}(y_{i-1}) - g'_{abs}(y_i)\}\\&= f'_{abs}(x_0) - f'_{abs}(x_n) + g'_{abs}(y_0) - g'_{abs}(y_n)\\&\leq f'_{abs}(x_0) + g'_{abs}(y_0).\end{aligned} \quad (22)$$

This implies that the series $\sum_{i=1}^{\infty} d(x_{i-1},x_i)$ is convergent. As the proof in Theorem 4, the sequence $\{x_n\}$ is a Cauchy sequence. Likewise, the sequence $\{y_n\}$ is a Cauchy sequence.

Since metric space X is complete, each of them is convergent, namely $x_n \to u \in X$ and $y_n \to v \in X$ as $n \to \infty$.

If $\lim_{n\to\infty} f(x_n) = u$ implies $\lim_{n\to\infty} f(fx_n) = fu$ and if $\lim_{n\to\infty} g(x_n) = v$ implies $\lim_{n\to\infty} g(gx_n) = gv$ by f and g are orbitally continuous. It allows the sequence $x_{n+1} \to f(u)$ and $y_{n+1} \to g(v)$ as $n \to \infty$. This gives that $fu = u$ and $gv = v$. So the point u is a fixed point of f and the point v is a fixed point of g.
By inequalities (21), we obtain

$$\begin{aligned} d(u,v) &\le d(u,v) + d(u,fu) + d(v,gv) \\ &\le f'_{abs}(u) - f'_{abs}(fu) + g'_{abs}(v) - g'_{abs}(gv) \\ &= f'_{abs}(u) - f'_{abs}(u) + g'_{abs}(v) - g'_{abs}(v) \\ &= 0. \end{aligned}$$

This means $u = v$. In the other words, the point u is a common fixed point of f and g ($u = fu = gu$).

Suppose f has the other fixed point $w \in X$ ($fw = w$). By applying (21), we have

$$\begin{aligned} d(w,u) &\le d(w,u) + d(w,fw) + d(u,gu) \\ &\le f'_{abs}(w) - f'_{abs}(fw) + g'_{abs}(u) - g'_{abs}(gu) \\ &= f'_{abs}(w) - f'_{abs}(w) + g'_{abs}(u) - g'_{abs}(u) \\ &= 0, \end{aligned}$$

which implies $w = u$ (unique). Hence, the point u is the unique fixed point of f. Similarly, we can show that u is also the unique fixed point of g. This completes the proof. □

Example 3.
Let $X = [0.6,1]$ be endowed by usual metrics. Let $f,g : [0.6,1] \to \mathbb{R}$ be a real function with $f(x) = x^2$ and $g(x) = x^3$ for all $x \in [0.6,1]$. It is clear that f and g orbitally continuous and metrically differentiable on $[0.6,1]$ with absolute derivative as follows

$$f'_{abs}(x) = |2x| = 2x \tag{23}$$

and

$$g'_{abs}(x) = |3x^2| = 3x^2, \tag{24}$$

respectively. From the equation (23) and (24) we obtain

$$f'_{abs}(fx) = |2x^2| = 2x^2 \tag{25}$$

and

$$g'_{abs}(gx) = |3x^6| = 3x^6, \tag{26}$$

respectively. Now, we investigate as follows: From (23) and (25) we have that

$$f'_{abs}(x) - f'_{abs}(fx) = 2x - 2x^2 \ge 0$$

for all $x \in [0.6,1]$.

From (24) and (26) we have that

$$g'_{abs}(y) - g'_{abs}(gy) = 3y^2 - 3y^6 \ge 0$$

for all $y \in [0.6,1]$.

Since $(x-x^2) \ge 0$ and $(y-y^3) \ge 0$ for all $x,y \in [0.6,1]$, we obtain

$$\begin{aligned} |x - fx| &= |x-x^2| = (x-x^2) \\ &\le 2(x-x^2) = f'_{abs}(x) - f'_{abs}(fx) \end{aligned} \tag{27}$$

and

$$\begin{aligned} |y - gy| &= |y-y^3| = (y-y^3) \\ &\le 3(y^2-y^6) = g'_{abs}(y) - g'_{abs}(gy) \end{aligned} \tag{28}$$

for all $x,y \in [0.6,1]$.

Further, we consider the form $|x-y| + |x-fx| + |y-gy| = |x-y| + (x-x^2) + (y-y^3)$ for all $x \ne y \in [0.6,1]$.
If $x-y > 0$, then we obtain

$$\begin{aligned} |x-y| + |x-fx| + |y-gy| &= (x-y) + (x-x^2) + (y-y^3) \\ &= (2x-x^2) - y^3 < (2x-2x^2) + (y^2-y^3) \\ &< (2x-2x^2) + (y^2-y^6) \\ &< 2(x-x^2) + 3(y^2-y^6) \\ &= f'_{abs}(x) - f'_{abs}(fx) + g'_{abs}(y) - g'_{abs}(gy), \end{aligned} \tag{29}$$

for all $x \ne y \in [0.6,1]$ by inequalities (27) and (28).

If $x-y < 0$, then we obtain

$$\begin{aligned} |x-y| + |x-fx| + |y-gy| &= (-x+y) + (x-x^2) + (y-y^3) \\ &= -x^2 + (2y-y^3) < (x-x^2) + (2y-y^3) \\ &< 2(x-x^2) + 3(y^2-y^6) \\ &= f'_{abs}(x) - f'_{abs}(fx) + g'_{abs}(y) - g'_{abs}(gy), \end{aligned} \tag{30}$$

for all $x \ne y \in [0.6,1]$ by inequalities (27) and (28). Thus, all of the calculations above were fulfilling the inequality (21) so that f and g have common fixed point $z = 1 = f(1) = g(1)$.

5 Common fixed-point for set-valued functions

Next, we consider common fixed-point theorems of Caristi-type mappings for set-valued mappings. To the proof of theorem below, we shall use the following Lemma.

Lemma 5. [15] Let (X,d) be a metric space and let $F : X \longrightarrow \mathscr{CL}(X)$ be an upper semi-continuous. Suppose $\{x_n\}$ is a sequence in X such that $x_{n+1} \in Fx_n$. If the sequence $\{x_n\}$ converges to $u \in X$, then $u \in Fu$.

Theorem 6. Let (X,d) be a complete metric space and $F,G : X \longrightarrow \mathscr{CL}(X)$ be two upper semi-continuous set-valued mappings on X. If there exists selection $f \in F$ and $g \in G$ are metrically differentiable on X such that the absolute derivative $f'_{abs}, g'_{abs} : X \longrightarrow [0,\infty)$ satisfying the following condition: For each two points $x,y \in X$ there exists $u \in Fx$ and $v \in Gy$ such that

$$d(u,v) \leq f'_{abs}(x) - f'_{abs}(u) + g'_{abs}(y) - g'_{abs}(v), \qquad (31)$$

then F and G have a unique common fixed point.

Proof We take two points $x_0 \in X$ and $y_0 \in X$ fixed. Thus, we can form sequences as follows.

$$x_1 \in Fx_0, x_2 \in Fx_1, \cdots x_k \in Fx_{k-1}, \cdots$$

and

$$y_1 \in Gy_0, y_2 \in Gy_1, \cdots y_k \in Gy_{k-1}, \cdots$$

for $k \in \mathbb{N}$. In general we have

$$x_n \in Fx_{n-1} \quad \textit{and} \quad y_n \in Gy_{n-1}$$

for all $n \in \mathbb{N}$.

Suppose two points x_{i-1}, y_{i-1} are arbitrary in X, we can choose a point $x_i \in Fx_{i-1}$ and a point $y_i \in Gy_{i-1}$. By inequalities (31), we obtain

$$d(x_i,y_i) \leq f'_{abs}(x_{i-1}) - f'_{abs}(x_i) + g'_{abs}(y_{i-1}) - g'_{abs}(y_i) \qquad (32)$$

for all $i \in \mathbb{N}$.

Suppose two points x_i, y_{i-1} are arbitrary in X, we can choose a point $x_{i+1} \in Fx_i$ and a point $y_i \in Gy_{i-1}$. By inequalities (31), we obtain

$$d(x_{i+1},y_i) \leq f'_{abs}(x_i) - f'_{abs}(x_{i+1}) + g'_{abs}(y_{i-1}) - g'_{abs}(y_i) \qquad (33)$$

for all $i \in \mathbb{N}$.

Suppose two points x_i, y_i is arbitrary in X, we can choose a point $x_{i+1} \in Fx_i$ and a point $y_{i+1} \in Gy_i$. By inequalities (31), we obtain

$$d(x_{i+1},y_{i+1}) \leq f'_{abs}(x_i) - f'_{abs}(x_{i+1}) + g'_{abs}(y_i) - g'_{abs}(y_{i+1}) \qquad (34)$$

for all $i \in \mathbb{N}$.

From inequality (32), (33) and (34) and similar way to proof of Theorem 4, both sequences $\{x_n\}$ and $\{y_n\}$ are Chauchy sequences.

Since X is complete metric spaces, each of them is convergent, namely, $x_n \to u \in X$ and $y_n \to v \in X$ as $n \to \infty$. Since F and G are upper semi-continuous, by Lemma 5, we have $u \in Fu$ and $v \in Gv$. From inequalities (31), we obtain

$$d(u,v) \leq f'_{abs}(u) - f'_{abs}(u) + g'_{abs}(v) - g'_{abs}(v) = 0.$$

This means $u = v$. Hence, $u \in Fu \cap Gu$.

Suppose F has the other fixed point $w \in X$ $(w \in Fw)$. By applying (31), we have

$$d(w,u) \leq f'_{abs}(w) - f'_{abs}(w) + g'_{abs}(u) - g'_{abs}(u) = 0.$$

So, $w = u$. In the other words, the point u is the only fixed point of F.

Suppose $t \in X$ satisfies $t \in Gt$. By applying (31) again, we have

$$d(u,t) \leq f'_{abs}(u) - f'_{abs}(u) + g'_{abs}(t) - g'_{abs}(t) = 0.$$

So, $t = u$. In the other words, the point u is the only fixed point of G. Thus the point u is a unique common fixed point of F and G. This completes the proof. □

Example 4. Let $X = [0,1]$ be endowed by usual metrics. Let $F,G : [0,1] \to \mathbb{R}$ be an interval-valued function with $Fx = [x^2 - x, x]$ and $Gx = [\frac{1}{2}x^2 + \frac{1}{2}, 1]$ for all $x \in [0,1]$. We choose selections $fx = (x^2 - x) \in Fx$ and $gx = (\frac{1}{2}x^2 + \frac{1}{2}) \in Gx$ It is clear that f and g are metrically differentiable on $[0,1]$ with absolute derivative

$$f'_{abs}x = |2x-1| = 2x-1, \quad g'_{abs}x = |x| = x \qquad (35)$$

since $x \in [1,2]$.

For each $x,y \in X$, we choose the points $u \in Fx = [x^2 - x, x]$ and $v \in Gy = [\frac{1}{2}y^2 + \frac{1}{2}, 1]$ such that

$$x^2 - x \leq u \leq x, \quad \frac{1}{2}y^2 + \frac{1}{2} \leq v \leq 1. \qquad (36)$$

Now, we calculate as follows.

Let $(u-v) \geq 0$, $v \leq u \leq x$, $x \leq y$. From (35) and (36) we obtain

$$\begin{aligned}|u-v| = u-v = (3u-2u) - v &\leq (3x-2u) - v \\ &\leq (2x-2u) + (x-v) \\ &\leq (2x-2u) + (y-v) \\ &= (2x-1) - (2u-1) + (y-v) \\ &= f'_{abs}x - f'_{abs}u + g'_{abs}y - g'_{abs}v.\end{aligned}$$

Let $(u-v) \leq 0$, $u \leq v \leq x$, $x \leq y$. From (35) and (36) we obtain

$$\begin{aligned}|u-v| = v-u = v + (u-2u) &\leq x + (x-2u) \\ &\leq (2x-2u) + (y-v) \\ &= (2x-1) - (2u-1) + (y-v) \\ &= f'_{abs}x - f'_{abs}u + g'_{abs}y - g'_{abs}v.\end{aligned}$$

Let $(u-v) \geq 0$, $v \leq u \leq y$, $y \leq x$. From (35) and (36) we obtain

$$\begin{aligned}|u-v| = u-v = (3u-2u) - v &\leq (3y-2u) - v \\ &\leq (2y-2u) + (y-v) \\ &\leq (2x-2u) + (y-v) \\ &= (2x-1) - (2u-1) + (y-v) \\ &= f'_{abs}x - f'_{abs}u + g'_{abs}y - g'_{abs}v.\end{aligned}$$

Let $(u-v) \leq 0$, $u \leq v \leq y$, $y \leq x$. From (35) and (36) we obtain

$$\begin{aligned} |u-v| = v-u = v+(u-2u) &\leq y+(y-2u) = 2y-2u \\ &\leq (2x-2u) \leq (2x-1)-(2u-1)+(y-v) \\ &= f'_{abs}x - f'_{abs}u + g'_{abs}y - g'_{abs}v. \end{aligned}$$

Thus, all of the calculations above are fulfilling the inequality (31) and the point $z = 1 \in F(1) \cap G(1)$ is unique common fixed point of set-valued F and G.

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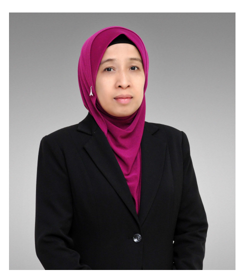

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